RATIONAL POINTS AND UNIFORMITY

WORKING GROUP, IMJ-PRG, TN

Practical information:

- Place: S.U., Campus Jussieu, room 15-16-413.
- Schedule: Thursday 11:00 - 13:00 (two 45 min talks with 30 min break).

The aim of the working group is to study recent uniformity results about rational points of varieties over number fields or geometric (often more tractable) analogues of these questions. In the first part of this working group, we will focus on two kinds of results, which have interpretations in terms of "paucity" of rational points on modular varieties and, this way, are closely related to celebrated conjectures of Lang [Lan91].
Let k be a number field,

- Genus ≥ 2 curves: Let X be a smooth, projective, geometrically connected curve of genus $g \geq 2$ over k. According to the Mordell conjecture [Fal83], $X(k)$ is finite. The weak form of the Lang conjecture about the non-Zariski density of k-rational points on varieties of general type predicts that $|X(k)|$ should be uniformly bounded only in terms of g and k and the strong form of this conjecture predicts that, up to finitely many exceptional X (depending on k), this uniform bound should depend only on g. Mazur asked for a weaker bound on $X(k)$, involving also the rank of the Jacobian of X. In a series of very recent works [DGH19, Gao20, DGH20b], Dimitrov, Gao and Habegger answered positively this question of Mazur. One part of the working group will be devoted to their proof.
- Torsion on abelian varieties and generalizations: Let X be an elliptic curve over k. Manin [Man69] proved that for every prime ℓ the k-rational ℓ-torsion on X is uniformly bounded only in terms of ℓ and k and Frey showed that this bound was depending only in ℓ and the degree $d = [k : \mathbb{Q}]$. The results of Manin (resp. Frey) is a direct consequence of the Mordell conjecture (resp. the Mordell-Lang conjecture [Fal91], [Fal94]) plus the fact that the genus (resp. the geometric gonality) of $Y_1(\ell^n)$ is ≥ 2 for $n \gg 0$ (resp. tends to infinity with n) and the Mordell-Weil theorem. The abc conjecture [Oes] predicts that the whole k-rational torsion of X should be uniformly bounded only in terms of k that is, $Y_1(n)(k)$ should be empty for $n \gg 0$. This was shown by Mazur [Maz77] for $k = \mathbb{Q}$ and extended to arbitrary k (with a bound depending only on the degree $d = [k : \mathbb{Q}]$) by Merel [Mer96], the latter stronger form does not seem to follow from any known form of the abc conjecture over number fields. But it follows from the Lang's conjecture [Maz98]. Moreover Lang's conjecture has similar consequences for the modular curves $X_0(N)$.

  Further uniformity consequences of Lang's conjecture. Given $g \geq 2$ and $d \geq 1$, a curve C of genus g defined over a number field K of degree d, one has $|C(K)| \leq B(g, d)$ ([Pac97]. In the same spirit, given integers d, e, $g \geq 1$, there exists $B(e, d, g)$ such that for any number field K of degree $\leq d$, and any curve C over K of genus g and gonality $> 2e$, the set of points of C of degree $\leq e$ over K is finite and of cardinality $\leq B(e, d, g)$. Concerning the torsion of elliptic curves, for any integer $d \geq 1$, there is $M(d) \geq 1$ such that for any number field K of degree$d$, the number of isomorphism classes of elliptic curves over K having a torsion subgroup (over K) of order > 18 is at most $M(d)$ (that does not follow from Merel). Similarly for isogenies, there is a number B (perhaps $B = 20$) such that for any number field K, the number of elliptic curves over K (up to $\bar{K}$-isomorphism) having $> B$ K-rational cyclic subgroups is finite.

  Cadoret and Tamagawa extended the results of Manin and Frey to one-dimensional families of higher-dimensional abelian varieties. Their approach actually allows to handle any kind of uniformity results that can be encoded in ℓ-adic representations of the étale fundamental group of the base curve (e.g. jumping locus of rank of Jacobian, Néron-Severi group, dimension of motivated Galois group, Brauer groups etc.) [ref]. Assuming the weak form of the Lang conjecture, it reduces such uniformity results to proving (seemingly very hard) geometric results along some towers of étale covers of the parametrizing variety which play the part of the modular curves $Y_1(\ell^n)$ in Manin and Frey's arguments. That it is reasonable to expect stronger uniformity results in the spirit of those of Mazur and Merel is much less clear though Vojta proposed an higher-dimensional analogue of the abc-conjecture that have some consequences in this direction [AVA17]. Another part of the working group will be devoted to the approach of Cadoret-Tamagawa.

Tentative program:

Part I: Consequences of the Lang conjectures on uniformity results for rational points
Part II: Dimitrov-Gao-Habegger
Part III: Cadoret-Tamagawa

- Session 1: Introduction:
  - (45 min) Expand the above introductory sketch explaining the heuristic (modular interpretation plus "hard" diophantine conjectures) motivating the uniformity problems. State the main results to be studied in the working group (Dimitrov–Gao–Habbeger, Cadoret–Tamagawa).
  - (45 min) Present the detailed plan of the working group.

Part I: Consequences of the Lang conjectures on uniformity results for rational points

- Session I-1: Quick recollection on basics about Kodaira dimension. State the various (arithmetic and geometric) forms of the Lang conjecture (See for instance [HS00] or [AVA18]). Review of what is known ([Fal94] for the arithmetic form; give the application (with proof!) to points of bounded degree on curves of high gonality [Fre94], [Bog77] for the geometric form in the case of surfaces, more ??). Maybe review what is known for classical moduli spaces ($\mathcal{M}_{g,++}$ - [Log03] and ref. therein, $\mathcal{A}_{g,++}$ - [AVA18] and ref. therein, locally symetric spaces - [Bru20] and ref. therein *etc.*)

- Sessions I-2, (3?): [CHM97]
  - [CHM97, 1.3] - correlation theorem + Lang conjecture imply uniformity theorem (Thm. 1.1, Thm. 1.2)
  - [CHM97, 1.4] - preparation for the proof of the correlation theorem (Thm. 1.3)
  - [CHM97, 3, 4? 5] - sketch of proof of the correlation theorem (Thm. 1.3)

- Session I-4, (5?): the aim of this session is to give an overview of the hyperbolicity results for moduli spaces of abelian varieties with level-ℓ structure (and more generally locally symmetric spaces). The main ref. are [AVA18], [Bru20] but the later uses possibly too much Hodge theory to be treated thoroughly so we propose to focus rather on [AVA18], which follows a strategy of Mumford [Mum77] (might be a tool for generalization of CT) and introduce basics about $\mathcal{A}_g$. Alternatively, could be treated in one session if we have already devoted one to basics about $\mathcal{A}_g$ (in Part II?).

- Session I-6: Explain the proof of the main theorem of [Bog77]: on a smooth projective surface of general type with $c_1^2 - c_2 > 0$, curves with fixed geometric genus form a bounded family (in particular, the geometric form of the Lang conjecture holds for such surfaces). See also the possible more accessible exposition in [Deb04, §7].

Part II: Dimitrov–Gao–Habegger

See the webpage of Ziyang Gao for the latest versions of [DGH20b], [DGH20a], [Gao20]. They may be different from the arXiv versions.

- Session II-1: Statement of DGH. Basics on the Height Machine (Weil height and Néron–Tate height); cf. [HS00, Part B, B1–B5]. Comparison of the Weil height and the Néron–Tate height for families of abelian varieties; cf. [Sil83, Thm A] and/or [DGH20b, Appendix A].

- Session II-2: Reduce Mordell conjecture to proving the finiteness of large points. Bound the number of large points on curves. Statement, basic setting-up, the uniform Mumford inequality and the uniform Vojta inequality *assuming the zero estimates (see Session II-3).* [HS00, Part E] and/or [BG06, Chapter 11], [dD97].

- Session II-3: The need to analyse "zero estimates in Diophantine approximation", *i.e.* one or all of Dyson's Lemma, Roth's Lemma, Faltings product Theorem.

- Session II-4: Reduce Mazur's conjectural bound to a height inequality in the universal abelian variety [DGH20a, Theorem 2.2]. The key point is the "new Gap Principle" [DGH20b, Proposition 7.1]. [DGH20b, §6, §7, §8] and/or [DGH20a, §2], ([DGH19, §2, §3] may help).

- Session II-5: Preparation to the proof of the height inequality: An analytic construction of the universal abelian variety $\mathfrak{A}_g$, the Betti map and the Betti form; see [DGH20b, §2]. Definition of non-degenerate subvarieties; see [DGH20b, Definition 1.5]. Proof of the height inequality on non-degenerate subvarieties *that are projective*; an extra note will be given as a reference for this last part.

- Session II-6: Proof of the height inequality for non-degenerate subvarieties of the universal abelian varieties. [DGH20b, §3, §4, §5].

- Session II-7: Non-degeneracy of $\mathcal{D}_M(\mathfrak{C}_S^{[M+1]})$ part 1: Reduction to unlikely intersections on $\mathfrak{A}_g$. Bi-algebraic system on $\mathfrak{A}_g$, weak Ax-Schanuel for $\mathfrak{A}_g$ and its application; cf. [Gao20, §5.1, §5.2, §6].[1] Special subvarieties and weakly special subvarieties, proof of the geometric description of bi-algebraic subvarieties of $\mathfrak{A}_g$; references given later.

- Session II-8: Non-degeneracy of $\mathcal{D}_M(\mathfrak{C}_S^{[M+1]})$ part 2: Solving the unlikely intersection problem. A finiteness result *à la Bogomolov*, upper bound and lower bound on the dimension, end of the proof. [Gao20, §7, §8, §10].

Part III: Cadoret–Tamagawa

- Session III-1: Statement of CT, first applications to uniform boundedness problems and general strategy.
  - (45 min) Statement of [CT13, Thm. 1.1]. Fundamental examples of GLP representations:
    - ℓ-adic cohomology (recall the smooth proper base change theorem). Mention briefly the applications of [Cad13], [CC20]. Treat in more details the application to uniform boundedness of ℓ-primary torsion and ℓ-primary cyclic subgroups (in particular in abelian varieties) [CT13, §4.1].
    - Extension of constant by semisimple. See [Cad20]. Mention briefly the applications there (maybe restrict to [Cad20, §6.2]).

    If times allow the Applications of [Cad13], [CC20], [Cad20] *etc.* could be treated in session III-?.

  - (45 min) General strategy: introduction of abstract modular schemes (*i.e.* the $\mathcal{X}_n$ of [CT13, §3.3]) [CT13, §3.1]. Explain the general strategy. Say somewhere for higher-dimensional base varieties, arithmetic results are conditional under projective variant of the weak Lang conjecture (a priori much weaker than the classical form). State [CT13, Thm. 3.3]. [CT13, §. 3.2.3, §3.3].

- Session III-2: Curves - Main geometric result [CT12].
  - (45 min) Proof of [CT13, Thm. 3.4]. [CT12, §3.3.1, §3.3.2 §2].
  - (45 min) Proof of [CT13, Thm. 3.4] assuming [CT13, Thm. 2.1]. State [CT13, Thm. 2.1] and treat [CT13, §3.2].

- Sessions III-3
  - (45 min) Proof of [CT13, Thm. 2.1];
  - (45 min) Proof of [CT13, Thm. 1.3].

- Sessions III-4 Complements. The following can be treated partially:
  - (45 min) More details about the applications in [Cad13], [CC20], [Cad20];
  - (45 min) Modulo-ℓ results: make a brief overview of [CT19b], [CT16], [CT19a], [EHK12].

- Session III-5 (and 6?) [Anna]: Higher-dimensional base varieties. Recall the general strategy is working for arbitrary dimensional base variety but arithmetic applications are conditional under Lang's conjectures.
  - Galois case with full level-ℓ structure for surfaces (based on Kodaira classification)

[1] For the bi-algebraic system on $\mathfrak{A}_g$, explain the geometric description of bi-algebraic subvarieties without giving the proof here. For weak Ax-Schanuel, explain the statement without giving the proof.

– Case of product-quotient surfaces.

Suggestion pour la chronologie des exposés (environ 18 séances):

Intro
I-1
I-2
II-1 à 8 (séance×7 ; sessions 5 et 6 faites en une seule séance)
II-2
II-3
II-4
II-5 et II-6
II-7
II-8

III-1 (séance×1)
I-4 (séance×1)
III-2 à 4 (séance×3)
I-6 (séance×1)
III-5 et 6 (séance×2)

References

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