Exercise Sheet 4

Let G be a group and let A be a G-module. Recall that $H_0(G, A) = A \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/I_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$. An explicite way of constructing $H_i(G, A)$ is as follows: Let

$$... \to P_1 \to P_0 \to \mathbb{Z} \to 0 \qquad (*)$$

be a projective resolution of $\mathbb{Z}$, i.e. $(*)$ is exact and all P_i's are projective G-modules. Then we get a complex

$$... \xrightarrow{\partial_1} P_1 \otimes_{\mathbb{Z}[G]} A \xrightarrow{\partial_0} P_0 \otimes_{\mathbb{Z}[G]} A \to 0,$$

and then

$$H_i(G, A) = \ker \partial_i / \mathrm{im} \partial_{i-1}$$

for any $i \geqslant 0$.

Exercise 1. A G-module is said to be relatively projective if it is isomorphic to a direct factor of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} X$ for some abelian group X (G acts on $\mathbb{Z}[G] \otimes_{\mathbb{Z}} X$ by multiplication on the left on $\mathbb{Z}[G]$). Prove that for any relatively projective G-module A, $H_i(G, A) = 0$ for any $i > 0$.