NOTES OF SESSION III-2

Contents

1. Étale cohomology

1.1. **The étale morphism.** A morphism of finite type between schemes $f : X \to Y$ is called *étale* if it is flat and the diagonal morphism $\Delta : X \to X \times_Y X$ is open and closed. This is understood as an analogue of local homeomorphism in topology.

Fact 1.1. *Suppose $f : X \to Y$ is a morphism of finite type. The followings are equivalent:*

(1) *f is étale;*
(2) *f is smooth of relative dimension 0;*
(3) *f is flat and unramified.*

Example 1.2. Open immersion is étale.

A *finite étale* morphism is an étale morphism which is also a finite morphism. This is understood as an analogue of finite covering map in topology.

Example 1.3. Suppose $Y = \operatorname{Spec} K$, where K is a field. Then $X \to Y$ is étale if and only if $X \cong \operatorname{Spec} R$, where R is a finite direct product of finite separable field extensions of K. In particular, if $K = \overline{K}$ is algebraically closed, then R is a finite direct product of K.

Example 1.4. Suppose L/K is a finite extension of number fields, $X = \operatorname{Spec} \mathcal{O}_L$, $Y = \operatorname{Spec} \mathcal{O}_K$, then $X \to Y$ is étale if and only if L/K is unramified at all finite places.

Fact 1.5. *Étale morphism has the following properties:*

(1) *Composition of étale morphisms is étale;*
(2) *Étale morphism is stable under base change, i.e. if $X \to Y$ is étale, $Z \to Y$ is any morphism, then the fiber product $X \times_Y Z \to Z$ is étale;*
(3) *If g and $g \circ f$ are étale, then f is étale.*

1.2. **The étale fundamental group.**

1.2.1. First let's recall the fundamental group in topology. Let X be a topological space which is path-connected, locally path-connected, and semi-locally simply connected. Fix a point $x \in X$, then there is the *topological fundamental group* $\pi_1^{\mathrm{top}}(X, x)$ with respect to the base point x, which consists of the paths on X starting and ending at x modulo homotopic equivalence. Since X is path-connected, different choices of x yield (non-canonically) isomorphic $\pi_1^{\mathrm{top}}(X, x)$, hence sometimes we simply write $\pi_1^{\mathrm{top}}(X)$ instead.

On the other hand, we may define the fundamental group in terms of deck transformation groups. For a connected topological space X' and a covering map $f : X' \to X$, the deck transformation group

$\mathrm{Aut}(X'/X)$ which consists of homeomorphisms $\sigma : X' \to X'$ such that $f \circ \sigma = f$. The f is called a *Galois cover*, if the $\mathrm{Aut}(X'/X)$-action on $f^{-1}(x)$ is transitive for some ($\Leftrightarrow$ for all) $x \in X$. If f is a finite covering map, it is equivalent to $\# \mathrm{Aut}(X'/X) = \deg f$.

Suppose $f_i : X_i' \to X$, $i = 1, 2$ are two Galois covers, and $g : X_1' \to X_2'$ is such that $f_1 = f_2 \circ g$. Then g is also a Galois cover, induces a surjective group homomorphism $\mathrm{Aut}(X_1'/X) \twoheadrightarrow \mathrm{Aut}(X_2'/X)$ whose kernel is $\mathrm{Aut}(X_1'/X_2')$, which maps an element $\sigma_1 \in \mathrm{Aut}(X_1'/X)$ to the unique element $\sigma_2 \in \mathrm{Aut}(X_2'/X)$ such that $\sigma_2(g(x_1')) = g(\sigma_1(x_1'))$ for some ($\Leftrightarrow$ for all) $x_1' \in X_1'$.

If $f : X' \to X$ is a Galois cover, fix a point $x \in X$ and its preimage $x' \in X'$, it induces a group homomorphism $\pi_1^{\mathrm{top}}(X, x) \to \mathrm{Aut}(X'/X)$ given by $[\gamma] \mapsto \sigma$ where $\sigma \in \mathrm{Aut}(X'/X)$ is the unique element which sends $\gamma'(0) = x'$ to $\gamma'(1)$, here γ' is the unique lifting of γ to X' such that $\gamma'(0) = x'$.

There is the universal cover $\widetilde{f} : \widetilde{X} \to X$, in the sense that for any covering map $f : X' \to X$, any point $x \in X$ and its preimages $x' \in X'$, $\widetilde{x} \in \widetilde{X}$, there exists a unique map $g : \widetilde{X} \to X'$ such that $\widetilde{f} = f \circ g$ and such that $g(\widetilde{x}) = x'$. It is a Galois cover, and the induced group homomorphism $\pi_1^{\mathrm{top}}(X, x) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{X}/X)$ is in fact an isomorphism.

Without introducing the universal cover of X, we may consider the category of finite Galois covers of X. It is cofiltered, but there are too many morphisms in it; to restrict the number of morphisms, we need to introduce base points.

Suppose $f_i : X_i' \to X$, $i = 1, 2$ are two finite covering maps, such that there exist maps $g : X_1' \to X_2'$ such that $f_1 = f_2 \circ g$. The number of such g may be ≥ 2, but if we fix a point $x \in X$ as well as its preimages $x_i' \in X_i'$, $i = 1, 2$, and require moreover that $g(x_1') = x_2'$, then the number of such g is ≤ 1, moreover, if f_1 is Galois, then the number of such g is exactly $= 1$. Therefore, if we fix a point $x \in X$ and consider the category of finite Galois covers of X endowed with a preimage of x, then for any two objects in it, there are at most one morphism between them, which allows us to define the *algebraic fundamental group* to be

$$\pi_1^{\mathrm{alg}}(X, x) := \varprojlim_{(X', x')/(X, x) \text{ finite Galois cover}} \mathrm{Aut}(X'/X).$$

For each (X', x') appeared in the above inverse limit, there is a map $\pi_1^{\mathrm{top}}(X, x) \to \mathrm{Aut}(X'/X)$, also, the map $g : (\widetilde{X}, \widetilde{x}) \to (X', x')$ induces $\mathrm{Aut}(\widetilde{X}/X) \to \mathrm{Aut}(X'/X)$. These maps are compatible with the inverse system, hence we obtain group homomorphisms

$$\pi_1^{\mathrm{top}}(X, x) \to \pi_1^{\mathrm{alg}}(X, x) \quad \text{and} \quad \mathrm{Aut}(\widetilde{X}/X) \to \pi_1^{\mathrm{alg}}(X, x),$$

these maps are compatible with the isomorphism $\pi_1^{\mathrm{top}}(X, x) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{X}/X)$, and that $\pi_1^{\mathrm{alg}}(X, x)$ is in fact the profinite completion of them.

It's clear that π_1^{top} is a functor, namely, the $\varphi : (X, x) \to (Y, y)$ induces $\pi_1^{\mathrm{top}}(X, x) \to \pi_1^{\mathrm{top}}(Y, y)$ by $[\gamma] \mapsto [\varphi \circ \gamma]$.

1.2.2. Now back to the algebraic geometry setting. Let X be a connected scheme. A morphism $f : X' \to X$ is called a *Galois cover* if X' is connected, f is finite étale, and $\# \mathrm{Aut}(X'/X) = \deg f$, where the automorphism group $\mathrm{Aut}(X'/X)$ consists of isomorphisms $\sigma : X' \to X'$ such that $f \circ \sigma = f$. If $x : \mathrm{Spec}\, \overline{k} \to X$ is a geometric point of X, by abuse of notation, denote by $f^{-1}(x)$ the geometric points $x' : \mathrm{Spec}\, \overline{k} \to X'$ of X' such that $x = f \circ x'$. If $f : X' \to X$ is a Galois cover, then the $\mathrm{Aut}(X'/X)$-action on $f^{-1}(x)$ is transitive for all geometric points x of X.

Similar to topology setting, if $f_i : X_i' \to X$, $i = 1, 2$ are two Galois covers, such that there exist morphisms $g : X_1' \to X_2'$ such that $f_1 = f_2 \circ g$, fix a geometry point x of X as well as its preimages x_i' in X_i', $i = 1, 2$, and require moreover that $g \circ x_1' = x_2'$, then the number of such g is exactly $= 1$. Therefore, if we fix a geometry point x of X, we can define the *étale fundamental group* to be

$$\pi_1^{\text{ét}}(X, x) := \varprojlim_{(X', x')/(X, x) \text{ Galois cover}} \mathrm{Aut}(X'/X).$$

Since X is connected, different choices of x yield (non-canonically) isomorphic $\pi_1^{\text{ét}}(X, x)$ (see [Mil80], Chapter I, Remark 5.1, and [FK88], Appendix A I.2), hence sometimes we simply write $\pi_1^{\text{ét}}(X)$ instead.

The $\pi_1^{\text{ét}}$ is a functor (see [FK88] Appendix A I.3), namely, a morphism $\varphi : (X, x) \to (Y, y)$ induces $\pi_1^{\text{ét}}(X, x) \to \pi_1^{\text{ét}}(Y, y)$, which is defined as follows. For any Galois cover $f : (Y', y') \to (Y, y)$, the $X \times_Y Y' \to X$ is a finite étale morphism, as well as $(X \times_Y Y')_0 \to X$, where $(X \times_Y Y')_0$ is the connected component of $X \times_Y Y'$ containing the geometric point $x \times_y y' : \mathrm{Spec}\, \overline{k} \to X \times_Y Y'$. There exists a Galois cover $g : (X', x') \to (X, x)$ such that it factors through $(X \times_Y Y')_0 \to X$ and such that $x \times_y y'$ factors through x', therefore we may define group homomorphism $\pi_1^{\text{ét}}(X,x) \twoheadrightarrow \operatorname{Aut}(X'/X) \to \operatorname{Aut}(Y'/Y)$. Taking inverse limit we obtain $\pi_1^{\text{ét}}(X,x) \to \pi_1^{\text{ét}}(Y,y)$.

Example 1.6. If $X = \operatorname{Spec} K$ where K is a field, then X'/X is a Galois cover if and only if $X' = \operatorname{Spec} K'$ where K' is a finite Galois extension of K, and $\operatorname{Aut}(X'/X) = \operatorname{Gal}(K'/K)$. Therefore $\pi_1^{\text{ét}}(\operatorname{Spec} K) = \operatorname{Gal}(\overline{K}/K)$ is the absolute Galois group of K.

Fact 1.7. *The category $X_{\text{fét}}$ of finite étale morphisms over X is equivalent to the category of finite $\pi_1^{\text{ét}}(X,x)$-sets, given by $(f : X' \to X) \mapsto f^{-1}(x)$.*

Fact 1.8. *If X is a geometrically connected variety over a field K, then there is an exact sequence of groups:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1^{\text{ét}}(X_{\overline{K}}) & \longrightarrow & \pi_1^{\text{ét}}(X) & \longrightarrow & \pi_1^{\text{ét}}(\operatorname{Spec} K) & \longrightarrow & 0. \\ & & & & & & \| & & \\ & & & & & & \operatorname{Gal}(\overline{K}/K) & & \end{array}$$

Moreover, if K is a subfield of $\mathbb{C}$, then a geometric point $x : \operatorname{Spec} \mathbb{C} \to X$ induces a natural isomorphism $\pi_1^{\text{ét}}(X_{\overline{K}}, x) \xrightarrow{\sim} \pi_1^{\text{alg}}(X(\mathbb{C}), x)$.

1.3. **The étale sheaf.**

1.3.1. *Étale topology.* Suppose X is a scheme. Define $X_{\text{ét}}$ to be the category of étale morphisms over X, more precisely, it has:

- Objects: $U \to X$ étale morphism,
- Morphisms: $U \to V$ such that

  $$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

  commutes,

with the étale topology

- For $U \in X_{\text{ét}}$, $\{\phi_i : U_i \to U\}$ is a cover of U if each ϕ_i is a morphism in $X_{\text{ét}}$ such that $U = \bigcup_i \phi_i(U_i)$ as topological spaces,

which satisfies the axioms of so-called Grothendieck topology:

(1) If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a cover of U;
(2) Pullback of cover is cover, i.e. if $\{U_i \to U\}$ is a cover of U, and $V \to U$ is a morphism in $X_{\text{ét}}$, then $\{U_i \times_U V \to V\}$ is a cover of V;
(3) Composition of covers is cover, i.e. if $\{U_i \to U\}$ is a cover of U, and for each i, $\{U_{ij} \to U_i\}$ is a cover of U_i, then $\{U_{ij} \to U_i \to U\}$ is a cover of U.

1.3.2. *Presheaf and sheaf.* A presheaf $\mathcal{F} \in \mathbf{Psh}(X_{\text{ét}}, \mathbf{Set})$ is just a contravariant functor $\mathcal{F} : X_{\text{ét}}^{\text{op}} \to \mathbf{Set}$, called an *étale presheaf*. If $V \to U$ is a morphism in $X_{\text{ét}}$, the induced morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is usually denoted by $|_V$. If $\mathcal{U} = \{U_i \to U\}$ is a cover, define the zeroth Čech cohomology

$$\check{H}^0(\mathcal{U}, \mathcal{F}) := \operatorname{Eq}\left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j) \right),$$
$$(s_i)_i \mapsto (s_i|_{U_i \times_U U_j})_{i,j},$$
$$(s_i)_i \mapsto (s_j|_{U_i \times_U U_j})_{i,j},$$

in other words,

$$\check{H}^0(\mathcal{U}, \mathcal{F}) := \left\{ (s_i) \in \prod_i \mathcal{F}(U_i) \;\middle|\; s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}, \forall i, j \right\}.$$

An étale presheaf $\mathcal{F}$ is a sheaf, denoted by $\mathcal{F} \in \mathbf{Sh}(X_{\text{ét}}, \mathbf{Set})$, called an *étale sheaf*, if for any $U \in X_{\text{ét}}$ and any cover $\{U_i \to U\}$, the natural map $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$ induces a natural isomorphism $\mathcal{F}(U) \xrightarrow{\sim} \check{H}^0(\mathcal{U}, \mathcal{F})$.

1.3.3. *Sheafification.* Suppose $\mathcal{F}$ is a presheaf. The *sheafification* of $\mathcal{F}$, denoted by $\mathcal{F}^+$, is a sheaf with a morphism $\mathcal{F} \to \mathcal{F}^+$ such that for any sheaf $\mathcal{G}$ and any morphism $\mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\mathcal{F}^+ \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{G}. \\ \downarrow & \nearrow & \\ \mathcal{F}^+ & & \end{array}$$

In other words, it is the left adjoint functor of the forgetful functor from the category of sheaves to the category of presheaves: $\mathrm{Hom}_{\mathbf{Sh}}(\mathcal{F}^+, \mathcal{G}) = \mathrm{Hom}_{\mathbf{Psh}}(\mathcal{F}, \mathcal{G})$. The sheafification of $\mathcal{F}$ is unique up to isomorphism. It have several constructions, for example, if define

$$\check{H}^0(U, \mathcal{F}) := \varinjlim_{\mathcal{U} \text{ cover of } U} \check{H}^0(\mathcal{U}, \mathcal{F}),$$

then

$$\mathcal{F}^+(U) = \check{H}^0\big(U, V \mapsto \check{H}^0(V, \mathcal{F})\big),$$

and an abstract nonsense construction:

$$\mathcal{F}^+ = \varprojlim_{\substack{\mathcal{G} \text{ étale sheaf} \\ \text{with morphism } \mathcal{F} \to \mathcal{G}}} \mathcal{G}.$$

1.3.4. There are some examples of étale sheaves:

Example 1.9 (The representable (pre)sheaf)**.** Consider the representable presheaf $\mathcal{F} = h_Y = Y(-) = \mathrm{Hom}_{\mathbf{Sch}/X}(-, Y)$ on $X_{\text{ét}}$, where Y is any scheme over X. It is easy to check that $\mathcal{F}$ is already a sheaf, using the adjoint property of global section functor.

There is an important class of étale sheaves, called *locally constant and constructible* (lcc for short) sheaves, one of the equivalent definition is that represented by a finite étale scheme Y over X.

Fact 1.10. *Conversely, every sheaf $\mathcal{F} \in \mathbf{Sh}(X_{\text{ét}}, \mathbf{Set})$ is representable by a (possibly non-separable and of infinite type) scheme over X.*

Example 1.11 (The constant sheaf)**.** Suppose S is a set, the constant sheaf $\underline{S}$ on $X_{\text{ét}}$ is the sheafification of the presheaf $U \mapsto S$. In fact, we have $\underline{S}(U) = S^{\pi_0^{\text{ét}}(U)}$, and $\underline{S}$ is represented by $S \times X$.

Example 1.12. Suppose $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-module on X, define

$$\begin{aligned} \mathcal{F}_{\text{ét}} : X_{\text{ét}} &\to \mathbf{Ab}, \\ (f : U \to X) &\mapsto (f^*\mathcal{F})(U), \end{aligned}$$

recall that $f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_U$. Then $\mathcal{F}_{\text{ét}}$ is an étale sheaf on $X_{\text{ét}}$.

Example 1.13. If $X = \operatorname{Spec} \overline{K}$, where $\overline{K}$ is a algebraically closed field, then we have an equivalence of categories $X_{\text{ét}} \cong \{\text{finite sets}\}$, and any étale sheaf $\mathcal{F}$ is determined by $\mathcal{F}(X)$. Therefore

$$\begin{aligned} \mathbf{Sh}(X_{\text{ét}}, \mathbf{Set}) &\cong \mathbf{Set}, \\ \mathcal{F} &\mapsto \mathcal{F}(X). \end{aligned}$$

1.4. **Direct and inverse image, stalks.**

1.4.1. *Direct image.* Suppose $f : X \to Y$ is a morphism of schemes, then we can define the *direct image functor* f_* to be

$$\begin{aligned} f_* : \mathbf{Sh}(X_{\text{ét}}, \mathbf{Set}) &\to \mathbf{Sh}(Y_{\text{ét}}, \mathbf{Set}), \\ \mathcal{F} &\mapsto f_*\mathcal{F} : (U \to Y) \mapsto \mathcal{F}(f^{-1}(U \to Y)), \end{aligned}$$

where $f^{-1}(U \to Y) := (X \times_Y U \to X)$ is the pullback of $U \to Y$ under $f : X \to Y$.

1.4.2. *Inverse image.* The direct image functor f_* has a left adjoint, called the *inverse image functor*, denoted by f^*:

$$f^* : \mathbf{Sh}(Y_{\text{ét}}, \mathbf{Set}) \to \mathbf{Sh}(X_{\text{ét}}, \mathbf{Set}),$$

which satisfies

$$\operatorname{Hom}(f^*\mathcal{G}, \mathcal{F}) = \operatorname{Hom}(\mathcal{G}, f_*\mathcal{F}).$$

The following result is easy to see by using Yoneda lemma.

Fact 1.14. *Suppose $\mathcal{G} \in \mathbf{Sh}(Y_{\text{ét}}, \mathbf{Set})$ is represented by a scheme G étale over Y, then $f^*\mathcal{G} \in \mathbf{Sh}(X_{\text{ét}}, \mathbf{Set})$ is represented by $f^{-1}(G) = X \times_Y G$.*

In the general case, $f^*\mathcal{G}$ is the sheafification of the following presheaf on $X_{\text{ét}}$:

$$(U \to X) \mapsto \varinjlim_{U \to V \to Y} \mathcal{G}(V),$$

where the direct limit is taken over the inverse system (note that $\mathcal{G}$ is contravariant) of $(U \to V, V \to Y)$ such that $V \to Y$ is étale, and the following diagram commutes:

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

This can be understood as an étale open V of Y such that "$U \subset f^{-1}(V)$", or we say "$f(U) \subset V$".

1.4.3. *Stalks.* For a geometric point $u : \xi \to X$ (i.e. $\xi = \operatorname{Spec} \overline{K}$ for some algebraically closed field $\overline{K}$), we define the *stalk of $\mathcal{F}$ at ξ* to be $\mathcal{F}_\xi := (u^*\mathcal{F})(\xi)$ (note that $u^*\mathcal{F}$ is an étale sheaf over an algebraically closed field, it is determined by its global section).

Fact 1.15. $\mathcal{F}_\xi = \varinjlim_{U \text{ étale neighborhood of } \xi} \mathcal{F}(U).$

By an *étale neighborhood of* ξ, we mean an étale morphism $U \to X$ with an inclusion $\xi \to U$ such that the following diagram commutes:

$$\begin{array}{ccc} \xi & \longrightarrow & U \\ & \searrow^{u} & \downarrow \\ & & X \end{array}$$

Fact 1.16. *If $f : X \to Y$ is a morphism of schemes and $\mathcal{G}$ is an étale sheaf on Y, then for any geometric point ξ of X, $(f^*\mathcal{G})_\xi = \mathcal{G}_{f(\xi)}$.*

Proposition 1.17. *Suppose $\mathcal{F} \to \mathcal{G}$ is a morphism between étale sheaves over X. Then it is injective (resp. surjective, isomorphism) if and only if for any geometric point $\xi \to X$, $\mathcal{F}_\xi \to \mathcal{G}_\xi$ is injective (resp. surjective, isomorphism).*

Example 1.18. If $Y \to Z$ is a smooth surjective morphism of X-schemes, then the induced morphism $h_Y \to h_Z$ of étale sheaves on X is surjective.

1.5. **Abelian sheaf.** Suppose X is a scheme. From now on, we consider the category $\mathbf{Sh}(X_{\text{ét}}, \mathbf{Ab})$.

Fact 1.19. $\mathbf{Sh}(X_{\text{ét}}, \mathbf{Ab})$ *is an abelian category.*

For example, if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of abelian sheaves on $X_{\text{ét}}$, then

$$\ker f = (U \mapsto \ker(f(U) : \mathcal{F}(U) \to \mathcal{G}(U))),$$
$$\operatorname{coker} f = (U \mapsto \operatorname{coker}(f(U) : \mathcal{F}(U) \to \mathcal{G}(U)))^+,$$

note that in $\operatorname{coker} f$, sheafification is required.

Example 1.20 (The constant sheaf)**.** Suppose A is an abelian group, the constant sheaf $\underline{A} \in \mathbf{Sh}(X_{\text{ét}}, \mathbf{Ab})$ is the sheafification of the presheaf $U \mapsto A$. It is represented by $A \times X$.

Note that A can be viewed as a constant group scheme over $\mathbb{Z}$. If A is a finite group, then the associated group scheme is finite flat. This partially explains why in étale cohomology, $A = \mathbb{Z}/n\mathbb{Z}$ is interesting, but $A = \mathbb{Z}$ is less interesting. Based on this observation, we have a generalized example:

Example 1.21. Suppose A is a commutative group scheme over $\mathbb{Z}$, then it induces an abelian sheaf $\underline{A} \in \mathbf{Sh}(X_{\text{ét}}, \mathbf{Ab})$, which is represented by $A \times X$.

In particular, if $A = \mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$, then $\underline{\mathbb{G}_m}$ is $U \mapsto \mathcal{O}_U(U)^\times$. Denote this sheaf by $\mathcal{O}^\times$. Suppose $n \in \mathbb{Z}$. Consider the morphism

$$n : \mathcal{O}^\times \to \mathcal{O}^\times, \qquad a \mapsto a^n.$$

Denote its kernel by $\underline{\mu_n}$. Then it is

$$U \mapsto \{a \in \mathcal{O}_U(U) \mid a^n = 1\},$$

and it is the étale sheaf associated to the group scheme $\mu_n := \operatorname{Spec} \mathbb{Z}[t]/(t^n - 1)$ over $\mathbb{Z}$.

Fact 1.22. *If n is invertible at $x \in X$ (which means n is invertible in the residue field of x), then $\underline{\mu_n}$ is locally isomorphic (under the étale topology) to $\underline{\mathbb{Z}/n\mathbb{Z}}$ at x.*

In particular, if n is invertible on X, then $\underline{\mu_n}$ is locally isomorphic to $\underline{\mathbb{Z}/n\mathbb{Z}}$ everywhere.

Proposition 1.23. *If n is invertible on X, then*

$$0 \to \underline{\mu_n} \to \mathcal{O}^\times \xrightarrow{n} \mathcal{O}^\times \to 0$$

is an exact sequence of étale sheaves on X, which is called the Kummer sequence.

This is because the morphism $n : \mathbb{G}_{m,X} \to \mathbb{G}_{m,X}$ is étale. Note that this sequence is not exact under Zariski topology in general.

There are some important classes of abelian étale sheaves:

- An abelian sheaf $\mathcal{F}$ is called *torsion sheaf* if it is the sheafification of a presheaf whose sections are all torsion. Equivalently, all of its stalks are torsion. If S is a scheme, the $\mathcal{F}$ is called with torsion orders invertible on S, if there is an integer $n \neq 0$ invertible on S, such that $n\mathcal{F} = 0$.
- An abelian sheaf $\mathcal{F}$ is called *locally constant sheaf* if there exists a cover $\{U_i \to X\}$ of X such that $\mathcal{F}|_{U_i}$ is a constant sheaf for all i.
- We omit the original definition of *constructible sheaf*, but only states that it is equivalent to a quotient of a sheaf representable by an étale commutative group scheme over X.
- An abelian sheaf $\mathcal{F}$ is called *locally constant and constructible* (lcc for short) if it is locally constant and is also constructible. It is equivalent to that representable by a finite étale commutative group scheme over X.

Clearly, lcc sheaves are automatically torsion. Concerning the Fact 1.7, assume that X is connected, we have an equivalence of categories:

$$\{\text{lcc abelian sheaves over } X_{\text{ét}}\} \xleftrightarrow{1:1} \left\{\text{finite abelian groups with continuous } \pi_1^{\text{ét}}(X)\text{-action}\right\}.$$

1.6. **The étale cohomology.**

1.6.1. *Étale cohomology.* Consider the global section functor

$$\Gamma(X, -) : \mathbf{Sh}(X_{\text{ét}}, \mathbf{Ab}) \to \mathbf{Ab}, \qquad \mathcal{F} \mapsto \mathcal{F}(X).$$

It is left exact, because the kernel of a morphism $f : \mathcal{F} \to \mathcal{G}$ between two sheaves is just $U \mapsto \ker(f(U) : \mathcal{F}(U) \to \mathcal{G}(U))$.

Fact 1.24. $\mathbf{Sh}(X_{\text{ét}}, \mathbf{Ab})$ *has enough injective objects.*

Therefore $\Gamma(X, -)$ has right derived functors $R^i\Gamma(X, -)$.

Definition 1.25. The *étale cohomology* is $H^i_{\text{ét}}(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$.

1.6.2. *Higher direct image.* For a morphism $f : X \to Y$ between two schemes, the direct image functor $f_* : \mathbf{Sh}(X_{\text{ét}}, \mathbf{Ab}) \to \mathbf{Sh}(Y_{\text{ét}}, \mathbf{Ab})$ is left exact, so it has right derived functors $R^i f_*$, called the *higher direct image functor*. The inverse image functor $f^* : \mathbf{Sh}(Y_{\text{ét}}, \mathbf{Ab}) \to \mathbf{Sh}(X_{\text{ét}}, \mathbf{Ab})$ is an exact functor.

Fact 1.26. *For an abelian sheaf $\mathcal{F}$ on $X_{\text{ét}}$, $R^i f_*\mathcal{F}$ is the sheafification of the presheaf $U \mapsto H^i_{\text{ét}}(f^{-1}(U), \mathcal{F})$ on $Y_{\text{ét}}$, where $f^{-1}(U) := U \times_Y X$.*

In particular, if $f : X \to \operatorname{Spec} \overline{K}$ is the structure morphism, then $R^i f_*\mathcal{F} = H^i_{\text{ét}}(X, \mathcal{F})$.

Fact 1.27. *If f is quasi-compact and quasi-separated (qcqs for short), then $R^i f_*$ preserves torsion sheaves.*

Theorem 1.28. *Suppose $\mathcal{F}$ is a quasi-coherent sheaf on a scheme X, and $\mathcal{F}_{\text{ét}}$ is the étale sheaf associated to $\mathcal{F}$. Then there is a canonical isomorphism $H^1(X, \mathcal{F}) \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathcal{F}_{\text{ét}})$.*

Therefore étale cohomology of quasi-coherent sheaf doesn't give any new information.

1.6.3. We consider the étale cohomology of the étale sheaf $\mathcal{O}^\times$, which is the étale sheaf associated to $\mathbb{G}_m$, and maps U to $\mathcal{O}_U(U)^\times$.

Fact 1.29. $H^1_{\text{ét}}(X, \mathcal{O}^\times) = \operatorname{Pic}(X)$.

In particular, if $X = \operatorname{Spec} K$, then we get Hilbert's theorem 90: $H^1(K, \overline{K}^\times) = 0$.

1.6.4. *Leray spectral sequence.* Suppose $f : X \to Y$ is a morphism of schemes and $\mathcal{F}$ is an abelian sheaf on $X_{\text{ét}}$. Then we have the *Leray spectral sequence*

$$E_2^{pq} = H^p_{\text{ét}}(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathcal{F}),$$

which is just the special case of Grothendieck spectral sequence. Similarly, if $g : Y \to Z$ is another morphism of schemes, then we have

$$E_2^{pq} = R^p g_*(R^q f_* \mathcal{F}) \Rightarrow R^{p+q}(gf)_* \mathcal{F}.$$

The above spectral sequence induces edge morphisms

$$R^p g_*(f_* \mathcal{F}) \to R^p (gf)_* \mathcal{F}, \tag{1.1}$$

and

$$R^p (gf)_* \mathcal{F} \to g_*(R^p f_* \mathcal{F}). \tag{1.2}$$

1.7. **Base change theorems.** Consider the following cartesian diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y, \end{array}$$

namely, $X' = X \times_Y Y'$ is the pullback of f and g. The goal is to study the behavior of $R^p f_*$ under the base change g. In the following we will construct a natural morphism

$$g^*(R^p f_* \mathcal{F}) \to R^p f'_*((g')^* \mathcal{F}), \tag{1.3}$$

where $\mathcal{F}$ is an étale sheaf on X, called the *base change morphism*. If it is an isomorphism, we say that $R^p f_*$ commutes with the base change g.

We have the composition of a series of morphisms

$$R^p f_* \mathcal{F} \to R^p f_* g'_*(g')^* \mathcal{F} \xrightarrow{(1.1)} R^p (fg')_*(g')^* \mathcal{F} = R^p (gf')_*(g')^* \mathcal{F} \xrightarrow{(1.2)} g_* R^p f'_*(g')^* \mathcal{F},$$

where the first morphism is $R^p f_*$ apply to the natural morphism $\mathcal{F} \to g'_*(g')^* \mathcal{F}$ given by the adjoint property of $(g')^*$ and g'_*. Now by the adjoint property of g^* and g_*, we obtain a morphism $g^*(R^p f_* \mathcal{F}) \to R^p f'_*((g')^* \mathcal{F})$, which is the base change morphism (1.3).

The base change morphism can also be defined in an explicit way as follows. Firstly, in $p = 0$ case we obtain a morphism $g^* f_* \mathcal{F} \to f'_*(g')^* \mathcal{F}$ by the above method, without introducing R^p or the map (1.1) or (1.2). In the general case, let $\mathcal{F} \overset{\text{qis}}{\hookrightarrow} \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$, and let $(g')^* \mathcal{I}^\bullet \overset{\text{qis}}{\hookrightarrow} \mathcal{J}^\bullet$ be an injective resolution of $(g')^* \mathcal{I}^\bullet$. Since $(g')^*$ is exact, we have $(g')^* \mathcal{F} \overset{\text{qis}}{\hookrightarrow} (g')^* \mathcal{I}^\bullet \overset{\text{qis}}{\hookrightarrow} \mathcal{J}^\bullet$ which means that $\mathcal{J}^\bullet$ is an injective resolution of $(g')^* \mathcal{F}$. Combine with $p = 0$ case we obtain the composition of a series of morphisms

$$g^* f_* \mathcal{I}^\bullet \to f'_*(g')^* \mathcal{I}^\bullet \to f'_* \mathcal{J}^\bullet,$$

taking p-th cohomology we obtain the morphism (1.3).

Theorem 1.30 (Proper base change theorem)**.** *If f is proper and $\mathcal{F}$ is torsion, then* (1.3) *is an isomorphism.*

Theorem 1.31 (Smooth base change theorem)**.** *If g is smooth, f is qcqs, $\mathcal{F}$ is torsion with torsion orders invertible on Y, then* (1.3) *is an isomorphism.*

The following result is an important consequence of the above base change theorems, which said that smooth and proper morphism preserves lcc torsion sheaves with orders invertible on base scheme, hence induces a functor between torsion representations (whose orders invertible on base scheme) of étale fundamental groups.

Corollary 1.32. *Suppose f is smooth and proper, $\mathcal{F}$ is torsion with torsion orders invertible on Y. If $\mathcal{F}$ is lcc, then so is $R^p f_* \mathcal{F}$ for any $p \geq 0$.*

Another consequence of proper base change theorem is as follows. Suppose $f : X \to Y$ is a morphism of schemes which is *compactifiable*, namely, there exist morphisms $j : X \to \overline{X}$ and $\overline{f} : \overline{X} \to Y$ satisfying $f = \overline{f} \circ j$, such that j is an open immersion and $\overline{f}$ is a proper morphism. Then we define the *higher direct image with proper support* $R^p f_! : \mathbf{Sh}(X_{\text{ét}}, \mathbf{Ab}) \to \mathbf{Sh}(Y_{\text{ét}}, \mathbf{Ab})$ to be $R^p f_! \mathcal{F} := R^p \overline{f}_*(j_! \mathcal{F})$, where for an open immersion (more generally, for an étale morphism) $j : X \to \overline{X}$, the $j_! \mathcal{F} \in \mathbf{Sh}(\overline{X}_{\text{ét}}, \mathbf{Ab})$ is defined to be the sheafification of the following presheaf on $\overline{X}_{\text{ét}}$:

$$(V \to \overline{X}) \mapsto \bigoplus_{\substack{V \to X \text{ étale such that} \\ (V \to X \to \overline{X}) = (V \to \overline{X})}} \mathcal{G}(V \to X).$$

The proper base change theorem implies that $R^p f_!$ is well-defined, namely, its definition is independent of the choice of j and $\overline{f}$.

1.8. The ℓ-adic cohomology.

Let X be a scheme and ℓ be a prime invertible on X.

1.8.1. *ℓ-adic sheaf.* A *$\mathbb{Z}_\ell$-sheaf* $\mathcal{F}$ on X is an inverse system $\mathcal{F} = (\mathcal{F}_n)_{n \geq 1}$ of abelian étale sheaves on X such that $\ell^n \mathcal{F}_n = 0$ for all n, and each transition map $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induces isomorphism $\mathcal{F}_{n+1}/\ell^n \mathcal{F}_{n+1} \xrightarrow{\sim} \mathcal{F}_n$. A *$\mathbb{Q}_\ell$-sheaf* is just understood as a formal symbol $\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ where $\mathcal{F}$ is a $\mathbb{Z}_\ell$-sheaf.

Obvious examples of ℓ-adic sheaves are $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell(1) = (\mu_{\ell^n})_{n \geq 1}$, $\mathbb{Q}_\ell(1)$, etc.

A $\mathbb{Z}_\ell$-sheaf or $\mathbb{Q}_\ell$-sheaf is called *locally constant and constructible* (lcc for short), or called *lisse* (= smooth), if each $\mathcal{F}_n$ is locally constant and constructible.

Assume that X is connected, we have the following equivalences of categories:

$$\{\text{lcc } \mathbb{Z}_\ell\text{-sheaves over } X_{\text{ét}}\} \xleftrightarrow{1:1} \left\{\text{finite generated } \mathbb{Z}_\ell\text{-modules with continuous } \pi_1^{\text{ét}}(X)\text{-action}\right\},$$

and

$$\{\text{lcc } \mathbb{Q}_\ell\text{-sheaves over } X_{\text{ét}}\} \xleftrightarrow{1:1} \left\{\text{finite dimensional } \mathbb{Q}_\ell\text{-vector spaces with continuous } \pi_1^{\text{ét}}(X)\text{-action}\right\}.$$

1.8.2. *The ℓ-adic cohomology.* If $\mathcal{F} = (\mathcal{F}_n)_{n \geq 1}$ is a $\mathbb{Z}_\ell$-sheaf, define its *ℓ-adic cohomology* $H^i_{\text{ét}}(X, \mathcal{F}) := \varprojlim_n H^i_{\text{ét}}(X, \mathcal{F}_n)$, where the $H^i_{\text{ét}}(X, \mathcal{F}_n)$ is the étale cohomology for torsion abelian sheaves. Define $H^i_{\text{ét}}(X, \mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) := H^i_{\text{ét}}(X, \mathcal{F}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Similarly, if f is a morphism of schemes, define $R^i f_* \mathcal{F} := \varprojlim_n R^i f_* \mathcal{F}_n$ and $R^i f_*(\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) := (R^i f_* \mathcal{F}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

The Corollary 1.32 tells us that if $f : X \to Y$ is smooth and proper, then $R^p f_*$ preserves lcc ℓ-adic sheaves, hence induces a functor from the category of the ℓ-adic representations of $\pi_1^{\text{ét}}(X)$ to the category of the ℓ-adic representations of $\pi_1^{\text{ét}}(Y)$.

Example 1.33. Consider the natural morphism $\pi : \mathcal{E} \to X$ where $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty, 0, 1728\}$ is the modular curve of level 1 over $\mathbb{Q}$ with some points removed, and $\mathcal{E}$ is the universal elliptic curve. Then π is proper and smooth, and $\mathcal{F} = R^1 \pi_* \mathbb{Q}_\ell$ is a lisse $\mathbb{Q}_\ell$-sheaf over X. At each geometry point x of X, we have $\mathcal{F}_x = H^1_{\text{ét}}(\mathcal{E}_x, \mathbb{Q}_\ell) = V_\ell(\mathcal{E}_x)^*$ where $\mathcal{E}_x$ is the elliptic curve with j-invariant corresponding to x.

1.9. The comparison theorem.

Theorem 1.34 (Comparison theorem)**.** *Suppose X is a smooth connected variety over $\overline{\mathbb{Q}}$, $\mathcal{F}$ is a constructible sheaf over $X_{\text{ét}}$, then there exists a canonical isomorphism $H^i_{\text{ét}}(X, \mathcal{F}) \xrightarrow{\sim} H^i(X(\mathbb{C}), \mathcal{F})$, where the latter H^i is the singular cohomology.*

1.10. **Semisimplicity theorem.** The first one is the semisimplicity theorem for complex analytic geometry.

Theorem 1.35 (Deligne)**.** *Let S be a smooth connected separated scheme over $\mathbb{C}$, and let $s : \operatorname{Spec}\mathbb{C} \to S$ be a base point. Let $f : X \to S$ be a morphism such that $R^i f_* \mathbb{Q}$ is a local system on S. Let G be the Zariski closure of the image of $\pi_1^{\text{top}}(S, s)$ in $\operatorname{Aut}_{\mathbb{C}}\left((R^i f_* \mathbb{C})_s\right)$ and G^0 be the connected component of G containing 1.*

(i) *If f is proper and smooth, then G^0 is semisimple.*

(ii) *In the general case, G^0 does not have any quotients of multiplicative type (namely, the radical of G^0 is unipotent).*

The following is the ℓ-adic analogue.

Theorem 1.36 (Weil)**.** *Let S be a smooth connected separated scheme over $K = \overline{\mathbb{Q}}$ or $\overline{\mathbb{F}}_p$ with $p \neq \ell$, and let $s : \operatorname{Spec} K \to S$ be a base point. Let $f : X \to S$ be a smooth and proper morphism. Let G be the Zariski closure of the image of $\pi_1^{\text{ét}}(S, s)$ in $\operatorname{Aut}_{\mathbb{Q}_\ell}\left((R^i f_* \mathbb{Q}_\ell)_s\right)$ and G^0 be the connected component of G containing 1. Then G^0 is semisimple.*

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