Exercise Sheet 9
Let K be a non-archimedean local field.

Exercise 1. Let L/K be a finite separable extension. Prove that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Br}(L) & \xrightarrow{\mathrm{Cor}} & \mathrm{Br}(K) \\ \Big\downarrow{\scriptstyle \mathrm{inv}_L} & & \Big\downarrow{\scriptstyle \mathrm{inv}_K} \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Q}/\mathbb{Z} \end{array}$$

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Exercise 2. Let L/K be a finite separable extension and let $N\colon L^* \to K^*$ be the norm map. Prove that $\ker(N)$ is compact and $N(L^*)$ is closed in K^*. (Hint: prove that $N(U_L)$ is closed of finite index in U_K and $N^{-1}(U_K) = U_L$. Note that U_K is open in K^* because K^* is with discrete absolute value).

Exercise 3. Let I be a subgroup of K^* of finite index that contains U_K. The goal of this exercise is to prove that $I = N_{L/K}(L^*)$ for some finite separable extension L/K. Fix a separable closure $\bar{K}$ of K.

1. Denote by $v\colon K^* \to \mathbb{Z}$ the order (i.e. $a \mapsto \log_{|\pi|}|a|$ for a uniformizer π of K). Prove that $I = v^{-1}(n\mathbb{Z})$ for some $n \in \mathbb{Z}_{>0}$;
2. For the rest of the exercise, let $n \in \mathbb{Z}_{>0}$ be as in (1). Prove that there exists a unique unramified extension K_n/K of degree n with $K_n \subset \bar{K}$;
3. Prove that $U_K \subset N_{K_n/K}(K_n^*)$ (Hint: use Hensel's Lemma or Newton's Lemma);
4. Prove $v(N_{K_n/K}(K_n^*)) = n\mathbb{Z}$;
5. Conclude.