Exercise Sheet 6

Exercise 1. The goal of this exercise is to compute the cohomology groups of the profinite group $G = \hat{\mathbb{Z}} = \varprojlim_{n\in\mathbb{N}^*} \mathbb{Z}/n$. Let A be a discrete G-module.

1. Let F be the automorphism of A defined by the canonical topological generator $1 \in \hat{\mathbb{Z}}$. Let A' be the subgroup of A consisting of those $a \in A$ for which there is a positive integer n satisfying $(1 + F + ... + F^{n-1})a = 0$. Prove that $H^1(G, A) = A'/(F - 1)A$.
2. Compute $\mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{Q}/\mathbb{Z})$ (recall that for profinite groups, we usually only consider continuous maps).
3. Suppose that A is finite. Prove that $H^2(G, A) = 0$.
4. Supoose that A is torsion. Prove that $H^2(G, A) = 0$. (Hint: use the fact that every torsion G-module can be written as the direct limit of finite G-modules).
5. Supoose that A is a divisible group, i.e. $A \xrightarrow{n\cdot} A$ is surjective for any $n \in \mathbb{Z}_{>0}$. Prove that $H^2(G, A) = 0$.
6. Give an example of A with $H^2(G, A) \neq 0$. (Hint: use 2).