Chapter 7

Toroidal compactification

7.1 Background knowledge on toric varieties

Let k be an algebraically closed field. All varieties are defined over k.

Let T be a k-split algebraic torus.

Definition 7.1.1. *A* **toric variety with torus** T *is a variety* V *equipped with an open immersion* $\phi\colon T \to V$ *and an action* T *on* V *such that* $t \cdot \phi(t') = \phi(tt')$ *for all* $t, t' \in T$*.*

7.1.1 Affine toric varieties

Assume V is affine. The action of T on V induces an action of T on the ring of regular functions $k[V]$. So $k[V]$ (as a vector space) is a representation of T. For each $\chi \in X^*(T)$, define

$$k[V]_\chi := \{f \in k[V] : t{\cdot}f = \chi(t)f\} = \{f \in k[V] : f(\chi(t)v) = \chi(t)f(v), \text{ for all } v \in V \text{ and } t \in T\}.$$

Then $k[V] = \bigoplus_{\chi \in X^*(T)} k[V]_\chi$.

Lemma 7.1.2. $k[V]_\chi \neq 0$ *if and only if* χ *extends to a regular function on* V*.*

Proof. This lemma is clearly true because $k[T]$ equals the group algebra $k[X^*(T)]$. □

Corollary 7.1.3. $S(V) := \{\chi \in X^*(T) : k[V]_\chi \neq 0\}$ *is a semi-group, with the identity being the trivial character* χ_0*.*

Proof. It is easy to check $k[V]_{\chi_0} = k$, so $\chi_0 \in S(V)$. For $\chi_1, \chi_2 \in S(V)$, by Lemma 7.1.2 both χ_1 and χ_2 extend to a regular functions on V, and so does the product $\chi_1\chi_2$. So $\chi_1\chi_2 \in S(V)$ by Lemma 7.1.2. □

The following theorem is then easy to check.

Theorem 7.1.4. *The following categories are equivalent:*

(i) sub-semi-groups S *of* $X^*(T)$ *of finite type which generate* $X^*(T)$ *as a group,*

(ii) affine toric varieties with torus T*.*

For (i) to (ii), S *is sent to* $\mathrm{Speck}[S]$*, with* $k[S] = \{\sum a_s s : a_s \in k,\ s \in S\}$*. For (ii) to (i),* V *is sent to* $S(V)$*.*

Among the sub-semi-groups of $X^*(T)$, the *saturate* ones (*i.e.* $(S \otimes \mathbb{Q}) \cap X^*(T) = S$) give rise to normal affine toric varieties.

Next, we want to turn to the *cocharacters* of T. Denote for simplicity by $X_* := X_*(T)$. Use $X_{*,\mathbb{Q}}$ (resp. $X_{*,\mathbb{R}}$) to denote $X_*(T) \otimes \mathbb{Q}$ (resp. $X_*(T) \otimes \mathbb{R}$).

Definition 7.1.5. *A subset $\sigma \subseteq X_{*,\mathbb{R}}$ is called a* **(rational) polyhedral cone** *if it satisfies one of the two equivalent conditions:*

- *σ is the intersection of finitely many rational semi-spaces, i.e. there exist $\lambda_1, \dots, \lambda_m \in X^*(T) \otimes \mathbb{Q}$ such that $\sigma = \{x \in X_{*,\mathbb{R}} : \lambda_j(x) \geq 0$ for all $j \in \{1, \dots, m\}\}$.*
- *there exist $x_1, \dots, x_m \in X_{*,\mathbb{Q}}$ such that $\sigma = \{\sum_{j=1}^m \alpha_j x_j : \alpha_j \in \mathbb{R}_{\geq 0}\}$.*

For any polyhedral cone σ, its dual is $\sigma^\vee = \{\lambda \in X^*(T) \otimes \mathbb{R} : \lambda(x) \geq 0 \text{ for all } x \in \sigma\}$. So σ contains a line in $X_{*,\mathbb{R}}$ if and only if $\sigma^\vee$ is contained in a hyperplane of $X^*(T) \otimes \mathbb{R}$.

Definition 7.1.6. *A* **face** *of a polyhedral cone σ is a subset of the form $\{x \in \sigma : \lambda(x) = 0\}$ for some $\lambda \in \sigma^\vee$.*

The intersection of two faces of σ is still a face, because $\{x \in \sigma : \lambda_1(x) = 0\} \cap \{x \in \sigma : \lambda_2(x) = 0\} = \{x \in \sigma : (\lambda_1 + \lambda_2)(x) = 0\}$.

Theorem 7.1.7. *The map $\sigma \mapsto V_\sigma := \mathrm{Spec} k[\sigma^\vee \cap X^*(T)]$ defines a bijection between:*

- *polyhedral cones in $X_{*,\mathbb{R}}$ which do not contain lines,*
- *isomorphic classes of normal affine toric varieties with torus T.*

Moreover, we have:

(1) For $\mu \in X_*$, we have $\mu \in \sigma \Leftrightarrow \lim_{t \to 0} \mu(t) \in V_\sigma$.

(2) V_σ is smooth if and only if $\sigma \cap X_*$ is generated by part of a $\mathbb{Z}$-bases of X_* (in which case $V_\sigma \simeq \mathbb{G}_{\mathrm{m}}^\bullet \times \mathbb{G}_{\mathrm{a}}^\bullet$).

(3) If $\sigma_1 \subseteq \sigma_2$, then there exists a morphism $V_{\sigma_1} \to V_{\sigma_2}$. This morphism is an open immersion if and only if σ_1 is a face of σ_2.

Example 7.1.8. *Consider the simplest example $T = \mathbb{G}_{\mathrm{m},k}$. Then $X_{*,\mathbb{R}} \simeq \mathbb{R}$. Polyhedral cones in $X_{*,\mathbb{R}}$ which do not contain lines are $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$, and $\{0\}$. In the first two cases $V_\sigma \simeq \mathbb{G}_{\mathrm{a},k}$ and in the third case $V_\sigma \simeq \mathbb{G}_{\mathrm{m},k}$.*

7.1.2 General toric varieties

Definition 7.1.9. *A* **fan** *Σ in $X_{*,\mathbb{R}}$ is a collection $\{\sigma\}$ of polyhedral cones such that:*

(i) If $\sigma \in \Sigma$ and $\sigma' \subseteq \sigma$ is a fact, then $\sigma' \in \Sigma$;

(ii) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma' \in \Sigma$ and is a common face of σ and of σ'.

The **trivial fan** *consists of only the trivial cone.*

Each fan Σ gives arise to a toric variety V_Σ as follows: To each $\sigma \in \Sigma$ we associate V_σ as in Theorem 7.1.7, and then glue V_σ and $V_{\sigma'}$ along the common open subset $V_{\sigma \cap \sigma'}$.

Theorem 7.1.10. *The map $\Sigma \mapsto V_\Sigma$ defines a bijection between*

- *fans in $X_{*,\mathbb{R}}$,*
- *isomorphic classes of normal toric varieties with torus T.*

Moreover, V_Σ is a complete variety if and only if $X_{*,\mathbb{R}} = \bigcup_{\sigma\in\Sigma}\sigma$.

Example 7.1.11. *Continue with Example 7.1.8. If the fan $\Sigma = \{\mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}, \{0\}\}$, then we get $V_\Sigma \simeq \mathbb{P}^1_k$ by glueing $\mathbb{G}_{\mathrm{a},k}$ and $\mathbb{G}_{\mathrm{a},k}$ along their intersection $\mathbb{G}_{\mathrm{m},k}$.*

Definition 7.1.12. *A* **refinement** *of a fan Σ is a fan Σ' such that*

(i) each $\sigma' \in \Sigma'$ is contained in some $\sigma \in \Sigma$,

(ii) each $\sigma \in \Sigma$ is a finite union of some $\{\sigma'\} \subseteq \Sigma'$.

Let Σ and Σ' be two fans in $X_{*,\mathbb{R}}$. Condition (i) above implies that there exists a T-equivariant morphism $V_{\Sigma'} \to V_\Sigma$. Then the valuative criterion of properness implies: this morphism is proper if and only if Σ' is a refinement of Σ.

Theorem 7.1.13. *Each fan Σ admits a refinement Σ' such that $V_{\Sigma'}$ is a resolution of singularities of V_Σ. If V_Σ is complete, then we can find such an Σ' that $V_{\Sigma'}$ is smooth and projective.*

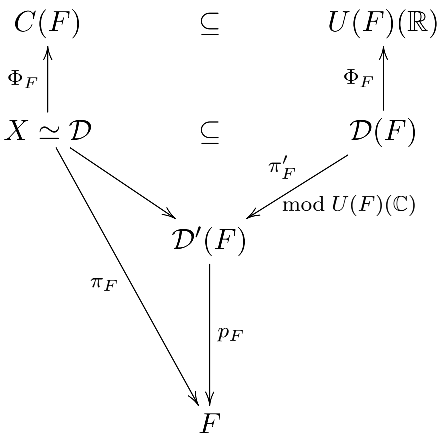
7.2 Toroidal compactifications of $\Gamma\backslash X$

Let $(\mathbf{G}, X)$ be a Shimura datum. By abuse of notation use X to denote a connected component. Let $\Gamma < \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup.

7.2.1 The algebraic torus associated with a rational analytic boundary component

Take a rational analytic boundary component F whose normalizer is $\mathbf{P}$.

Recall the diagram from Theorem 6.3.7, with $C(F)$ a cone in $U(F)(\mathbb{R})$ such that $X \simeq \mathcal{D} = \Phi_F^{-1}(C(F))$:



(7.2.1)

Let $\Gamma_U := \Gamma \cap \mathbf{U}(F)(\mathbb{Q})$, and let

$$T_F := \Gamma_U \backslash U(F)(\mathbb{C}). \tag{7.2.2}$$

Then T_F is an algebraic torus, and $X_*(T_F) = \Gamma_U$ and $X_*(T_F)_\mathbb{R} = U(F)(\mathbb{R})$. Thus $C(F)$ is a cone in $X_*(T_F)_\mathbb{R}$.

7.2.2 The fibration on each rational analytic boundary

Take a rational analytic boundary component F whose normalizer is $\mathbf{P}$. It is tempting to take the quotient of $\mathcal{D}(F)$ by $\Gamma_F := \Gamma \cap \mathbf{P}(\mathbb{Q})$. It turns out that Γ_F too large! Instead, we consider the following short exact sequence

$$1 \to \Gamma_F^\circ \to \Gamma_F \to \overline{\Gamma}_F \to 1, \tag{7.2.3}$$

where $\Gamma_F^\circ := \{\gamma \in \Gamma_F : \gamma u \gamma^{-1} = u \text{ for all } u \in U(F)(\mathbb{R})\}$. We will do the quotient in two steps: quotient by Γ_F° and then by $\overline{\Gamma}_F$.

By Lemma 6.3.2 and Proposition 6.3.3, we have

$$\Gamma_F^\circ = \Gamma \cap (\mathbf{W}(F)\mathbf{G}_h(F)\mathbf{M}(F))(\mathbb{Q}) = \Gamma \cap (W(F)G_h(F))(\mathbb{R}).$$

Hence $\overline{\Gamma}_F$ is canonically isomorphic to (a finite-indexed subgroup of)

$$\Gamma_{l,F} := \Gamma \cap \mathbf{G}_l(F)(\mathbb{Q}) = \Gamma \cap G_l(F)(\mathbb{R}).$$

Quotient by Γ_F°

From (7.2.1) we obtain

$$\Gamma_F^\circ \backslash \mathcal{D}(F) \longrightarrow \mathcal{A}_\Gamma \longrightarrow \Gamma_F \backslash F =: S_F, \tag{7.2.4}$$

where $\mathcal{A}_\Gamma = (\Gamma_F^\circ/\Gamma_U)\backslash \mathcal{D}'(F)$ is an abelian scheme over S_F (which is an algebraic variety since it is a connected component of a Shimura variety).

The fibration $\Gamma_F^\circ \backslash \mathcal{D}(F) \longrightarrow \mathcal{A}_\Gamma$ is easily seen to be a T_F-torsor. We can "compactify" T_F using a fan in $X_*(T_F)_\mathbb{R} = U(F)(\mathbb{R})$ as in §7.1.2 (in particular Theorem 7.1.10 and 7.1.13). In our case, this fan must satisfy some properties so that we can do the quotient by $\overline{\Gamma}_F$.

$\Gamma_{l,F}$-admissible polyhedral decomposition of $C(F)$ and further quotient by $\Gamma_{F,l}$

Definition 7.2.1. *A* $\Gamma_{l,F}$**-admissible polyhedral decomposition of** $C(F)$ *is a fan* Σ_F *in* $X_*(T_F)_\mathbb{R} = U(F)(\mathbb{R})$ *satisfying the following properties:*

(i) Each polyhedral cone in Σ_F *is contained in* $\overline{C(F)}$ *and is strongly convex.*

(ii) $C(F) \subseteq \bigcup_{\sigma\in\Sigma_F} \sigma$, *i.e.* $C(F) = \bigcup_{\sigma\in\Sigma_F}(C(F)\cap\sigma)$.

(iii) For any $\gamma \in \Gamma_{l,F}$ *and any cone* $\sigma \in \Sigma_F$, *we have* $\gamma\sigma \in \Sigma_F$.

(iv) There are only finitely many classes of cones in Σ_F *modulo* $\Gamma_{l,F}$.

Now take Σ_F to be a $\Gamma_{l,F}$-admissible polyhedral decomposition of $C(F)$. By Theorem 7.1.10, we get a toric variety V_{Σ_F} which torus T_F. Consider

$$(\Gamma_F^\circ \backslash \mathcal{D}(F)) \times^{T_F} V_{\Sigma_F} \tag{7.2.5}$$

which is the quotient of $(\Gamma_F^\circ \backslash \mathcal{D}(F)) \times V_{\Sigma_F}$ by the diagonal action of T_F. Finally set

$$(\Gamma_F^\circ \backslash \mathcal{D}(F))_{\Sigma_F}$$

to be the interior of the closure of $\Gamma_F^\circ \backslash \mathcal{D}(F)$ in (7.2.5). Now $X \simeq \mathcal{D} = \Phi_F^{-1}(C(F))$ allows us to define

$$(\Gamma_F^\circ \backslash \mathcal{D})_{\Sigma_F} \subseteq (\Gamma_F^\circ \backslash \mathcal{D}(F))_{\Sigma_F}. \tag{7.2.6}$$

Finally, the $\Gamma_{l,F}$-admissibility of Σ_F allows to do the quotients

$$(\Gamma_F \backslash \mathcal{D})_{\Sigma_F} := \frac{(\Gamma_F^\circ \backslash \mathcal{D})_{\Sigma_F}}{\Gamma_{l,F}} \subseteq \frac{(\Gamma_F^\circ \backslash \mathcal{D}(F))_{\Sigma_F}}{\Gamma_{l,F}}. \tag{7.2.7}$$

7.2.3 Final conclusion

Definition 7.2.2. *A* **Γ-admissible polyhedral decomposition** *is a collection* $\{\Sigma_F\}_F$ *of* $\Gamma_{l,F}$*-admissible polyhedral decomposition of* $C(F)$ *for all rational analytic boundary components* F *satisfying the following properties:*

(i) If $F_1 = \gamma \cdot F_2$ *for* $\gamma \in \Gamma$*, then* $\Sigma_{F_1} = \gamma\Sigma_{F_2}$*.*

(ii) If F_2 *is contained in the boundary of* F_1 *(i.e.* $F_2 \subseteq \overline{F_1}$ *which implies* $C(F_1) \subseteq \overline{C(F_2)}$*), then* $\Sigma_{F_1} = \{\sigma \cap \overline{C(F_1)} : \sigma \in \Sigma_{F_2}\}$*.*

Now take a Γ-admissible polyhedral decomposition $\{\Sigma_F\}_F$, and set

$$\overline{\Gamma\backslash X}_{\Sigma}^{\mathrm{tor}} := \frac{\coprod(\Gamma_F\backslash\mathcal{D})_{\Sigma_F}}{\sim}. \tag{7.2.8}$$

Here the equivalence $\sim$ is defined as follows: Two points

$$x_1 \in (\Gamma_{F_1}\backslash\mathcal{D})_{\Sigma_{F_1}} \quad \text{and} \quad x_2 \in (\Gamma_{F_2}\backslash\mathcal{D})_{\Sigma_{F_2}}$$

are equivalent (*i.e.* $x_1 \sim x_2$) if and only if

(a) there exists a rational analytic boundary component F and some $\gamma \in \Gamma$ such that

$$F_1 \subseteq \overline{F} \quad \text{and} \quad \gamma F_2 \subseteq \overline{F};$$

(b) there exists a point $x \in (\Gamma_F\backslash\mathcal{D})_{\Sigma_F}$ which projects to x_1 and x_2 respectively under the natural projections.

Theorem 7.2.3. $\overline{\Gamma\backslash X}_{\Sigma}^{\mathrm{tor}}$ *is a compactification of* $\Gamma\backslash X$*, which dominants* $\overline{\Gamma\backslash X}^{\mathrm{BB}}$*. More precisely, there exists a natural morphism*

$$\overline{\Gamma\backslash X}_{\Sigma}^{\mathrm{tor}} \longrightarrow \overline{\Gamma\backslash X}^{\mathrm{BB}}$$

which is identity on $\Gamma\backslash X$*.*

Moreover, there exists a refinement of Σ *such that the morphism above is a resolution of singularities.*