Chapter 5

(Reductive) Borel–Serre compactification

Let $(\mathbf{G}, X)$ be a Shimura datum. By abuse of notation use X to denote a connected component. Let $\Gamma < \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup.

Throughout the whole chapter, $\mathbf{P}$ will denote a proper parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$.

5.1 General discussion on compactifications of $\Gamma\backslash X$

5.1.1 General philosophy of compactifying $\Gamma\backslash X$

In general, here is what we get a compactification of $\Gamma\backslash X$ in the following steps:

(i) Consider a certain set of proper rational parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$. To each such $\mathbf{P}$, we associate a boundary component $e(\mathbf{P})$. Set $X^* := X \sqcup \coprod e(\mathbf{P})$.

(ii) Extend the action of Γ on X to X^*.

(iii) Endow a nice topology on X^* (often called the *Satake topology*), such that the action of Γ on X^* is continuous and proper[1] and that $\Gamma\backslash X^*$ is Hausdorff and compact.

In practice, *Siegel sets* play a crucial role in defining the topology on X^* and in showing the compactness of $\Gamma\backslash X^*$. Let us review its definition in the next subsection.

5.1.2 Revision on the rational symmetric spaces and Siegel sets

Recall the rational Langlands decomposition of $\mathbf{P}$ from (4.5.9)

$$P(\mathbb{R})^+ \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times M_{\mathbf{P}}(\mathbb{R})$$

and the induced rational horospherical decomposition (4.5.10)

$$h\colon X \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}}$$

with the rational boundary symmetric space

$$X_{\mathbf{P}} = M_{\mathbf{P}}(\mathbb{R})^+/(M_{\mathbf{P}} \cap K_\infty).$$

[1]Namely, any $x \in X^*$ has an open neighborhood W such that $\{\gamma \in \Gamma : \gamma W \cap W \neq \emptyset\}$ is a finite set.

Let $\Delta(A_{\mathbf{P}}, P) = \{\alpha_1, \ldots, \alpha_r\}$ be the subset of simple roots defined as in (4.5.8); they are characters of a maximal $\mathbb{Q}$-split torus $\mathbf{A_P}$ contained in $\mathbf{P}$. Then we have an isomorphism

$$A_{\mathbf{P}}(\mathbb{R})^+ \xrightarrow{\sim} \mathbb{R}^r_{>0}, \qquad a \mapsto (\alpha_1(a)^{-1}, \ldots, \alpha_r(a)^{-1}). \tag{5.1.1}$$

A Siegel set in X associated with $\mathbf{P}$ is of the form

$$\Sigma_{\mathbf{P},U,t,V} := h^{-1}(U \times A_{\mathbf{P},t} \times V) \subseteq X \tag{5.1.2}$$

with $U \subseteq N_P(\mathbb{R})$ and $V \subseteq X_{\mathbf{P}}$ bounded and

$$A_{\mathbf{P},t} := \{a \in A_{\mathbf{P}}(\mathbb{R})^+ : \alpha(a)^{-1} \leq t, \ \forall \alpha \in \Delta(A_{\mathbf{P}}, P)\}.$$

If $\mathbf{P} < \mathbf{Q}$ are parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$, then $A_{\mathbf{P}} > A_{\mathbf{Q}}$. Moreover, it can be shown that the followings are equivalent: (i) $X_{\mathbf{P}}$ is compact, (ii) $M_{\mathbf{P}}$ has $\mathbb{Q}$-rank 0, (iii) $\mathbf{P}$ is minimal parabolic in $\mathbf{G}^{\mathrm{der}}$. Furthermore, *reduction theory* asserts the following: If $\mathbf{P}$ is a minimal parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$, then there exist a finite subset $J \subseteq \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and a Siegel set $\mathfrak{S} := U \times A_{\mathbf{P},t} \times V$ of $\mathbf{P}$ such that

$$\Sigma := J \cdot \mathfrak{S} \tag{5.1.3}$$

is a fundamental set for the Γ-action on X.

5.2 Borel–Serre compactification

5.2.1 Borel–Serre partial compactification: definition

For any $\mathbf{P}$, define the *Borel–Serre boundary component*

$$e(\mathbf{P}) := N_P(\mathbb{R}) \times X_{\mathbf{P}}. \tag{5.2.1}$$

Since N_P is a normal subgroup of P, the boundary component $e(\mathbf{P}) \simeq P(\mathbb{R})^+/A_{\mathbf{P}}(\mathbb{R})^+(M_{\mathbf{P}} \cap K_\infty)$ is then an $N_P(\mathbb{R})$-principle bundle over the rational boundary symmetric space $X_{\mathbf{P}} \simeq P(\mathbb{R})^+/N_P(\mathbb{R})A_{\mathbf{P}}(\mathbb{R})^+(M_{\mathbf{P}} \cap K_\infty)$. Another visualization of $e(\mathbf{P})$ is given in (5.2.8), where we see that $e(\mathbf{P})$ is in some way the quotient of X by $A_{\mathbf{P}}(\mathbb{R})^+$.

The *Borel–Serre partial compactification* $\overline{X}^{\mathrm{BS}}$ is defined, as a set, to be

$$\overline{X}^{\mathrm{BS}} := X \cup \bigsqcup_{\mathbf{P}} e(\mathbf{P}). \tag{5.2.2}$$

The extension of the Γ-action, or more generally the $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action, to $\overline{X}^{\mathrm{BS}}$ will be given in (5.2.11).

Let us define the topology on $\overline{X}^{\mathrm{BS}}$, for which we only need to define the neighborhoods of the boundary points. For this purpose, we need to analyze boundary components $e(\mathbf{Q})$ and $e(\mathbf{P})$ for two parabolic subgroups $\mathbf{P} < \mathbf{Q}$ of $\mathbf{G}^{\mathrm{der}}$.

For the reductive subgroup $\mathbf{M_Q}$ of $\mathbf{Q}$, set $\mathbf{P'} := \mathbf{P} \cap \mathbf{M_Q}$. Then $\mathbf{P'}$ is a parabolic subgroup of $\mathbf{M_Q}$ such that, by looking at the root system construction,

$$\mathbf{M_{P'}} = \mathbf{M_P}, \ \mathbf{A_P} = \mathbf{A_Q} \times \mathbf{A_{P'}}, \ N_P = N_Q \rtimes N_{P'}. \tag{5.2.3}$$

Thus the horospherical decomposition of $X_{\mathbf{Q}}$ associated with $\mathbf{P'}$ is

$$X_{\mathbf{Q}} \simeq N_{P'}(\mathbb{R}) \times A_{\mathbf{P'}}(\mathbb{R})^+ \times X_{\mathbf{P'}} = N_{P'}(\mathbb{R}) \times A_{\mathbf{P'}}(\mathbb{R})^+ \times X_{\mathbf{P}}. \tag{5.2.4}$$

Next, we find another $\mathbb{Q}$-split torus $\mathbf{A}_{\mathbf{P},\mathbf{Q}}$ of $\mathbf{A}_{\mathbf{P}}$ which is isomorphic to $\mathbf{A}_{\mathbf{P}'}$. We start with the case where $\mathbf{P}$ is a standard parabolic subgroup. Namely, we fix a basis ${}_{\mathbb{Q}}\Delta$ of the relative root system ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{A}, \mathbf{G}^{\mathrm{der}})$ for some maximal $\mathbb{Q}$-split torus $\mathbf{A}$ in $\mathbf{G}^{\mathrm{der}}$, and then we obtain a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$ as below (4.4.2), and assume $\mathbf{P} = \mathbf{P}_I$ for some subset $I \subseteq {}_{\mathbb{Q}}\Delta$ as in Theorem 4.4.6. Since $\mathbf{Q} > \mathbf{P}$, we have $\mathbf{Q} > \mathbf{P}_0$ and hence $\mathbf{Q} = \mathbf{P}_J$ for some $J \subseteq {}_{\mathbb{Q}}\Delta$ by Theorem 4.4.6, and it is clear that $I \subseteq J$. By Lemma 4.4.7, we have then $\mathbf{A}_I > \mathbf{A}_J$. Better, using definitions of $\mathbf{A}_I$ and $\mathbf{A}_J$ we get that $\mathbf{A}_I = \mathbf{A}_{I,J} \times \mathbf{A}_J$, with $\mathbf{A}_{I,J} := \left(\bigcap_{\alpha' \in J \setminus I} \mathrm{Ker}\alpha'\right)^{\circ}$. Notice that in this case, $\Delta(A_{\mathbf{P}}, P) = {}_{\mathbb{Q}}\Delta \setminus I$, and hence $J \setminus I \subseteq \Delta(A_{\mathbf{P}}, P)$. In general, $\mathbf{P}$ is conjugate to a unique $\mathbf{P}_I$, and then the conjugation of $\mathbf{Q}$ by the same element in $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ is standard (*i.e.* contains $\mathbf{P}_0$), and hence $\mathbf{Q} = \mathbf{P}_J$ for some $J \subseteq {}_{\mathbb{Q}}\Delta$. Let $\mathbf{A}_{\mathbf{P},\mathbf{Q}} < \mathbf{P}$ be the suitable conjugation of $\mathbf{A}_{I,J}$, and let $I_{\mathbf{P},\mathbf{Q}} \subseteq \Delta(A_{\mathbf{P}}, P)$ be the suitable conjugation of $J \setminus I$. Then we have $\mathbf{A}_{\mathbf{P}} = \mathbf{A}_{\mathbf{Q}} \times \mathbf{A}_{\mathbf{P},\mathbf{Q}}$. Thus $\mathbf{A}_{\mathbf{P}'} \simeq \mathbf{A}_{\mathbf{P},\mathbf{Q}}$ by the second equality in (5.2.3). So (5.2.4) becomes

$$X_{\mathbf{Q}} \simeq N_{P'}(\mathbb{R}) \times A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ \times X_{\mathbf{P}}. \tag{5.2.5}$$

Therefore by (5.2.3) and (5.2.5), we have

$$e(\mathbf{Q}) = N_Q \times X_{\mathbf{Q}} \simeq N_P(\mathbb{R}) \times A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ \times X_{\mathbf{P}}. \tag{5.2.6}$$

Definition 5.2.1. *The topology on $\overline{X}^{\mathrm{BS}}$ is defined as follows: (i) on X it is the natural one, (ii) for each parabolic subgroup $\mathbf{P}$ of $\mathbf{G}^{\mathrm{der}}$, the neighborhoods of a point $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$ is $\bigsqcup_{\mathbf{Q} > \mathbf{P}} U \times A_{\mathbf{P},\mathbf{Q},t} \times V$ for all neighborhoods U of n in $N_P(\mathbb{R})$, all neighborhoods V of z in $X_{\mathbf{P}}$, and all $t > 0$, with*

$$A_{\mathbf{P},\mathbf{Q},t} := \{a \in A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ : \alpha(a)^{-1} < t, \ \forall \alpha \in I_{\mathbf{P},\mathbf{Q}}\}.$$

A better description is given in Corollary 5.2.4.

5.2.2 Borel–Serre partial compactification: corners and Hausdorff property

Recall the isomorphism $A_{\mathbf{P}}(\mathbb{R})^+ \simeq \mathbb{R}^r_{>0}$ from (5.1.1). Use $\overline{A_{\mathbf{P}}}$ to denote the closure of $A_{\mathbf{P}}(\mathbb{R})^+$ in $\mathbb{R}^r$ under the natural inclusion $\mathbb{R}^r_{>0} \subseteq \mathbb{R}^r$. The discussion on the topology of $\overline{X}^{\mathrm{BS}}$ in the previous subsection yields easily the following results.

Lemma 5.2.2. *We have a disjoint decomposition*

$$\overline{A_{\mathbf{P}}} = A_{\mathbf{P}}(\mathbb{R})^+ \cup \bigsqcup_{\mathbf{Q} > \mathbf{P}} (A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ \times 0_{\mathbf{Q}})$$

where $0_{\mathbf{Q}}$ is the origin of the real vector space $\mathbb{R}^{r'}$ arising from $A_{\mathbf{Q}}(\mathbb{R})^+ \simeq \mathbb{R}^{r'}_{>0} \subseteq \mathbb{R}^{r'}$.

Proposition 5.2.3. *The embedding $N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}} \simeq X \subseteq \overline{X}^{\mathrm{BS}}$ can be naturally extended to an open embedding $N_P(\mathbb{R}) \times \overline{A_{\mathbf{P}}} \times X_{\mathbf{P}} \hookrightarrow \overline{X}^{\mathrm{BS}}$. Moreover, the image of $N_P(\mathbb{R}) \times \overline{A_{\mathbf{P}}} \times X_{\mathbf{P}}$ in $\overline{X}^{\mathrm{BS}}$ is equal to the subset*

$$X \cup \bigsqcup\nolimits_{\mathbf{Q} > \mathbf{P}} e(\mathbf{Q}) \tag{5.2.7}$$

in $\overline{X}^{\mathrm{BS}}$.

We will call (5.2.7) the *corner associated with* $\mathbf{P}$ and denote it by $X(\mathbf{P})$. Then we have

$$X(\mathbf{P}) \simeq X \times_{A_{\mathbf{P}}(\mathbb{R})^+} \overline{A_{\mathbf{P}}}, \quad e(\mathbf{P}) = N_P(\mathbb{R}) \times \{(0, \cdots, 0)\} \times X_{\mathbf{P}}, \quad X(\mathbf{P}) \simeq e(\mathbf{P}) \times [0, \infty)^r. \quad (5.2.8)$$

Another corollary of Lemma 5.2.2 is the following description of neighborhood bases of points in the boundaries.

Corollary 5.2.4. *For any point* $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$*, a neighborhood basis in* $\overline{X}^{\mathrm{BS}}$ *is given by* $U \times \overline{A_{\mathbf{P}}}_{,t} \times V \subseteq X(\mathbf{P})$*, where* $n \in U, z \in V$ *are bases of neighborhoods of* n *and* z *respectively, and* $t > 0$ *with*

$$\overline{A_{\mathbf{P}}}_{,t} := \{a \in \overline{A_{\mathbf{P}}} : \alpha(a)^{-1} < t, \ \forall \alpha \in \Delta(A_{\mathbf{P}}, P)\}.$$

This neighborhood basis is highly related to the Siegel sets (5.1.2). This relation is the key to the proof of the following proposition.

Proposition 5.2.5. $\overline{X}^{\mathrm{BS}}$ *is a Hausdorff space.*

Proof. Take two distinct points $y_1, y_2 \in \overline{X}^{\mathrm{BS}} \setminus X$, with $y_j \in e(\mathbf{P}_j)$.

If $\mathbf{P}_1 = \mathbf{P}_2$, then $e(\mathbf{P}_1) = e(\mathbf{P}_2)$ and clearly there are open neighborhoods of y_1 and y_2 which are disjoint.

From now on assume $\mathbf{P}_1 \neq \mathbf{P}_2$. Assume that y_1 and y_2 have open neighborhoods which are non-disjoint. By Corollary 5.2.4, we may assume that the neighborhoods are $U_1 \times \overline{A_{\mathbf{P}_1}}_{,t} \times V_1$ and $U_2 \times \overline{A_{\mathbf{P}_2}}_{,t} \times V_2$ with $t > 0$. We may furthermore assume that U_1, V_1, U_2, V_2 are bounded. Call the intersection W. Then W is open in $U_j \times \overline{A_{\mathbf{P}_j}}_{,t} \times V_j$.

Because $U_j \times A_{\mathbf{P}_j,t} \times V_j$ is open and dense in $U_j \times \overline{A_{\mathbf{P}_j}}_{,t} \times V_j$, we have that $W \cap (U_j \times A_{\mathbf{P}_j,t} \times V_j)$ is open and dense in W. So $W \cap (U_1 \times A_{\mathbf{P}_1,t} \times V_1) \cap (U_2 \times A_{\mathbf{P}_2,t} \times V_2)$ is non-empty.

But $\mathbf{P}_1 \neq \mathbf{P}_2$, so general theory of Siegel sets says that $(U_1 \times A_{\mathbf{P}_1,t} \times V_1) \cap (U_2 \times A_{\mathbf{P}_2,t} \times V_2) = \emptyset$ for $t \gg 1$ (say $t \geq t_0$ for some fixed $t_0 \in \mathbb{R}$). Therefore by the previous paragraph, $t < t_0$. Hence we find open neighborhoods $U_1 \times \overline{A_{\mathbf{P}_1}}_{,t_0} \times V_1$ of y_1 and $U_2 \times \overline{A_{\mathbf{P}_2}}_{,t_0} \times V_2$ of y_2 which are disjoint. We are done. $\square$

5.2.3 Extension of $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action

For any element $\gamma \in \mathbf{G}^{\mathrm{der}}(\mathbb{C})$, write ${}^{\gamma}(\cdot)$ for the conjugate $\gamma(\cdot)\gamma^{-1}$.

We start by explaining the action of $P(\mathbb{R})^+$ on the boundary component $e(\mathbf{P})$. Recall $P = N_P \rtimes (A_{\mathbf{P}} M_{\mathbf{P}})$. Let $p \in P(\mathbb{R})^+$, which decomposes into $p = n_0 a_0 m_0$ with $n_0 \in N_P(\mathbb{R})$, $a_0 \in A_{\mathbf{P}}(\mathbb{R})^+$ and $m_0 \in M_{\mathbf{P}}(\mathbb{R})$. Then for $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$, set

$$p \cdot (n, z) := (n_0 \cdot {}^{a_0 m_0} n, m_0 z) \in N_P(\mathbb{R}) \times X_{\mathbf{P}} = e(\mathbf{P}). \quad (5.2.9)$$

We can rewrite this action in the following way. Instead of decomposing $p = n_0 a_0 m_0$, we can also decompose it into $p = m' a' n'$ with $m' \in M_{\mathbf{P}}(\mathbb{R})$, $a' \in A_{\mathbf{P}}(\mathbb{R})^+$ and $n' \in N_P(\mathbb{R})$. Indeed (since $A_{\mathbf{P}}$ and $M_{\mathbf{P}}$ commute), we can take $m' = m_0$, $a' = a_0$, and $n' = {}^{(a_0 m_0)^{-1}} n_0$. Then

$$p \cdot (n, z) = ({}^{m' a'}(n' n), m' z). \quad (5.2.10)$$

Next we extend this action to the action of $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ on $\overline{X}^{\mathrm{BS}}$ as follows. Let $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$. Then we can decompose $g = kp$ for $k \in K_\infty$ and $p \in P(\mathbb{R})$, and moreover $p = m' a' n'$ with $m' \in M_{\mathbf{P}}(\mathbb{R})$, $a' \in A_{\mathbf{P}}(\mathbb{R})^+$ and $n' \in N_P(\mathbb{R})$. Notice that both k and m' are not uniquely determined by g, but determined up to an element in $K_\infty \cap P = K_\infty \cap M_{\mathbf{P}}$. In particular, the product km' is uniquely determined by g. Notice that ${}^k\mathbf{P} = {}^g\mathbf{P}$ is a $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$. Set

$$g \cdot (n, z) := ({}^{km'a'}(n'n), k \cdot m'z) \in N_{{}^kP}(\mathbb{R}) \times X_{{}^k\mathbf{P}} = e({}^k\mathbf{P}) = e({}^g\mathbf{P}). \tag{5.2.11}$$

We need to explain the notation $k \cdot m'z$. Denoting by $K_{\mathbf{P}} := K_\infty \cap M_{\mathbf{P}}$, the point $m'z \in X_{\mathbf{P}} \simeq M_{\mathbf{P}}(\mathbb{R})^+/K_{\mathbf{P}}$ can be written as $mK_{\mathbf{P}}$ for some $m \in M_{\mathbf{P}}(\mathbb{R})^+$. Then $k \cdot m'z \in X_{{}^k\mathbf{P}} = X_{{}^g\mathbf{P}}$ is ${}^k m K_{{}^k\mathbf{P}} = {}^k m K_{{}^g\mathbf{P}}$.

Proposition 5.2.6. *The action of $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ on $\overline{X}^{\mathrm{BS}}$ defined above is continuous.*

Proof. It suffices to prove the following: Let $\{y_j\}$ be a sequence of points in $\overline{X}^{\mathrm{BS}}$ which converges to y_∞, then $\{g \cdot y_j\}$ converges to $g \cdot y_\infty$ for any $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$. This is clearly true if $y_\infty \in X$. Thus we may assume $y_\infty \in e(\mathbf{P})$ for some $\mathbf{P}$.

Now there are two cases to consider: either $\{y_j\} \subseteq X$, or $\{y_j\} \subseteq e(\mathbf{Q})$ for some $\mathbf{Q} > \mathbf{P}$. Indeed, by passing to a subsequence we can always reduce to one of these two cases. In the first case, write each y_j under the horospherical decomposition associated with $\mathbf{P}$. In the second case, write the $X_{\mathbf{Q}}$-component of each y_j under the relative horospherical decomposition (5.2.5). We omit the details of the computation. $\square$

Finally, let $\Gamma < \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ be an arithmetic subgroup. We have:

Corollary 5.2.7. *Γ acts properly on $\overline{X}^{\mathrm{BS}}$, i.e. any point $x \in \overline{X}^{\mathrm{BS}}$ has an open neighborhood W such that*

$$\{\gamma \in \Gamma : \gamma W \cap W \neq \emptyset\}$$

is a finite set.

Proof. It is known that Γ acts properly on X. So it suffices to prove the result for $x \in e(\mathbf{P})$ for any $\mathbf{P}$. By Corollary 5.2.4, we may take $W = U \times \overline{A_{\mathbf{P}t}} \times V$, with $U \times A_{\mathbf{P},t} \times V$ a Siegel set in X. Since W is open in $\overline{X}^{\mathrm{BS}}$ and that Γ acts continuously on $\overline{X}^{\mathrm{BS}}$ (Proposition 5.2.6), we have:

$$\gamma(U \times A_{\mathbf{P},t} \times V) \cap (U \times A_{\mathbf{P},t} \times V) \neq \emptyset \Rightarrow \gamma W \cap W \neq \emptyset$$

with an argument similar to Proposition 5.2.5. Hence the desired finiteness follows from general theory of Siegel sets. $\square$

5.2.4 Quotient by Γ and conclusion

Theorem 5.2.8. *The quotient $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ is a compact Hausdorff space. If Γ is torsion-free, then $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ has a canonical structure of a real analytic manifold with corners.*

Moreover, there are finitely many Γ-conjugacy classes of proper rational parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$. Taking a set of representatives $\{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$, we have

$$\Gamma\backslash\overline{X}^{\mathrm{BS}} = \Gamma\backslash X \cup \coprod_{j=1}^{m} \Gamma_{\mathbf{P}_j}\backslash e(\mathbf{P}_j) \tag{5.2.12}$$

with $\Gamma_{\mathbf{P}_j} := \Gamma \cap \mathbf{P}_j(\mathbb{Q})$.

We also use $\overline{\Gamma\backslash X}^{\mathrm{BS}}$ to denote $\Gamma\backslash\overline{X}^{\mathrm{BS}}$.

Proof. $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ by Proposition 5.2.5 and Corollary 5.2.7.

For the proper rational parabolic subgroups $\mathbf{P}_1, \ldots, \mathbf{P}_m$ of $\mathbf{G}^{\mathrm{der}}$, reduction theory says that the images of some associated Siegel sets

$$U_1 \times A_{\mathbf{P}_1,t_1} \times V_1, \ldots, U_m \times A_{\mathbf{P}_m,t_m} \times V_m$$

under $X \to \Gamma\backslash X$ cover the whole space. Clearly we can take all the U_j, V_j's to be compact. By Proposition 5.2.3, the closure of $U_j \times A_{\mathbf{P}_j,t_j} \times V_j$ in $\overline{X}^{\mathrm{BS}}$ is $U_j \times \overline{A_{\mathbf{P}_j,t_j}} \times V_j$, which is a compact set. The Γ-translates of these compact sets cover $\overline{X}^{\mathrm{BS}}$ because X is dense in $\overline{X}^{\mathrm{BS}}$. So we prove the compactness of $\Gamma\backslash\overline{X}^{\mathrm{BS}}$.

Next we show that $\overline{X}^{\mathrm{BS}}$ has a canonical structure of real semi-algebraic manifolds with corners. Indeed, this is clearly true for $X(\mathbf{P}) \simeq N_P(\mathbb{R}) \times \overline{A_{\mathbf{P}}} \times X_{\mathbf{P}}$ for each $\mathbf{P}$, and it is not hard to check that the real semi-algebraic structures of different $X(\mathbf{P})$'s are compatible (it suffices to check for $\mathbf{Q} > \mathbf{P}$, for which we can use (5.2.5)). The $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action is easily seen to be given by real semi-algebraic diffeomorphisms.

Now if Γ is torsion-free, then the action of Γ on $\overline{X}^{\mathrm{BS}}$ has no fixed points. So $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ has a canonical structure of a real analytic manifold with corners.

Finally to get (5.2.12), it suffices to show that $\Gamma_{\mathbf{P}}$ is the stabilizer of $e(\mathbf{P})$ in Γ for each $\mathbf{P}$. This is true because: for any $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$, either $g \in P(\mathbb{R})^+$ and $ge(\mathbf{P}) = e(\mathbf{P})$, or $g \notin P(\mathbb{R})^+$ and $ge(\mathbf{P}) \cap e(\mathbf{P}) = \emptyset$; see (5.2.11). We are done. $\square$

Example 5.2.9. *For the Poincaré upper half plan $\mathfrak{H}$ and the group $\mathbf{SL}_2$, consider the parabolic subgroup*

$$\mathbf{P} = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{Q}^\times,\ b \in \mathbb{Q} \right\}.$$

We have $N_P = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{R} \right\} \simeq \mathbb{G}_{\mathrm{a},\mathbb{R}}$. *So* $e(\mathbf{P}) \simeq \mathbb{R}$, *by adding a "real axis" at the point* ∞. *Then* $\Gamma_{\mathbf{P}}\backslash e(\mathbf{P}) \simeq \mathbb{Z}\backslash\mathbb{R}$ *is a loop, with* $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

5.3 Reductive Borel–Serre compactification

It often occurs that the Borel–Serre compactification is too large. In this section we define the *reductive Borel–Serre compactification.* For each $\mathbf{P}$, define the *reductive Borel–Serre boundary component* to be

$$e(\mathbf{P}) := X_{\mathbf{P}}. \tag{5.3.1}$$

Then clearly it is obtained from the Borel–Serre boundary component (5.2.1) by collapsing $N_P(\mathbb{R})$. Define the *reductive Borel–Serre partial compactification* to be

$$\overline{X}^{\mathrm{RBS}} := X \cup \bigsqcup_{\mathbf{P}} e(\mathbf{P}), \tag{5.3.2}$$

with the topology as follows. Recall, for each $\mathbf{Q} > \mathbf{P}$, (5.2.5)

$$X_{\mathbf{Q}} \simeq N_{P'}(\mathbb{R}) \times A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ \times X_{\mathbf{P}}.$$

So $e(\mathbf{P})$ is attached to $e(\mathbf{Q})$ at infinity (here, we use the isomorphism $A_{\mathbf{P}}(\mathbb{R})^+ \simeq \mathbb{R}^r_{>0}$ which is componentwise the inverse of (5.1.1)). In particular for $\mathbf{Q} = \mathbf{G}^{\mathrm{der}}$, we retain the horospherical decomposition $X \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}}$. Now for any $z \in e(\mathbf{P}) = X_{\mathbf{P}}$, a basis of neighborhood system of of z in $\overline{X}^{\mathrm{RBS}}$ is given by

$$(N_P(\mathbb{R}) \times A_{\mathbf{P},t} \times W) \cup \bigsqcup\nolimits_{\mathbf{Q} > \mathbf{P}} (N_{P'}(\mathbb{R}) \times A_{\mathbf{P},\mathbf{Q},t} \times W)$$

with W a neighborhood of z in $e(\mathbf{P})$ and $t > 0$. Observe that if W is open, then the union above is the interior of the closure of $N_P(\mathbb{R}) \times A_{\mathbf{P},t} \times W$ in $\overline{X}^{\mathrm{RBS}}$.

Similarly to the discussion on Borel–Serre compactifications, we have:

Theorem 5.3.1. *$\overline{X}^{\mathrm{RBS}}$ is Hausdorff, and the $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action on X extends continuously to $\overline{X}^{\mathrm{RBS}}$.*

The quotient $\Gamma \backslash \overline{X}^{\mathrm{RBS}}$ is a compact Hausdorff space containing $\Gamma \backslash X$ as an open dense subset. If we let $\{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$ be a set of representatives of the Γ-conjugacy classes of proper parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$, then we have

$$\Gamma \backslash \overline{X}^{\mathrm{RBS}} = \Gamma \backslash X \cup \bigsqcup_{j=1}^{m} \Gamma_{\mathbf{M}_{\mathbf{P}_j}} \backslash X_{\mathbf{P}_j} \tag{5.3.3}$$

with $\Gamma_{\mathbf{M}_{\mathbf{P}_j}} := \Gamma \cap \mathbf{M}_{\mathbf{P}_j}(\mathbb{Q})$.

We also use $\overline{\Gamma \backslash X}^{\mathrm{RBS}}$ to denote $\Gamma \backslash \overline{X}^{\mathrm{RBS}}$.

Theorem 5.3.2. *The identity map on X extends to a continuous surjective $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-equivariant map $\overline{X}^{\mathrm{BS}} \to \overline{X}^{\mathrm{RBS}}$.*

The identity map on $\Gamma \backslash X$ extends to a continuous map $\overline{\Gamma \backslash X}^{\mathrm{BS}} \to \overline{\Gamma \backslash X}^{\mathrm{RBS}}$.