Exercise Sheet 3

Exercise 1. Let K be a non-archimedean local field and let L/K be a finite extension. Let G_0 be the inertia group and let G_1 be the ramification group. Let $p = \mathrm{char} k$ where k is the residue field of K. Prove:

1. G_1 is the unique p-Sylow subgroup of G_0;
2. Prove that the field L^{G_1} is the largest tamely ramified extension of K in L.

The goal of the next two exercises is a step to understand the difference between tamely and widely ramifications in more geometric terms. Let A be a Dedekind domain with fraction field K, let L/K be a finite separable extension and let B be the integral closure of A in L. Define the **different** by the formula

$$\mathcal{D}_{L/K}^{-1} := \{x \in L | Tr(xB) \subset A\}.$$

Exercise 2. Assume $B = A[\alpha]$. Let $f(X) = a_0 + a_1 X + ... + a_n X^n$ be the minimal polynomial of α and let

$$f(X)/(X - \alpha) = b_0 + b_1 X + ... + b_{n-1} X^{n-1}.$$

Let $\alpha_1, ..., \alpha_n$ be the roots of f.

1. Prove that
$$\sum_{i=1}^{n} \frac{f(X)}{X - \alpha_i} \frac{\alpha_i^r}{f'(\alpha_i)} = X^r$$
for any $0 \leqslant r \leqslant n - 1$.
2. Prove that the dual basis of $1, \alpha, ..., \alpha^{n-1}$ with respect to $Tr(xy)$ is given by
$$b_0/f'(\alpha), ..., b_{n-1}/f'(\alpha).$$
3. Prove $\mathcal{D}_{L/K} = (f'(\alpha))$.

Exercise 3. Assume that K is a number field. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$ and let $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_K$. Let e be the ramification index of $\mathfrak{P}$ over $\mathfrak{p}$, i.e. $\mathfrak{p}\hat{\mathcal{O}}_{L,\mathfrak{P}} = (\mathfrak{P}\hat{\mathcal{O}}_{L,\mathfrak{P}})^e$. Assume the following fact:

$$\mathcal{D}_{L/K}\hat{\mathcal{O}}_{L,\mathfrak{P}} = \mathcal{D}_{\hat{\mathcal{O}}_{L,\mathfrak{P}}/\hat{\mathcal{O}}_{K,\mathfrak{p}}}.$$

Prove that

$$v_{\mathfrak{P}}(\mathcal{D}_{L/K}) = e - 1 \qquad \text{if } \mathfrak{P} \text{ is tamely ramified}$$

and

$$e \leqslant v_{\mathfrak{P}}(\mathcal{D}_{L/K}) \leqslant e - 1 + v_{\mathfrak{P}}(e) \qquad \text{if } \mathfrak{P} \text{ is wildly ramified.}$$