Exercise Sheet 10
Let K be a non-archimedean local field and let $\bar{K}$ be a separable closure. For any positive integer m, let μ_m denote the set of m-th roots of unity in $\bar{K}$.

Exercise 1. The goal of this exercise is to prove that there is a bijection

$$\begin{array}{ccc} \{\text{finite abelian extensions of } K\} & \longrightarrow & \{\text{open subgroups of } K^*\} \\ L & \mapsto & N(L^*) \end{array}.$$

In the rest of the exercise, the fields L, L' will be finite abelian extensions of K.

1. Prove $L \subset L' \Rightarrow N(L^*) \supset N(L'^*)$;

2. Prove $N((LL')^*) = N(L^*) \cap N(L'^*)$ (Hint: use the map $\phi_K \colon K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$);

3. Use (2) to deduce $N(L^*) \supset N(L'^*) \Rightarrow L \subset L'$;

4. Conclude.

Exercise 2. Let $p = \mathrm{char} K$. Prove that

$$K_{\mathrm{ur}} = \bigcup_{(m,p)=1} K(\mu_m).$$

Deduce that $K_{\mathrm{ur}} \subset K^{\mathrm{ab}}$. (Hint: compare the residue fields of both sides of the equation).

Exercise 3. Let $K = \mathbb{Q}_p$ and let $\pi = p$ be a uniformizer. The goal of this exercise is to compute K_π for this case.

1. Prove that $N(\mathbb{Q}_p(\mu_{p^n})^*) = U_{\mathbb{Q}_p}^{(n)} \cdot p^{\mathbb{Z}}$;

2. Prove that $(\mathbb{Q}_p)_p = \bigcup_{n \geqslant 1} \mathbb{Q}_p(\mu_{p^n})$ (Hint: use Exercise 1).

Exercise 4. Use the previous two exercises to prove the following statement (which is also called *the local Kronecker-Weber theorem*): any finite abelian extension of $\mathbb{Q}_p$ is contained in a cyclotomic extension.