Chapter 3

Shimura data and Shimura varieties

3.1 Basic definitions

3.1.1 Shimura data

Definition 3.1.1. *A* **Shimura datum** *is a pair* $(\mathbf{G}, X)$ *where*

- $\mathbf{G}$ *is a reductive group defined over* $\mathbb{Q}$*,*
- X *is a* $\mathbf{G}(\mathbb{R})$*-orbit in* $\mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$

such that for one (and hence all) $h \in X$*, we have*

(SV1) the Hodge structure $\mathrm{Ad} \circ h$ *on* $\mathrm{Lie}\mathbf{G}$ *has type* $(-1,1)+(0,0)+(1,-1)$*,*

(SV2) $\mathrm{Int}(h(\sqrt{-1}))$ *is a Cartan involution of* $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$*,*

(SV3) for every $\mathbb{Q}$*-simple factor* $\mathbf{H}$ *of* $\mathbf{G}^{\mathrm{ad}}$*, the morphism* $\mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \to \mathbf{H}_{\mathbb{R}}$ *is non-trivial.*

A **(Shimura) morphism** *between two Shimura data* $\rho\colon (\mathbf{G}', X') \to (\mathbf{G}, X)$ *is a morphism* ρ *on the underlying groups such that* $\rho \circ h \in X$ *for all* $h \in X'$*. In particular, we call the image of such a Shimura morphism to be a* **sub-Shimura datum** *of* $(\mathbf{G}, X)$*.*

The main difference of a Shimura datum and the pair (G, X^+) from §2.3 is the definition field of the group (over $\mathbb{Q}$ or over $\mathbb{R}$). A similar assumption to (SV3) for (G, X^+) has been discussed in Remark 2.3.3. By Theorem 2.3.1, each connected component X^+ of X is a Hermitian symmetric domain (and the complex structure on X is $\mathbf{G}(\mathbb{R})$-invariant). By Proposition 1.3.5, each representation V of $\mathbf{G}$ gives rise to a $\mathbb{Q}$-VHS on X^+ by (SV1), which furthermore carries $\mathbb{R}$-polarization by Proposition 2.2.6 and (SV2).[1]

The following two further assumptions guarantee that this $\mathbb{Q}$-VHS carries a $\mathbb{Q}$-polarization. Notice that they may not be satisfied by an arbitrary Shimura datum.

(SV4) the morphism $w_h\colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to Z(\mathbf{G})_{\mathbb{R}}$ is defined over $\mathbb{Q}$,

(SV2') $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of $\mathbf{G}_{\mathbb{R}}/w_h(\mathbb{G}_{\mathrm{m},\mathbb{R}})$.

Example 3.1.2 (0-dimensional Shimura datum)**.** *The set* X *is a finite set if and only if* $\mathbf{G}$ *is abelian (and hence an algebraic torus). This case shows up when we study CM abelian varieties.*

[1](SV1) implies that $w_h\colon \mathbb{G}_{\mathrm{m}} \xrightarrow{w} \mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}}$ factors through $Z(\mathbf{G})_{\mathbb{R}}$, so we can apply Proposition 2.2.6.

Example 3.1.3 (Siegel Shimura datum)**.** *Let us take the example of the Siegel case from Example 2.3.1, except that the vector space and the groups are defined over* $\mathbb{Q}$*. More precisely,* $V = \mathbb{Q}^{2d}$ *and* $\psi\colon V \times V \to \mathbb{Q}$ *is* $(x, y) \mapsto x^{\mathrm{t}} Jy$ *with* $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$*. The* $\mathbb{Q}$*-group is*

$$\begin{aligned}\mathbf{GSp}(\psi) = \mathbf{GSp}_{2d} &:= \left\{g \in \mathrm{GL}(V) : \psi(gx, gy) = c\psi(x, y) \text{ for some } c \in \mathbb{Q}^{\times}\right\} \\ &= \left\{g \in \mathrm{GL}_{2d,\mathbb{Q}} : gJg^{\mathrm{t}} = cJ \text{ for some } c \in \mathbb{Q}^{\times}\right\},\end{aligned}$$

and $h_0\colon \mathbb{S} \to \mathbf{GSp}_{2d,\mathbb{R}}$ *maps* $a + b\sqrt{-1} \mapsto aI_{2d} + bJ$*. The derived subgroup is* $\mathbf{Sp}(\psi) = \mathbf{Sp}_{2d}$ *by requesting the* $c \in \mathbb{Q}^{\times}$ *in the definition to be* 1*.*

The $\mathbf{G}(\mathbb{R})$*-orbit is* $\mathbf{GSp}_{2d}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{GSp}_{2d,\mathbb{R}})$*. Under the identification similar to (2.3.1), we have*

$$\mathbf{GSp}_{2d}(\mathbb{R})h_0 \xrightarrow{\sim} \mathfrak{H}_d^{\pm} := \left\{\tau \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}},\ \text{either } \mathrm{Im}\tau > 0 \text{ or } \mathrm{Im}\tau < 0\right\}. \qquad (3.1.1)$$

Then $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$ *is a Shimura datum by Example 2.3.1 (see Remark 2.3.3 for (SV3)). It is called the* **Siegel Shimura datum***. Moreover, (SV4) and (SV2') are easily seen to be also satisfied. In fact,* V *is a representation of* $\mathbf{GSp}_{2d}$*, and* ψ *is the desired* $\mathbb{Q}$*-polarization on the induced* $\mathbb{Q}$*-VHS.*

Next we present an example where (SV4) and (SV2') are not satisfied. We also see in this example that two Shimura data may have the same underlying $\mathbb{R}$-group and the same underlying space, but the $\mathbb{Q}$-groups are different.

Example 3.1.4 (Shimura curves)**.** *Let* B *be a simple quaternion algebra over a totally real number field* F*. Assume that* B *is split at exactly one real place of* F*, i.e. there exists a real embedding* $\sigma\colon K \to \mathbb{R}$ *such that*

$$B_\sigma \simeq \begin{cases} \mathrm{M}_2(\mathbb{R}) & \text{if } \sigma = \sigma_0 \\ \mathbb{H} & \text{otherwise} \end{cases}$$

for all read embeddings $\sigma\colon K \to \mathbb{R}$*, where* $\mathbb{H}$ *is the Hamilton quaternion algebra over* $\mathbb{R}$*.*

Define the $\mathbb{Q}$*-group* $\mathbf{G}$

$$\mathbf{G}(R) := (B \otimes_{\mathbb{Q}} R)^{\times} \quad \text{for all } \mathbb{Q}\text{-algebra } R,$$

and let

$$h_0\colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \simeq \mathrm{GL}_{2,\mathbb{R}} \times \mathbb{H}^{\times} \times \cdots \times \mathbb{H}^{\times}, \qquad a + b\sqrt{-1} \mapsto \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, 1, \ldots, 1\right).$$

Thus all but the first components of $\mathbf{G}(\mathbb{R})h_0$ *are the identity map, and so* $\mathbf{G}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$ *is isomorphic to* $\mathfrak{H}_1^{\pm}$*, via an isomorphism similar to (3.1.1) (with* $d = 1$*). Both (SV1) and (SV2) hold true for the pair* $(\mathbf{G}, \mathfrak{H}_1^{\pm})$ *similarly to the Siegel case. To see (SV3), it suffices to observe that* $\mathbf{G}^{\mathrm{ad}}$ *is a simple group because* B *is a simple quaternion algebra over* F*.*

So $(\mathbf{G}, \mathfrak{H}_1^{\pm})$ *is a Shimura datum. However, if* $[F : \mathbb{Q}] > 1$*, then (SV4) and (SV2') do not hold true, by looking at the action of* $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$*.*

And even in the case $F = \mathbb{Q}$*, the group* $\mathbf{G}$ *is not necessarily* $\mathbf{GL}_2$*. So* $(\mathbf{G}, \mathfrak{H}_1^{\pm})$ *needs not be the Siegel Shimura datum in this case.*

3.1.2 Shimura varieties

Denote by $\mathbb{A}_{\mathrm{f}} \subseteq \prod_{p \in M_{\mathbb{Q},\mathrm{f}}} \mathbb{Q}_p$ the ring of finite adèles over $\mathbb{Q}$, and by $\widehat{\mathbb{Z}} := \prod \mathbb{Z}_p$. Then $\widehat{\mathbb{Z}}$ is a (maximal) compact open subgroup of $\mathbb{A}_{\mathrm{f}}$, and $\mathbb{Q}$ is dense in $\mathbb{A}_{\mathrm{f}}$.

Let $(\mathbf{G}, X)$ be a Shimura datum. Then $\mathbf{G}(\mathbb{Q})$ acts on X by definition of Shimura data, and consider the action of $\mathbf{G}(\mathbb{Q})$ on $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ by multiplication on the left.

Definition 3.1.5. *Let* $(\mathbf{G}, X)$ *be a Shimura datum. A* **Shimura variety** *is a double coset*

$$\mathrm{Sh}_K(\mathbf{G}, X) := \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$$

where $K \subseteq \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ *is a compact open subset. Here* $\mathbf{G}(\mathbb{Q})$ *acts on both* X *and* $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ *on the left as above, and* K *acts on* $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ *by the multiplication on the right; i.e.* $q(x,g)k = (q \cdot x, qgk)$ *for all* $q \in \mathbf{G}(\mathbb{Q})$, $(x,g) \in X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ *and* $k \in K$.

We will prove in this course that the double coset $\mathrm{Sh}_K(\mathbf{G}, X)$ is the set of $\mathbb{C}$-points of an algebraic variety. This justifies the name of Shimura variety.

Example 3.1.6. *In the Siegel example above, the group* $\mathbf{GSp}_{2d}$ *is defined over* $\mathbb{Z}$*; indeed we can take* V *to be* $\mathbb{Z}^{2d}$ *and* ψ *maps* $V \times V$ *to* $\mathbb{Z}$*. Then* $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$ *is a compact open subgroup of* $\mathbf{GSp}_{2d}(\mathbb{A}_{\mathrm{f}})$*. Other compact open subgroups include* gKg^{-1} *for any* $g \in \mathbf{GSp}_{2d}(\mathbb{A}_{\mathrm{f}})$ *and any finite-indexed subgroup* K *of* $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$*. We will come back to this example in §3.3 and prove that the Siegel Shimura varieties are moduli spaces of abelian varieties.*

Definition 3.1.7. *A* **(Shimura) morphism** $[\rho]\colon \mathrm{Sh}_{K'}(\mathbf{G}', X') \to \mathrm{Sh}_K(\mathbf{G}, X)$ *between two Shimura varieties is a morphism of Shimura data* $\rho\colon (\mathbf{G}', X') \to (\mathbf{G}, X)$ *such that* $\rho(K') \subseteq K$.

Example 3.1.8. *Let* $\mathrm{Sh}_K(\mathbf{G}, X)$ *be a Shimura variety.*

Let $K' \subseteq K$ *be another compact open subgroup of* $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$*. Then the identity map on* $(\mathbf{G}, X)$ *induces a Shimura morphism* $\mathrm{Sh}_{K'}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$*, with finite fibers since* K' *must have finite index in* K*. In fact, this is finite morphism in the category of algebraic varieties.*

Let $g \in \mathbf{G}(\mathbb{A}_{\mathrm{f}})$*. Then* gKg^{-1} *is a compact open subgroup of* $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$*, and we have a Shimura morphism* $[g\cdot]\colon \mathrm{Sh}_{gKg^{-1}}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$*, sending* $[x, g'] \mapsto [x, gg']$*. More generally, if* K' *is a compact open subgroup of* $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ *such that* $K' \subseteq gKg^{-1}$*, then we have a Shimura morphism* $[g\cdot]\colon \mathrm{Sh}_{K'}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$ *which is a finite morphism.*

Example 3.1.9 (Hecke operator)**.** *Let* $\mathrm{Sh}_K(\mathbf{G}, X)$ *be a Shimura variety.*

Any $g \in \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ *induces a correspondence on* $\mathrm{Sh}_K(\mathbf{G}, X)$ *as follows. Write* $K' := K \cap gKg^{-1}$ *for simplicity; it is a compact open subgroup of* $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ *and* $K' \subseteq gKg^{-1}$*. We have Shimura morphisms*

$$\begin{array}{ccc} & \mathrm{Sh}_{K'}(\mathbf{G}, X) & \\ {\scriptstyle [g\cdot]}\swarrow & & \searrow{\scriptstyle [1]} \\ \mathrm{Sh}_K(\mathbf{G}, X) & & \mathrm{Sh}_K(\mathbf{G}, X) \end{array}$$

where the right one is induces by identity on $(\mathbf{G}, X)$*. Both are finite morphisms, so we have a correspondence on* $\mathrm{Sh}_K(\mathbf{G}, X)$*, which is called the* **Hecke correspondence/operator** *and denoted by* T_g.

Definition 3.1.10. *Let* $\mathrm{Sh}_K(\mathbf{G}, X)$ *be a Shimura variety. We call any irreducible component of* $(T_g \circ [\rho])(\mathrm{Sh}_{K'}(\mathbf{G}', X'))$*, where* $[\rho]$ *is a Shimura morphism and* $g \in \mathbf{G}(\mathbb{A}_{\mathrm{f}})$*, to be a* **special subvariety** *of* $\mathrm{Sh}_K(\mathbf{G}, X)$*. A special subvariety of dimension* 0 *is called a* **special point**.

Of course in the definition of special subvarieties, it suffices to consider the Shimura morphisms arising from sub-Shimura data of $(\mathbf{G}, X)$. Thus special points arise from sub-Shimura data $(\mathbf{T}, X_{\mathbf{T}})$ of $(\mathbf{G}, X)$ where $\mathbf{T}$ is an algebraic torus.

3.2 Decomposition of Shimura varieties into Hermitian locally symmetric domains

Let $(\mathbf{G}, X)$ be a Shimura datum. Then any connected component X is a Hermitian symmetric domain. Fix one such X^+.

Let K be a compact open subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f})$. Then we have a Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ defined as the double coset $\mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_\mathrm{f})/K$. We wish to prove that this double coset is the $\mathbb{C}$-points of an algebraic variety.

In this section, we start with the first step, by endowing $\mathrm{Sh}_K(\mathbf{G}, X)$ with a structure of complex varieties.

Theorem 3.2.1. *There exists a finite-indexed subgroup K' of K such that*

$$\mathrm{Sh}_{K'}(\mathbf{G}, X) \simeq \bigsqcup_{g \in \mathcal{C}} \Gamma_g \backslash X^+, \tag{3.2.1}$$

for a finite set $\mathcal{C} \subseteq \mathbf{G}(\mathbb{A}_\mathrm{f})$, with each Γ_g a torsion-free discrete group acting on X^+.

The actual decomposition will be given later on (3.2.3), where the definitions of $\mathcal{C}$ and Γ_g are given. At this stage, let us make the following observation: since Γ_g is torsion-free discrete, the quotient $\Gamma_g\backslash X^+$ has a natural structure of complex manifolds and even more is a Hermitian locally symmetric domain. So $\mathrm{Sh}_{K'}(\mathbf{G}, X)$ is a finite disjoint union of Hermitian locally symmetric domains. As for $\mathrm{Sh}_K(\mathbf{G}, X)$, the finite-to-1 map $\mathrm{Sh}_{K'}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$ then makes $\mathrm{Sh}_K(\mathbf{G}, X)$ into a finite union of complex orbifolds.

3.2.1 Two approximation theorems for algebraic groups

Let $\mathbf{H}$ be an algebraic group defined over $\mathbb{Q}$. We will use the following approximation theorems.

- (*Real Approximation*) $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{R})$.
- (*Strong Approximation*) If $\mathbf{H}$ is semi-simple and simply-connected, then $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{A}_\mathrm{f})$.

The definition of "simply-connected" will be recalled later in §3.2.5.

3.2.2 Preparation and adjoint Shimura data

Now let us introduce the *adjoint Shimura datum* $(\mathbf{G}^{\mathrm{ad}}, \overline{X})$ of $(\mathbf{G}, X)$. Take $h \in X^+$. Then h induces a morphism

$$\overline{h} \colon \mathbb{S} \xrightarrow{h} \mathbf{G}_\mathbb{R} \to \mathbf{G}^{\mathrm{ad}}_\mathbb{R}.$$

Hence we obtain a $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$-orbit $\overline{X} := \mathbf{G}^{\mathrm{ad}}(\mathbb{R})\overline{h}$ in $\mathrm{Hom}(\mathbb{S}, \mathbf{G}^{\mathrm{ad}}_\mathbb{R})$, with a natural map $X \to \overline{X}$. The image of X^+ in $\overline{X}$ is connected, and the following lemma (applied to $G = \mathbf{G}(\mathbb{R})$)[2] easily implies that this image is again a connected component of $\overline{X}$. So by abuse of notation, we will also use X^+ to denote a connected component of $\overline{X}$.

Lemma 3.2.2. *For any algebraic group G over $\mathbb{R}$, the adjoint quotient $G^+ \to (G^{\mathrm{ad}})^+$ is surjective when restricted to the identity component.*

[2]Here is a background for this lemma. Let $\varphi \colon H \to H'$ be a morphism of algebraic groups defined over k. Assume $\mathrm{char}(k) = 0$. Then φ is called *surjective* if $\varphi(H(\overline{k})) = H'(\overline{k})$. If φ is surjective, it may happen that $\varphi(H(k)) \neq H'(k)$!

We omit the proof of this lemma. Define

$$\begin{aligned}\mathbf{G}(\mathbb{R})_+ &:= \text{inverse image of } \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+ \text{ in } \mathbf{G}(\mathbb{R})\\ \mathbf{G}(\mathbb{Q})_+ &:= \mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})_+.\end{aligned} \tag{3.2.2}$$

Lemma 3.2.3. $\mathbf{G}(\mathbb{R})_+$ *is the stabilizer of* X^+, *i.e.* $\mathbf{G}(\mathbb{R})_+ = \{g \in \mathbf{G}(\mathbb{R}) : gX^+ = X^+\}$.

With Lemma 3.2.3, we can complete our more precise version of (3.2.1):

$$\mathrm{Sh}_K(\mathbf{G}, X) \simeq \coprod_{[g]\in\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K} \Gamma_g\backslash X^+, \tag{3.2.3}$$

with $\Gamma_g := gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$; replacing K by a suitable finite-indexed subgroup K' guarantees that Γ_g is torsion-free, see §3.2.4. The finiteness of the double coset $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$ will be proved in §3.2.5; the proof uses the *Strong Approximation Theorem*.

Proof of Lemma 3.2.3. Consider the action of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ on $\overline{X}$, and recall that X^+ is a connected component of $\overline{X}$. It suffices to prove that $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+ = \{g \in \mathbf{G}^{\mathrm{ad}}(\mathbb{R}) : gX^+ = X^+\}$. This follows from general theory of Hermitian symmetric domains (and some knowledge on $\mathbb{R}$-algebraic groups *v.s.* real Lie groups) which we will not cover in this course. ☐

3.2.3 Proof of (3.2.3)

We start by showing that there is a bijection

$$\mathbf{G}(\mathbb{Q})_+\backslash X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}}) \xrightarrow{\sim} \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}}), \qquad [x, g] \mapsto [x, g]. \tag{3.2.4}$$

Injectivity: Assume $(x, g), (x', g') \in X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ are mapped to the same point on the right hand side. Then there exists $q \in \mathbf{G}(\mathbb{Q})$ such that $(x', g') = q(x, g) = (qx, qg)$. Hence $qX^+ \cap X^+$ is non-empty as it contains $qx = x'$. So $qX^+ = X^+$. So $q \in \mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})_+$. This proves the injectivity of the map above.

Surjectivity: Assume $(x, g) \in X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$. By the *Real Approximation* in §3.2.1, $\mathbf{G}(\mathbb{Q})x$ is dense in $\mathbf{G}(\mathbb{R})x = X$. So $\mathbf{G}(\mathbb{Q})x \cap X^+ \neq \emptyset$, and hence there exists $q \in \mathbf{G}(\mathbb{Q})$ such that $qx \in X^+$. Then $(qx, qg) \in X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$, and its image under (3.2.4) is $[x, g]$. We are done for the surjectivity of (3.2.3).

Now let us prove the bijectivity of the map

$$\coprod_{[g]\in\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K} \Gamma_g\backslash X^+ \to \mathbf{G}(\mathbb{Q})_+\backslash X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K, \qquad \Gamma_g x \mapsto [x, g]. \tag{3.2.5}$$

Injectivity: If $[x', g'] = [x, g]$, then $(qx, qgk) = (x', g')$ for some $q \in \mathbf{G}(\mathbb{Q})_+$ and $k \in K$. So $[g] = [g']$ in $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$. Hence it suffices to prove the injectivity of $\Gamma_g\backslash X^+ \to \mathbf{G}(\mathbb{Q})_+\backslash X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$. Now if $[x', g] = [x, g]$, then $(qx, qgk) = (x', g)$ for some $q \in \mathbf{G}(\mathbb{Q})_+$ and $k \in K$. So $q = gk^{-1}g^{-1} \in gKg^{-1}$. So $q \in \Gamma_g = gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$. Thus we have proved the injectivity of (3.2.5).

Surjectivitity: Let $[x, g]$ be an element of the right hand side. Then it is the image of $\Gamma_g x$.

We have thus proved (3.2.3). ☐

3.2.4 Torsion-free subgroup

Here is a choice of K' so that $gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ is torsion-free for all $g \in \mathbf{G}(\mathbb{A}_f)$. Take a faithful representation V of $\mathbf{G}$. Then there exists a lattice L in V such that $\widehat{L} := L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is fixed by K. Equivalently, we are embedding $\mathbf{G}$ as a closed subgroup of $\mathbf{GL}_N$ over $\mathbb{Q}$ such that K is a subgroup of $\mathbf{GL}_N(\widehat{\mathbb{Z}})$. Let $\ell \geq 3$ be an integer. Take K' to be the subgroup of K which acts trivially on $\widehat{L}/\ell\widehat{L}$, or equivalently

$$K' := \{g \in K < \mathbf{GL}_N(\widehat{\mathbb{Z}}) : g \equiv I_N \bmod \ell\}.$$

Then any element $\gamma \in \Gamma_g := gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+ < \mathrm{GL}(V)$ acts trivially on $\widehat{gL}/\ell\widehat{gL}$, so all the eigenvalues of γ are 1 (as they are 1 modulo $\ell \geq 3$). So $\gamma = 1$ if γ is torsion. So Γ_g is torsion-free.

3.2.5 The group of connected components of a Shimura variety

In this subsection, we prove the finiteness of the double coset $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_f)/K$. This finishes the proof of Theorem 3.2.1, and shows that $\pi_0(\mathrm{Sh}_K(\mathbf{G}, X)) \simeq \mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_f)/K$.

Case: simply-connected derived subgroup

The result in this case is better, with a clear understanding of the group $\pi_0(\mathrm{Sh}_K(\mathbf{G}, X))$. Consider the short exact sequence of $\mathbb{Q}$-groups

$$1 \to \mathbf{G}^{\mathrm{der}} \to \mathbf{G} \to \mathbf{T} := \mathbf{G}/\mathbf{G}^{\mathrm{der}} \to 1$$

with $\mathbf{T}$ an algebraic torus defined over $\mathbb{Q}$.

Definition 3.2.4. *An algebraic group H defined over a field k of characteristic 0 is said to be* **simply-connected** *if any central isogeny $H' \to H$ (i.e. a surjective morphism whose kernel is finite and contained in the center of H') is an isomorphism.*

Theorem 3.2.5. *Assume $\mathbf{G}^{\mathrm{der}}$ is simply-connected. Then $\nu(\mathbf{G}(\mathbb{Q})_+)$ has finite index in $\mathbf{G}(\mathbb{Q})$, $\nu(K)$ is a compact open subgroup of $\mathbf{T}(\mathbb{A}_f)$, and $\nu(\mathbf{G}(\mathbb{Q})_+)\backslash\mathbf{T}(\mathbb{A}_f)/\nu(K)$ is a finite abelian group. Moreover, ν induces a natural isomorphism of groups*

$$\pi_0(\mathrm{Sh}_K(\mathbf{G}, X)) \simeq \nu(\mathbf{G}(\mathbb{Q})_+)\backslash\mathbf{T}(\mathbb{A}_f)/\nu(K).$$

Before proving this theorem, we point out without proof that

$$\nu(\mathbf{G}(\mathbb{Q})_+) = \mathbf{T}(\mathbb{Q}) \cap \nu(Z(\mathbf{G})(\mathbb{R})) =: \mathbf{T}(\mathbb{Q})^{\dagger}. \tag{3.2.6}$$

Proof. General theory of semi-simple simply-connected $\mathbb{Q}$-groups asserts that $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ is connected. Therefore $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ stabilizes X^+ and hence is contained in $\mathbf{G}(\mathbb{R})_+$ by Lemma 3.2.3. So $\mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \subseteq \mathbf{G}(\mathbb{Q})_+$. By the *Strong Approximation Theorem* from §3.2.1, $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ is dense in $\mathbf{G}^{\mathrm{der}}(\mathbb{A}_f)$. Hence

$$\mathbf{G}^{\mathrm{der}}(\mathbb{A}_f) = \mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \cdot (K \cap \mathbf{G}^{\mathrm{der}}(\mathbb{A}_f)) \subseteq \mathbf{G}(\mathbb{Q})_+ \cdot (K \cap \mathbf{G}^{\mathrm{der}}(\mathbb{A}_f)). \tag{3.2.7}$$

Because $\mathbf{G}^{\mathrm{der}}$ is simply-connected, the short exact sequence of groups above Theorem 3.2.5 induces a short exact sequence

$$1 \to \mathbf{G}^{\mathrm{der}}(\mathbb{A}_f) \to \mathbf{G}(\mathbb{A}_f) \xrightarrow{\nu} \mathbf{T}(\mathbb{A}_f) \to 1.$$

Here we use the knowledge on semi-simple simply-connected $\mathbb{Q}$-groups that $H^1(\mathbb{Q}_p, \mathbf{G}^{\mathrm{der}}) = 0$ for any prime p.

Now ν induces a map

$$\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K \to \nu(\mathbf{G}(\mathbb{Q})_+) \backslash \mathbf{T}(\mathbb{A}_{\mathrm{f}})/\nu(K), \tag{3.2.8}$$

which, by (3.2.7), is a bijection. The right hand side is an abelian group because $\mathbf{T}$ is an algebraic torus (hence abelian).

Now to prove the theorem, it remains to prove:

(i) $\nu(\mathbf{G}(\mathbb{Q}))$ has finite index in $\mathbf{T}(\mathbb{Q})$.

(ii) $\nu(K)$ is a compact open subgroup of $\mathbf{T}(\mathbb{A}_{\mathrm{f}})$.

(iii) The right hand side of (3.2.8) is finite.

Let us prove (i). The Hasse Principle for simply-connected $\mathbb{Q}$-groups says that the natural map $H^1(\mathbb{Q}, \mathbf{G}^{\mathrm{der}}) \to \prod_{p \leq \infty} H^1(\mathbb{Q}_p, \mathbf{G}^{\mathrm{der}}) = H^1(\mathbb{R}, \mathbf{G}^{\mathrm{der}})$ is injective; here we used again the fact that $H^1(\mathbb{Q}_p, \mathbf{G}^{\mathrm{der}}) = 0$ for any prime number p (as $\mathbf{G}^{\mathrm{der}}$ is furthermore semi-simple). So by the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{G}^{\mathrm{der}}(\mathbb{Q}) & \longrightarrow & \mathbf{G}(\mathbb{Q}) & \longrightarrow & \mathbf{T}(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q}, \mathbf{G}^{\mathrm{der}}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathbf{G}^{\mathrm{der}}(\mathbb{R}) & \longrightarrow & \mathbf{G}(\mathbb{R}) & \longrightarrow & \mathbf{T}(\mathbb{R}) & \longrightarrow & H^1(\mathbb{R}, \mathbf{G}^{\mathrm{der}})
\end{array}$$

we get that $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q})) \to \mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$ is injective. But $\nu(\mathbf{G}(\mathbb{R})^+) = \mathbf{T}(\mathbb{R})^+$. So $\mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$ is finite, and hence $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q}))$ is finite. This establishes the claim.

For (ii), we extend $\mathbf{G} \to \mathbf{T}$ to a morphism of group schemes over $\mathbb{Z}[1/N]$ for some integer N, and prove that $\mathbf{G}(\mathbb{Z}_p) \to \mathbf{T}(\mathbb{Z}_p)$ is surjective for almost all prime p. We first work on $\mathbb{F}_p$ and then list using an argument similar to Newton's Lemma. We omit this proof.

Now we prove (iii). It suffices to prove that $\mathbf{T}(\mathbb{Q}) \backslash \mathbf{T}(\mathbb{A}_{\mathrm{f}})/\nu(K)$ is finite, and up to replacing $\nu(K)$ by a smaller compact open subgroup we may assume $\nu(K) \subseteq \mathbf{T}(\widehat{\mathbb{Z}})$. As $[\mathbf{T}(\widehat{\mathbb{Z}}) : \nu(K)]$ is finite (since $\mathbf{T}(\widehat{Z})$ is compact and $\nu(K)$ is open), it suffices to prove that

$$\mathbf{T}(\mathbb{Q}) \backslash \mathbf{T}(\mathbb{A}_{\mathrm{f}})/\mathbf{T}(\widehat{\mathbb{Z}})$$

is finite. This is exactly the *class group* of the algebraic torus $\mathbf{T}$ which is known to be finite by classical theory (and this number is called the *class number* of $\mathbf{T}$). In the case where $\mathbf{T} = \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_{\mathrm{m}}$ for a number field K, this is exactly the class group of K. □

General case

Let $\widetilde{\mathbf{G}}$ be the universal cover of $\mathbf{G}^{\mathrm{der}}$, *i.e.* $\widetilde{\mathbf{G}}$ is simply-connected with a central isogeny (surjective with finite kernel contained in the center) $u \colon \widetilde{\mathbf{G}} \to \mathbf{G}^{\mathrm{der}}$. Then we have a surjective morphism of $\mathbb{Q}$-groups

$$\varphi \colon \mathbf{G}' := Z(\mathbf{G}) \times \widetilde{\mathbf{G}} \to \mathbf{G}, \qquad (z, g) \mapsto zu(g)$$

which is a central isogeny. Thus to prove the finiteness of $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$, it suffices to prove the finiteness of

$$\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_{\mathrm{f}})/K'$$

for K' a compact open subgroup of $\mathbf{G}'(\mathbb{A}_{\mathrm{f}})$. But the derived subgroup of $\mathbf{G}'$ is $\widetilde{\mathbf{G}}$ which is simply-connected. So we are back to the previous case, and hence $\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_{\mathrm{f}})/K'$ is finite. So $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$ is finite.

3.2.6 An upshot of Theorem 3.2.1 on special subvarieties

Special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)$ can be better understood via Theorem 3.2.1 as follows. Let S be a special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)$, arising from the sub-Shimura datum $(\mathbf{G}', X') \subseteq (\mathbf{G}, X)$ and the Hecke operator given by $g \in \mathbf{G}(\mathbb{A}_f)$. Then under the decomposition (3.2.1), S is the image of $u((X')^+)$ under the uniformization $u\colon X^+ \to \Gamma_g \backslash X^+$ for the suitable connected component $(X')^+$ of X'. Moreover, the sub-Shimura data can be constructed as follows. Take $h \in X$, and let $\mathrm{MT}(h)$ be the smallest $\mathbb{Q}$-subgroup of $\mathbf{G}$ such that $h\colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ factors through $\mathrm{MT}(h)_{\mathbb{R}}$. Then take $\mathbf{G}' := \mathrm{MT}(h)$ and $X' := \mathbf{G}'(\mathbb{R})h$.

3.3 Siegel modular variety

Take the example of Siegel case in Example 3.1.3 and Example 3.1.6. In particular $V = \mathbb{Q}^{2d}$, $\psi\colon V \times V \to \mathbb{Q}$ is $(x, y) \mapsto x^{\mathrm{t}} J y$ with $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. Thus there is a lattice L in V such that ψ restricts to $L \times L \to \mathbb{Z}$. To simplify notation, denote by $L = V(\mathbb{Z})$.

The Siegel Shimura datum is $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$. For each N, set

$$\begin{aligned} K(N) &:= \left\{ g \in \mathbf{GSp}_{2d}(\mathbb{A}_{\mathrm{f}}) : gV(\widehat{\mathbb{Z}}) \subseteq V(\widehat{\mathbb{Z}}) \text{ and acts trivially on } V(\widehat{\mathbb{Z}})/NV(\widehat{\mathbb{Z}}) \right\} \\ &= \left\{ g \in \mathbf{GSp}_{2d}(\widehat{\mathbb{Z}}) : g \equiv I_{2d} \pmod{N} \right\}. \end{aligned}$$

Then we have the Shimura variety $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$.

Theorem 3.3.1. *Assume $N \geq 3$. Then $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$ is the fine moduli space of principally polarized abelian varieties of dimension d with a level-N-structure, i.e. there is a canonical bijection between*

- *the $\mathbb{C}$-points of $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$,*
- *and the isomorphism classes of the triples (A, λ, η_N) where A is a complex abelian variety of dimension d, λ is a principal polarization on A, and η_N is a level-N-structure on A.*

When $N = 1, 2$, the Shimura variety is a coarse moduli space.

Let us explain the meaning of this theorem. Let A be an abelian variety defined over $\mathbb{C}$.

(i) A *principal polarization* on A is a polarization on the Hodge structure $H_1(A, \mathbb{Z})$ with determinant 1, *i.e.* an alternating pairing $\lambda\colon H_1(A, \mathbb{Z}) \times H_1(A, \mathbb{Z}) \to \mathbb{Z}$, which under suitable $\mathbb{Z}$-basis of $H_1(A, \mathbb{Z})$ is $\begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. In more geometric terms, it is an isomorphism $\lambda\colon A \xrightarrow{\sim} A^{\vee}$.

(ii) A *(symplectic) level-N-structure* on A is a basis of $H_1(A, \mathbb{Z}/N\mathbb{Z})$ which is symplectic with respect to λ. In more geometric terms, it is a basis of the $\mathbb{Z}/N\mathbb{Z}$-module $A[N]$ which is symplectic under $e_N\colon A[N] \times A[N] \xrightarrow{(1,\lambda)} A[N] \times A^{\vee}[N] \to \mu_N$ where last map is the Weil pairing. Or more concretely, it is an isomorphism

$$\eta_N\colon A[N] \xrightarrow{\sim} H_1(A, \mathbb{Z}/N\mathbb{Z})$$

such that the two composites

$$A[N] \times A[N] \xrightarrow{(\eta_N, \eta_N)} H_1(A, \mathbb{Z}/N\mathbb{Z}) \times H_1(A, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\bar{\lambda}} \mathbb{Z}/N\mathbb{Z}$$
$$\text{and} \quad A[N] \times A[N] \xrightarrow{e_N} \mu_N \xrightarrow{e^{2\pi\sqrt{-1}a/N} \mapsto [a]} \mathbb{Z}/N\mathbb{Z}$$

differ from the multiplication by an element in $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^\times$, and we say that this level-N-structure *has twist* $[\ell]$.

Proof. Recall that each point in $\mathfrak{H}_d^\pm$ parametrizes a $\mathbb{Q}$-Hodge structure on V of type $(-1,0)+(0,-1)$; see §2.3.1.

We shall use Theorem 3.2.1 and the more precise version (3.2.3), and better, Theorem 3.2.5 because $\mathbf{Sp}_{2d}$ is simply-connected. One can compute that $\mathbf{GSp}_{2d}(\mathbb{R})_+ = \mathbf{GSp}_{2d}(\mathbb{R})^+ = \{g \in \mathbf{GSp}_{2d}(\mathbb{R}) : \det(g) > 0\}$. So $\mathbf{GSp}_{2d}(\mathbb{Q})_+ = \{g \in \mathbf{GSp}_{2d}(\mathbb{Q}) : \det(g) > 0\}$. Thus for the quotient

$$1 \to \mathbf{Sp}_{2d} \to \mathbf{GSp}_{2d} \xrightarrow{\nu} \mathbb{G}_{\mathrm{m}} \to 1,$$

we have $\nu(\mathbf{GSp}_{2d}(\mathbb{Q})_+) = \mathbb{Q}_{>0}$.[3] It is not hard to compute that $\nu(K(N)) = \{z \in \widehat{\mathbb{Z}} : z \equiv 1 \pmod N\} = 1 + N\widehat{\mathbb{Z}}$. Thus

$$\pi_0\left(\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)\right) \simeq \mathbb{Q}_{>0}\backslash\mathbb{A}_{\mathrm{f}}^\times/(1+N\widehat{\mathbb{Z}}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.$$

Write $\Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$ for the connected component of $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ indexed by $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^\times$. Below we only give the constructions of the two directions, without proving that they are inverse to each other.

Given a triple (A, λ, η_N). Assume that the level-N-structure has twist $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^\times$. First $H_1(A, \mathbb{Z})$ is a $\mathbb{Z}$-Hodge structure of type $(-1,0)+(0,-1)$, and hence under suitable isomorphism $(H_1(A,\mathbb{Z}), \lambda) \simeq (V(\mathbb{Z}), \psi)$ we obtain a point $\tau \in \mathfrak{H}_d^+$. Then we get a point in $\Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$ as the image of τ under $\mathfrak{H}_d^+ \to \Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$.

Conversely let $x \in \Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$. Let τ be a pre-image of x under the quotient $\mathfrak{H}_d^+ \to \Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$. Recall that τ parametrizes a $\mathbb{Q}$-Hodge structure on V of type $(-1,0)+(0,-1)$, and thus we can endow $V(\mathbb{R})$ with a complex structure by the bijection $V(\mathbb{R}) \subseteq V(\mathbb{C}) \to V(\mathbb{C})/V_\tau^{0,-1}$. This makes $A_\tau := V(\mathbb{R})/V(\mathbb{Z})$ into a compact complex torus of dimension d, with $H_1(A_\tau, \mathbb{Z}) = V(\mathbb{Z})$. Thus ψ induces a principle polarization via $H_1(A_\tau, \mathbb{Z})$. Hence A_τ is an abelian variety with a principal polarization which by abuse of notation we still use ψ to denote. The level-N-structure on A_τ is given as follows. We have $A_\tau[N] = \frac{1}{N}V(\mathbb{Z})/V(\mathbb{Z}) = V(\mathbb{Z})/NV(\mathbb{Z}) = V(\mathbb{Z}/N\mathbb{Z})$. Take $g \in \mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$ such that $\nu(g) \in \widehat{\mathbb{Z}}^\times$ is congruent to ℓ modulo $1+N\widehat{\mathbb{Z}}$. Then g induces an isomorphism $g\colon V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) \xrightarrow{\sim} V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})$. But $V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) = V(\mathbb{Z}/N\mathbb{Z}) = H_1(A_\tau, \mathbb{Z}/N\mathbb{Z})$. Thus we have $A_\tau[N] = V(\mathbb{Z}/N\mathbb{Z}) = V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) \xrightarrow{g} V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) = H_1(A_\tau, \mathbb{Z}/N\mathbb{Z})$. This is the desired level-N-structure because $\psi(gx, gy) = \nu(g)\psi(x,y)$ by definition of $\mathbf{GSp}_{2d}$. □

More generally, we can take any symplectic pairing ψ on V, *i.e.* $\psi\colon V \times V \to \mathbb{Q}$ is non-degenerate bilinear anti-symmetric. Then we have the symplectic group $\mathbf{GSp}(\psi)$ which is the subgroup of $\mathrm{GL}(V)$ preserving ψ (up to a number in $\mathbb{Q}^\times$) and a $\mathbf{GSp}(\psi)(\mathbb{R})$-orbit in $\mathrm{Hom}(\mathbb{S}, \mathbf{GSp}(\psi)_\mathbb{R})$ which can still be identified with $\mathfrak{H}_d^\pm$. This gives a Shimura datum $(\mathbf{GSp}(\psi), \mathfrak{H}_d^\pm)$. The associated Shimura varieties are then moduli spaces of abelian varieties *polarized by* ψ of dimension d with suitable level structures.

[3] In fact $\nu(g) = (\det g)^{1/d}$.

Definition 3.3.2. *A Shimura variety is called a* **Siegel modular space** *if the associated Shimura datum is isomorphic to* $(\mathbf{GSp}(\psi), \mathfrak{H}_d^{\pm})$ *for some* ψ *and* d *as above.*

A Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ *is called* **of Hodge type** *if there exists an injective Shimura morphism* $(\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}(\psi), \mathfrak{H}_d^{\pm})$.

A Shimura variety is called **of abelian type** *if it admits a finite covering (as an algebraic variety) by a Shimura variety variety of Hodge type.*

By the construction in §3.2.6 and Proposition 1.2.12, Shimura varieties of Hodge type are moduli spaces of abelian varieties A with some prescribed Hodge tensors of $H_1(A, \mathbb{Q})$.

Shimura varieties of abelian type can be detected purely on the underlying group $\mathbf{G}$, and they may not parametrize abelian varieties. As an example, all Shimura varieties associated with the Shimura data from Example 3.1.4 are of abelian type, but they do not parametrize abelian varieties unless $F = \mathbb{Q}$.

3.4 CM abelian varieties and special points

Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety. In Definition 3.1.10 we defined *special points* on $\mathrm{Sh}_K(\mathbf{G}, X)$. They are of particular importance. For example, there exists a natural number field $E(\mathbf{G}, X)$, called the *reflex field* of $(\mathbf{G}, X)$, on which $\mathrm{Sh}_K(\mathbf{G}, X)$ is "naturally" defined (or in more vigorous terms, has a canonical model), characterized by the action of the Galois group of $E(\mathbf{G}, X)$. This action is explicitly defined for special points on $\mathrm{Sh}_K(\mathbf{G}, X)$ via the class field theory, and is uniquely determined in this way by the following theorem whose proof we omit:

Theorem 3.4.1. *The set of special points is dense in* $\mathrm{Sh}_K(\mathbf{G}, X)$.

Here "dense" is true even for the usual topology. The hard part of this theorem is to prove the existence of one special point. Indeed, assume $\mathrm{Sh}_K(\mathbf{G}, X) \simeq \bigsqcup \Gamma_g \backslash X^+$ has a special point $[x]$. Then its inverse image x in X^+ gives rise to a morphism $x \colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ which factors through $\mathbf{T}_{\mathbb{R}}$ for an algebraic torus $\mathbf{T} < \mathbf{G}$. But then the morphism given by $g \cdot x$ for any $g \in \mathbf{G}(\mathbb{Q})$ factors through $(g\mathbf{T}g^{-1})_{\mathbb{R}}$, with $g\mathbf{T}g^{-1}$ clearly an algebraic torus in $\mathbf{G}$ (since it is abelian), and hence defines a Shimura datum $(g\mathbf{T}g^{-1}, g \cdot \mathbf{T}(\mathbb{R})x)$. But $\mathbf{T}(\mathbb{R})x$ is a finite set of points since $\mathbf{T}$ is abelian. So the image of $\mathbf{G}(\mathbb{Q})x$ under the quotient $X^+ \to \Gamma_g \backslash X^+$ consists of special points of $\mathrm{Sh}_K(\mathbf{G}, X)$. Notice that $X^+ = \mathbf{G}(\mathbb{R})^+ x$. Now it suffice to use the Real Approximation that $\mathbf{G}(\mathbb{Q})$ is dense in $\mathbf{G}(\mathbb{R})$ to conclude.

For the existence of special points, we shall focus on the Siegel modular variety, for which we have:

Theorem 3.4.2. *Take* $[x] \in \mathrm{Sh}_K(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})(\mathbb{C})$. *Then* $[x]$ *is a special point if and only if the abelian variety* A_x *parametrized by* $[x]$ *is CM, i.e.* $\mathrm{End}(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$ *contains a commutative* $\mathbb{Q}$*-subalgebra of dimension* $2d$. *Equivalently, an abelian variety* A *defined over* $\mathbb{C}$ *is CM if and only if the Mumford–Tate group of the* $\mathbb{Q}$*-Hodge structure* $H_1(A, \mathbb{Q})$ *is an algebraic torus.*

We will not give a full proof of this theorem, but only recall the definition of CM abelian varieties and give a brief explanation why the associated Mumford–Tate group (which we call the Mumford–Tate group of A) is an algebraic torus.

Assume A is a simple abelian variety. Then A is CM if and only if $E := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a CM field, *i.e.* there exists a totally real field F such that E/F is a totally imaginary quadratic extension. Write $\overline{(\cdot)}$ for the complex conjugation with respect to E/F. Then there exists an element $\iota \in E$ such that $\bar{\iota} = -\iota$ (totally imaginary element). Then E can be endowed with the $\mathbb{Q}$-symplectic form

$$\langle x, y \rangle := \mathrm{Tr}_{E/\mathbb{Q}}(\overline{x}\iota y).$$

This makes $(E, \langle , \rangle) \simeq (\mathbb{Q}^{2d}, \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix})$ into a symplectic space. Set $\mathbf{GU}_E$ to be the subgroup of $\mathbf{GSp}_{2d}$ generated by $\mathbb{G}_{\mathrm{m}} = Z(\mathbf{GSp}_{2d})$ and

$$\mathbf{U}_E := \{x \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{\mathrm{m}} : x\overline{x} = 1\}.$$

Then one can check that $\mathbf{GU}_E$ is an algebraic torus which contains the Mumford–Tate group of A. Thus the Mumford–Tate group of A is abelian, and hence must be an algebraic torus. In fact, one can check that $\mathbf{GU}_E$ is a maximal torus of $\mathbf{GSp}_{2d}$.