Exercise Sheet 11

Exercise 1. Let G be a profinite group and let n be a non-negative integer. Let p be a prime number. Prove that the followings are equivalent:

1. $\mathrm{cd}_p(G) \leqslant n$.
2. $H^{n+1}(H, \mathbb{Z}/p\mathbb{Z}) = 0$ for every closed subgroup H of G.
3. $H^{n+1}(U, \mathbb{Z}/p\mathbb{Z}) = 0$ for every open subgroup U of G.

Exercise 2. The goal of this exercise is to prove the following proposition: Let G be a profinite group and let $G(p) = G/N$ be the largest quotient of G which is a pro-p-group. If $\mathrm{cd}_p(N) \leqslant 1$, then the canonical maps

$$H^q(G(p), \mathbb{Z}/p\mathbb{Z}) \to H^q(G, \mathbb{Z}/p\mathbb{Z})$$

are isomorphisms. In particular, $\mathrm{cd}(G(p)) \leqslant \mathrm{cd}_p(G)$.

1. Prove that every morphism from N to a pro-p-group is trivial (Hint: such a morphism induces a map $N \to N/M$ with N/M a pro-p-group).
2. Prove $H^i(N, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $i \geqslant 1$.
3. Conclude.

Exercise 3. Let k be a field and let k' be a purely inseparable extension of k. Prove that the canonical map $\mathrm{Br}(k) \to \mathrm{Br}(k')$ is surjective. (Hint: Let $\bar{k}'$ be a separable closure of k' and let $\bar{k}$ be the separable closure of k in $\bar{k}'$, then the group $\bar{k}'^*/\bar{k}^*$ is p-primary torsion where $p = \mathrm{char}(k)$).

Exercise 4. (Bonus) The goal is to prove that the dualizing module I for $G := \mathrm{Gal}(\bar{K}/K)$, where K is a p-adic field, is $\mu := \cup \mu_n$.

1. Prove by definition of I that $\mathrm{Hom}_H(\mu_n, I) = \mathbb{Z}/n\mathbb{Z}$ for any open subgroup H of G;
2. Deduce from (1) that $\mathrm{Hom}_G(\mu_n, I) = \mathrm{Hom}(\mu, I) = \mathbb{Z}/n\mathbb{Z}$;
3. Let $\varphi_n \colon \mu_n \to I$ be the element of $\mathrm{Hom}_G(\mu_n, I)$ corresponding to $\bar{1}$. Prove that φ_n is injective;
4. Let $I[n] := \ker(I \xrightarrow{\cdot n} I)$. Prove that φ_n in (3) induces an isomorphism $\mu_n \cong I[n]$ (as G-modules);
5. Conclude.