Exercise Sheet 8

Exercise 1. Give an example of a finite group G and a G-module A such that there exists $q \in \mathbb{Z}$ with $\hat{H}^q(G, A) = \hat{H}^{q+1}(G, A) = 0$ but A is not cohomologically trivial.

Exercise 2. (Bonus) The goal of this exercise is to give a proof of Theorem 5.2 we saw in class using Theorem 5.1. In particular, please do not use Theorem 5.2 when doing this exercise. Let G be a finite group and let A be a G-module.

1. Assume that G is a p-group and A has no p-torsion. Prove the equivalence of:
   (a) there exists $q \in \mathbb{Z}$ such that $\hat{H}^q(G, A) = \hat{H}^{q+1}(G, A) = 0$.
   (b) A is cohomologically trivial.
   (c) the $\mathbb{F}_p[G]$-module A/pA is free.
2. Assume that A is $\mathbb{Z}$-free and let G_p be a Sylow-p-subgroup of G for each prime number p. Prove that the followings are equivalent:
   (a) For every prime number p, the G_p-module A satisfies the equivalent conditions of 1.
   (b) A is $\mathbb{Z}[G]$-projective.
3. Prove Theorem 5.2.

Exercise 3. For a field k, prove that the followings are equivalent:

1. $\mathrm{Br}(K) = 0$ for any finite seperable extension K of k;
2. If K/k is finite separable and L/K is finite Galois, then the $\mathrm{Gal}(L/K)$-module L^* is cohomologically trivial;
3. If K/k is finite separable and L/K is finite Galois, then the norm $N_{L/K} \colon L^* \to K^*$ is surjective.

The following theorem will be useful for the next exercise:

Theorem For any finite cyclic group G of order n and any G-module A, we have $\hat{H}^q(G, A) \cong \hat{H}^{q+2}(G, A)$ for all $q \in \mathbb{Z}$.

Exercise 4. The goal of this exercise is to show that there is a 4th equivalent condition to the three ones in the previous exercise: If K/k is finite separable and L/K is finite Galois such that $\mathrm{Gal}(L/K)$ is a cyclic group of prime degree, then the norm $N_{L/K} \colon L^* \to K^*$ is surjective.

Let K/k be a finite separable extension. Assume this 4th condition for K.

1. Prove: if L/K is finite Galois such that $\mathrm{Gal}(L/K)$ is solvable, then $\mathrm{Br}(L/K) = 0$;
2. Fix a prime number l. Let L/K be any finite Galois extension. Prove: $\mathrm{Br}(L/K)\{l\} = 0$ (Hint: note that any Sylow-l-subgroup of $\mathrm{Gal}(L/K)$ is solvable);
3. Let L/K be any finite Galois extension. Prove: $\mathrm{Br}(L/K) = 0$;
4. Prove: $\mathrm{Br}(K) = 0$.