Exercise Sheet 2

Correction: In Hensel's Lemma we have seen in the lecture today, it is enough to assume $\deg u < \deg f$ instead of $\deg u < \deg \bar{f}$. This can be seen by the proof.

Exercise 1. Find a square root of 2 in $\mathbb{Q}_7$.

Exercise 2. Let $(K, |\cdot|)$ be a complete non-archimedean field and let $P = \sum_{i=0}^{n} a_i X^i \in K[X]$ be an irreducible polynomial. Prove that the Gauss norm $|P|_{\mathcal{G}} = \max(|a_0|, |a_n|)$.

Exercise 3. (Complete the proof of Newton's Lemma) Let $(K, |\cdot|)$ be a complete non-archimedean field and let A be its valuation ring. Let $P \in A[X]$, let $\alpha \in A$ and let $\epsilon \in (0, 1)$. Suppose $|P(\alpha)| \leqslant \epsilon |P'(\alpha)|^2$. Newton's Lemma claims that there exists a unique $\tilde{\alpha} \in A$ s.t. $P(\tilde{\alpha}) = 0$ and $|\tilde{\alpha} - \alpha| \leqslant \epsilon |P'(\alpha)|$. Complete the sketch of its proof we saw in class:

1. Define the map $\Phi \colon x \mapsto x - P'(\alpha)^{-1} P(x)$. Show that $\Phi(\alpha) \in \bar{B}(\alpha, \epsilon |P'(\alpha)|)$;
2. Show that $|\Phi(y) - \Phi(x)| \leqslant \epsilon |y - x|$ for any $x, y \in \bar{B}(\alpha, \epsilon |P'(\alpha)|)$;
3. Conclude the proof using Picard's fixed point theorem.

Exercise 4. (Teichmüller representatives) Prove that the natural map $\pi \colon \mathbb{Z}_p^* \to \mathbb{F}_p^*$ has a unique section, i.e. a map $e \colon \mathbb{F}_p^* \to \mathbb{Z}_p^*$ with $\pi \circ e = 1_{\mathbb{F}_p^*}$, which is a homomorphism of multiplicative groups.