Chapter 4. Local Tate Duality

§1. Cohomological dimension.

§1.1 Basic facts

G = profinite group.

Def. 1) The p-cohomological dimension of G, denoted by $cd_p(G)$, is the smallest non-negative integer n. s.t.

(*) $\forall$ discrete torsion G-module A, $\forall q > n$, we have $H^q(G, A) = 0$.

2) The cohomological dimension of G $cd(G) := \sup_p cd_p(G)$.

Rmk It is possible that $cd_p(G) = \infty$.

e.g. $G = \hat{\mathbb{Z}}$, Exercise sheet 6 $\Rightarrow cd_p(\hat{\mathbb{Z}}) = 1$. $\forall p$.

Prop TFAE:

① $cd_p(G) \leq n$

② $H^q(G, A) = 0$, $\forall q > n$, $\forall$ discrete G-module A which is p-primary torsion

③ $H^{n+1}(G, A) = 0$, $\forall$ A simple discrete G-module killed by p.

Pf $\forall$ discrete torsion G-module A, we have $A = \bigoplus_p A\{p\}$

$H^q(G, A\{p\}) = H^q(G, A)\{p\}$ (not hard to check)

Now ① $\Leftrightarrow$ ②.

② $\Rightarrow$ ③ ✓

③ $\Rightarrow$ ②.

- $\forall A$ finite st $p^m A = 0$ for some m. we have $\exists A = A_0 \supset A_1 \supset \cdots \supset A_l = 0$. with $p \cdot (A_i / A_{i+1}) = 0$ and A_i / A_{i+1} simple

$0 \to A_{i+1} \to A_i \to A_i/A_{i+1} \to 0$

$\Rightarrow H^{n+1}(G, A_i) = 0$. $(\forall i)$

$\Rightarrow H^{n+1}(G, A) = 0$. $(A_0 = A)$

- $\forall A$ p-primary torsion, we have
$$A = \varinjlim A_\alpha \quad \text{with } A_\alpha \text{ finite } \& \ p^{m_\alpha} A_\alpha = 0.$$
$$\text{So } H^{n+1}(G,A) = \varinjlim H^{n+1}(G,A_\alpha) = 0.$$

Now $0 \to A \to \mathrm{Coind}_G(A) \to \mathrm{Coind}_G(A)/A \to 0.$

$\Rightarrow H^{n+2}(G,A) = 0$ since $H^{n+1}(G, \mathrm{Coind}_G(A)/A) = 0$ by the previous argument.

etc.

$\square$

Prop. $H < G$ closed group. Then $cd_p(H) \le cd_p(G)$ with equality in each of the following cases:

① $p \nmid [G:H]$

② H open in G and $cd_p(G) < \infty$

Pf. $\forall$ discrete torsion H-module A, $\mathrm{Coind}_G^H(A)$ is a discrete torsion G-module.

and $H^q(G, \mathrm{Coind}_G^H(A)) = H^q(H, A)$.

So $cd_p(H) \le cd_p(G)$.

In case ①, $\mathrm{Res}: H^q(G,A)\{p\} \hookrightarrow H^q(H,A)\{p\}$.

In case ②, $\mathrm{Cor}: H^n(H,A)\{p\} \twoheadrightarrow H^n(G,A)\{p\}$. with $n = cd_p(G)$.

So we have equality in both cases.

$\square$

Cor. G_p = Sylow-p-subgp of G, then
$$cd_p(G) = cd_p(G_p) = cd(G_p)$$

Cor $cd_p(G)=0 \iff$ the order of G is prime to p.

Pf. $(\Leftarrow)$ ✓

$(\Rightarrow)$ WMA $G=G_p$

$G\neq\{1\} \Rightarrow \exists\, G \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$

$\Rightarrow H^1(G, \mathbb{Z}/p\mathbb{Z})\neq 0$

$\Rightarrow cd_p(G)\geq 1$

$\square$

Cor $cd_p(G)\neq 0,\infty \Rightarrow$ the exponent of p in the order of G is

Pf. WMA $G=G_p$.

G finite $\Rightarrow$ $\{1\}$ is open in G

$\xRightarrow{cd_p(G)<\infty} cd_p(G)=cd_p(\{1\})=0$. Contradiction.

$\square$

Prop. G p-torsion free, $H<G$ open.
Then $cd_p(H)=cd_p(G)$.

Prop. $H\triangleleft G$ closed, $cd_p(H)=n$, $cd_p(G/H)=m$, then

$$H^{n+m}(G,A)\{p\} = H^m(G/H, H^n(H,A))\{p\}.$$

In particular, $cd_p(G)\leq cd_p(H)+cd_p(G/H)$.

"=" if $\begin{cases}\text{either } H \text{ is } p\text{-gp } \& \ H^n(H,\mathbb{Z}/p\mathbb{Z}) \text{ finite}\\ \text{or } H<Z(G)\end{cases}$

Rmk. $cd_p(G)\leq cd_p(H)+cd_p(G/H)$ is always true for closed $H\triangleleft G$. (i.e. no need to assume the finiteness of $cd_p(H)$ or $cd_p(G/H)$).

§1.2. p-groups

Prop. $G = G_p$. Then every discrete simple G-module killed by p is $\simeq \mathbb{Z}/p\mathbb{Z}$ (trivial action).

Pf. Let A be such a module, the A is obviously finite.
So A is a G/U-module for some $U \triangleleft G$ open.
So WMA G finite. Then this is true (cf. "Local Fields" P146).
$\square$

Prop. $G = G_p$. Then
$cd_p(G) \leq n \iff H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$.

Pf. This follows directly from the first prop. of §1.1 & §1.2.
$\square$

Cor. $G = G_p$. $cd(G) = n$.
If A is a discrete finite p-primary G-module, then $H^n(G, A) \neq 0$.

Pf. $\exists\ A = A_0 \supset A_1 \supset \cdots \supset A_l = 0$ as in the proof of the first prop of §1.1.
In particular, $\exists\ A \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$.
$cd(G) \leq n \Rightarrow H^n(G, A) \twoheadrightarrow H^n(G, \mathbb{Z}/p\mathbb{Z})$.
$cd(G) \geq n \Rightarrow H^n(G, \mathbb{Z}/p\mathbb{Z}) \neq 0$.
So $H^n(G, A) \neq 0$.
$\square$

§13. Criteria for fields. (Galois gp)

k = field $\bar{k}$ = a separable closure & $G_k := \mathrm{Gal}(\bar{k}/k)$

Lemma G profinite, $G(p) = G/N$ the largest quotient of G which is a p-group (N closed).

Assume $\mathrm{cd}_p(N) \le 1$, then the canonical maps

$$H^q(G(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^q(G, \mathbb{Z}/p\mathbb{Z})$$

are iso. In particular, $\mathrm{cd}(G(p)) \le \mathrm{cd}_p(G)$

Prop. ($p = \mathrm{char}\, k$) $\mathrm{cd}_p(G_k) \le 1$ and $\mathrm{cd}(G_k(p)) \le 1$.

Pf. $0 \to \mathbb{Z}/p\mathbb{Z} \longrightarrow \bar{k} \longrightarrow \bar{k} \longrightarrow 0$.

$x \mapsto x^p - x$.

$\forall$ Sylow-p-subgp $(G_k)_p$ of G_k,

$L := (\bar{k})^{(G_k)_p}$ has $\bar{k}$ as separable closure.

so $H^1(L, \bar{k}) \longrightarrow H^2(L, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2(L, \bar{k})$

$H^1(L,\bar{k}) = 0$, $H^2(L,\bar{k}) = 0$

so $H^2(L, \mathbb{Z}/p\mathbb{Z}) = 0$, so $\mathrm{cd}_p((G_k)_p) \le 1$.

So $\mathrm{cd}_p(G_k) = \mathrm{cd}((G_k)_p) = 1$

$N := \ker(G_k \to G_k(p))$, then similar argument $\Rightarrow \mathrm{cd}_p(N) \le 1$

So $\mathrm{cd}(G_k(p)) \le 1$.

□

Prop. ($p \ne \mathrm{char}\, k$) TFAE:

① $\mathrm{cd}_p(G_k) \le n$

② $\forall$ alg. ext. K/k, $H^{n+1}(K, \bar{K}^*)[p] = 0$ and $H^n(K, \bar{K}^*)$ is p-divisible

③ same assertion as ② but with K/k finite separable and of degree prime to p.

Pf. $1 \to \mu_p \to \bar{K}^* \longrightarrow \bar{K}^* \to 1$

$x \mapsto x^p$

So ② $\iff H^{n+1}(K, \mu_p) = 0, \ \forall K$ in ②

③ $\iff H^{n+1}(K, \mu_p) = 0, \ \forall K$ in ③

Now

<u>① ⇒ ②</u> $G_K \simeq$ a closed subgp of G_k.

So $cd_p(G_k) \le n \Rightarrow cd_p(G_K) \le n \Rightarrow H^{n+1}(K, \mu_p) = 0$.

<u>② ⇒ ③</u> ✓

<u>③ ⇒ ①</u> $K := (\bar{k})^{(G_k)_p}$, $(G_k)_p$ = sylow-p-subgp of G_k

then $K = \varinjlim K_\alpha$, with K_α/k finite separable and of degree prime to p.

③ $\Rightarrow H^{n+1}(K_\alpha, \mu_p) = 0$. $(\forall \alpha)$

$\Rightarrow H^{n+1}((G_k)_p, \mu_p) = H^{n+1}(K, \mu_p) = \varinjlim H^{n+1}(K_\alpha, \mu_p) = 0$

$(G_k)_p \subset G_k \curvearrowright \mu_p \Rightarrow (G_k)_p \to \mathrm{Aut}(\mu_p) \simeq \mathbb{Z}/(p-1)\mathbb{Z}$.

must be trivial

$\Rightarrow (G_k)_p \curvearrowright \mu_p$ trivial

$\Rightarrow \mu_p \simeq \mathbb{Z}/p\mathbb{Z}$ as $(G_k)_p$-module

So $cd_p(G_k) = cd_p((G_k)_p) \le n$ by the first prop of §1.1 & §1.2

□.

§1.4 Local fields

Prop k = field. TFAE:

① $cd(G_k) \leq 1$ and $Br(K)[p] = 0$, $\forall K/k$ alg ext with $p = char(k)$.
② $Br(K) = 0$. $\forall K/k$ separable finite.
③ $\forall K/k$ finite separable & L/K finite Galois, the $Gal(L/K)$-mod. $L^\times$ is cohomologically trivial.

④ $\forall K, L$ as in ③, $N_{L/K}(L^*) = K^*$.
⑤ $\forall K, L$ as in ③ st L/K is of prime degree, $N_{L/K}(L^*) = K^*$.

Pf. Ex Sheet 8 ex 3&4 $\Rightarrow$ equivalence of ②③④⑤.
①$\Leftrightarrow$② follows from the two prop. of §1.3.
$\square$

Prop K complete field with residue field k.
Then $\forall p$, $cd_p(G_K) \leq 1 + cd_p(G_k)$.
"=" if $cd_p(G_k) < \infty$ and $p \neq char\, K$.

Pf. Recall $Br(K_{ur}) = 0$ and by the same proof $Br(L) = 0$
for any finite separable extension L/K_{ur}

so $cd(Gal(\bar{K}/K_{ur})) \leq 1$.
so $cd_p(G_K) \leq cd_p(Gal(\bar{K}/K_{ur})) + ~~Gal~~\; cd_p(\underbrace{Gal(K_{ur}/K)}_{\substack{\parallel \\ Gal(\bar{k}/k)}})$.

$\leq 1 + cd_p(G_k)$.

Now if $d = cd_p(G_k) < \infty$ and $p \neq char\, K$, then
WMA G_k to a p-group (replace G_k by $(G_k)_p$ and
replace k, K, G_K accordingly)

Compute $H^{d+1}(G_K, \mu_p)$.

$$H^{d+1}(G_K,\mu_p) = H^d(G_k, H^1(Gal(\bar{K}/K_{ur}), \mu_p))$$
$$= H^d(G_k, K_{ur}^* / K_{ur}^{*p}).$$

But $K_{ur}^*/K_{ur}^{*p} \xrightarrow{v} \mathbb{Z}/p\mathbb{Z}$ (v = valuation) splits.

so $H^d(G_k, K_{ur}^*/K_{ur}^{*p}) \twoheadrightarrow \underset{\neq 0 \text{ by } cd_p(G_k)=d.}{H^d(G_k, \mathbb{Z}/p\mathbb{Z})}$

Done

$\square$.

<u>Thm</u>. K local field non-archimedean. Then

$cd_p(G_K) = 2$, $\forall p \neq \operatorname{char} K$.

and $cd(G_K) = 2$.

Pf. In this case k is finite, so $G_k \simeq \hat{\mathbb{Z}}$.

So $cd_p(G_k) = 1$.

Now the conclusion follows from the previous prop.

$\square$.

§2. Local Tate duality.

§2.1 Dualizing module.

Fact $\forall$ torsion abelian group A its dual $A^* := \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ is a commutative profinite group. If A is finite, then so is A^*.

Consider G = profinite gp, $cd(G) = n < \infty$. Consider the functor

$$\{\text{finite } G\text{-modules}\} \longrightarrow \mathbf{Ab}$$
$$A \longmapsto H^n(G, A)^*.$$

Thm Assume $H^n(G, A)$ is finite for any finite G-module A. Then the functor above is representable by a discrete torsion G-module I, i.e.

$$H^n(G, A)^* \simeq \mathrm{Hom}_G(A, I), \quad \forall A \text{ finite } G\text{-module}.$$

Rmk I is unique up to unique isomorphism.

Def. This I is called the dualizing module of G.

Lemma $H < G$ open. I is the dualizing module of G, then I is also the dualizing module of H.

Pf. $\left.\begin{array}{l} H \text{ open in } G \\ cd(G) < \infty \end{array}\right\} \Rightarrow cd(H) = cd(G) = n.$

$\forall$ finite H-module A,

$$H^n(H, A) = H^n(G, \mathrm{coind}_G^H(A))$$

$$\Rightarrow H^n(H, A)^* \simeq \mathrm{Hom}_G(\mathrm{Coind}_G^H(A), I) = \mathrm{Hom}_H(A, I)$$

so I is the dualizing module of H by def.

$\square$

§2.2. Tate duality for p-adic fields

K = p-adic field, i.e. a finite ext. of $\mathbb{Q}_p$. $G_K = \mathrm{Gal}(\overline{K}/K)$.

<u>Prop</u>. $\forall$ finite G_K-module A, $H^i(K, A)$ is finite ($\forall i \geq 0$)

<u>Sketch</u>. $\forall$ finite extension L/K,

$$H^i(L, \mu_n) = \begin{cases} \mu_n \cap L & i=0 \\ L^*/L^{*n} & i=1 \\ \mathbb{Z}/n\mathbb{Z} & i=2 \\ 0 & i \geq 3 \end{cases} \quad \text{finite}$$

- reduce to this by spectral sequence.

□.

Now: - $cd(G_K) = 2$.

- $H^i(G_K, A)$ is finite, $\forall$ finite G_K-module A.

So $\exists$ dualizing module I.

<u>Thm</u> $I = \mu = \bigcup_{n \geq 1} \mu_n$ (see HW).

<u>Thm</u>. $\forall$ finite G_K-module A, we have: the cup-product

$$H^i(K, A) \times H^{2-i}(K, A') \longrightarrow H^2(K, \mu) = \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing (so gives a duality), where

$$A' = \mathrm{Hom}(A, \overline{K}^*) = \underbrace{\mathrm{Hom}(A, \mu)}_{\text{since } A \text{ is finite}}, \quad i = 0, 1, 2.$$

<u>Pf</u>. <u>$i=2$</u>. is the def. of dualizing module.

<u>$i=0$</u> same as $i=2$ (since $(A')' = A$)

<u>$i=1$</u>. enough to prove $H^1(K, A) \hookrightarrow H^1(K, A')^*$

Take $0 \to A \to B := \mathrm{Coind}_G(A) \to C \to 0$, then

$$\begin{array}{ccccccc} H^0(K, B) & \longrightarrow & H^0(K, C) & \xrightarrow{\delta} & H^1(K, A) & \longrightarrow & H^1(K, B) = 0 \\ \downarrow & & \downarrow & & \downarrow i & & \\ H^2(K, B')^* & \longrightarrow & H^2(K, C')^* & \longrightarrow & H^1(K, A')^* & & \end{array}$$

The two maps on the left are iso. by the previous steps, and δ is surjective. So i is injective.

□.