Exercise Sheet 7

Exercise 1. Let G be a finite group ane let

$$0 \to A_1 \to A_2 \to \ldots \to A_n \to 0$$

be an exact sequence of G-modules. Let $j \in \{1, \ldots, n\}$. Prove that if A_i is cohomologically trivial for all $i \neq j$, then A_j is also cohomologically trivial.

Exercise 2. Let G be a profinite group and let p be a prime number. Let $n \in \mathbb{Z}_{>0}$. Assume

(*) For all $q > n$ and all discret G-module X which is p-primary torsion, $H^q(G, X) = 0$.

Now let U be an open normal subgroup of G and let A be a G-module which is p-primary torsion, i.e. for any $a \in A$, there exists an integer n_a such that $p^{n_a} a = 0$.

1. Prove that (*) still holds if we replace G by any closed subgroup H of G.
2. Let $0 \to A \to X^0 \to \ldots \to X^n \to \ldots$ be a resolution of coinduced p-primary G-modules and let $A_n := \ker(X^n \to X^{n+1})$. Prove that A_n is a cohomologically trivial G-module.
3. Prove that there exists a commutative diagram with exact lines, where N means the norm $N_{G/U}$:

$$\begin{array}{ccccccc}
((X^{n-1})^U)_{G/U} & \longrightarrow & (A_n^U)_{G/U} & \longrightarrow & H^n(U, A)_{G/U} & \longrightarrow & 0 \\
\downarrow N & & \downarrow N & & \downarrow \mathrm{Cor} & & \\
(X^{n-1})^G & \longrightarrow & A_n^G & \longrightarrow & H^n(G, A) & \longrightarrow & 0
\end{array}$$

4. Prove that the left vertical arrow is surjective.
5. Prove that the middle vertical arrow is injective and that the corestriction Cor in the diagram is an isomorphism.

Exercise 3. (Shapiro's Lemma for $\hat{H}$) Let G be a finite group, let H be a subgroup of G and let A be an H-module. Prove that the natural map

$$\hat{H}^i(G, \mathrm{Coind}_G^H(A)) \to \hat{H}^i(H, A)$$

is an isomorphism for any $i \in \mathbb{Z}$. (Hint: since G is finite, $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], A) = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A$. This may be useful to prove for $i < 0$.)