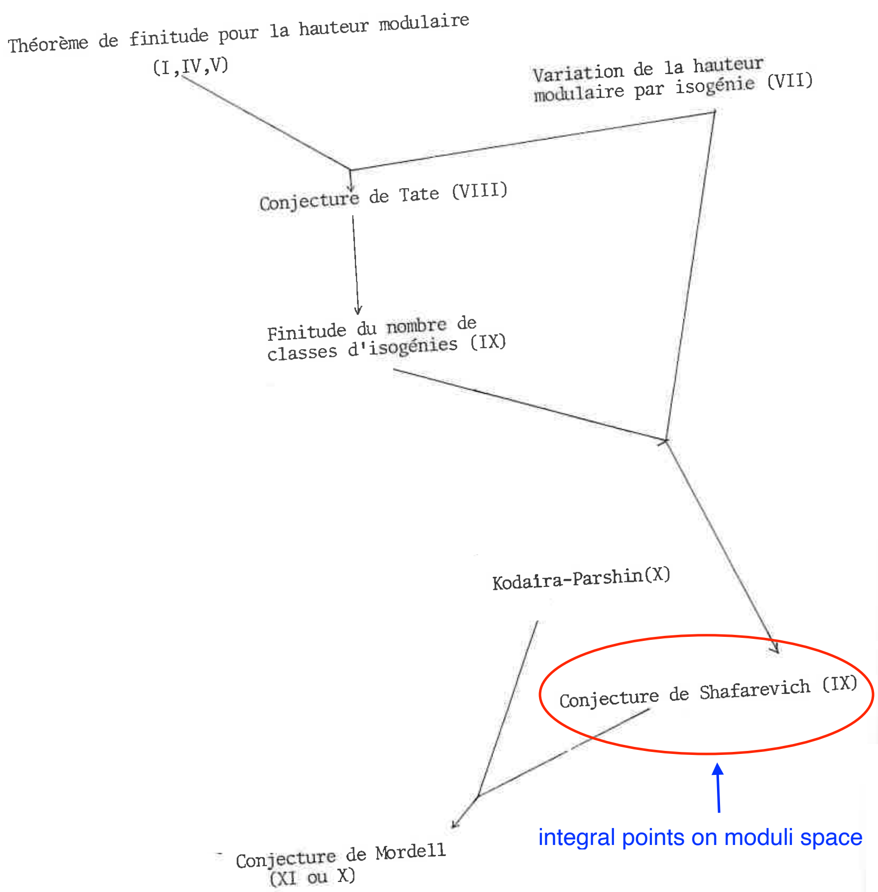
Mordell Conjecture, after Faltings 1983

UCLA, Winter and Spring 2025

Introduction

The aim of this participating seminar is to go through Faltings's proof of the Mordell Conjecture in 1983. Here is the sketch of the proof, extracted from [Ast127].



Inspired by this, we will divide the seminar into the following themes:

- Finiteness theorem for Faltings height;
- Isogeny and Tate Conjecture;
- Shafarevich Conjecture;
- Kodaira–Parshin construction and conclusion.

Schedule

1. Finiteness theorem for Faltings height:

    - 01/14 (Tom Han): Introduction to height theory. Define the logarithmic Weil height on projective spaces and state the Northcott property ([Gao22, §1.2.1], [HS00, §B.2]). Introduce the Height Machine ([Gao22, §5.1] or [HS00, §B.3]). Introduce Arakelov height defined by Hermitian line bundles on arithmetic varieties ([HS00, pp. 248 of §B.10]). Define the (stable) Faltings height of an abelian variety [Ast127, Exp. I, §3.3, Defn. 2] if time permits (assume the existence of the Néron model).
    - 01/21 (Jacob Swenberg): Define the height associated with a Hermitian line bundle with logarithmic singularities, and prove the "Northcott property" in this case. See [FW84, Chapter I, §4] and [Ast127, Exp. 1, §3.2]. Recall the definition of the (stable) Faltings height of an abelian variety [Ast127, Exp. 1, §3.3, Defn. 2] (assume the existence of the Néron model). Prove [Ast127, Exp. I, Thm. 3.2].
    - 01/28 (Jas Singh): Introduction to moduli spaces. Explain the general theory (without proof) of the moduli spaces of curves $\mathcal{M}_g$ and principally polarized abelian varieties $\mathcal{A}_g$ (over $\mathbb{Z}$).
    - 02/04 (Zach Baugher): Summary of the analytic theory of $\mathcal{A}_g$ over $\mathbb{C}$ and Baily–Borel compactification. Should cover [FW84, Chapter I, §5]. See also [Nam, §1 and 5].
    - 02/11 (John Zhou): Summary of the toroidal compactification of $\mathcal{A}_g$ over $\mathbb{C}$. Should cover [FW84, Chapter I, §6]. See also [AMRT10, I.1, I.3, I.4] and [Nam, §6, 7].
    - 02/18 (Tom Han): Comparison of the Faltings height and the theta height. We take the analytic approach [FW84, Chapter 2, Thm. 3.1]. An alternative (more algebraic) approach is provided by [Ast127, Exp. IV].

2. Isogeny and Tate Conjecture:

    - 02/25, 03/04, 03/11: Introduction to finite group schemes and Raynaud's result on group schemes of type $(p, \ldots, p)$. Need to cover [FW84, Chapter III, §2 and §4].
    - 04/01, 04/08: Introduction to p-divisible groups. See [FW84, Chapter III, §3 and §5].
    - 04/15: Prove that the Faltings height is bounded within an isogeny class. See [FW84, Chapter III, §3] and/or [Ast127, Exp. VII].
    - 04/22: Prove Tate Conjecture and the finiteness of isomorphism classes in an isogeny class. See [Ast127, Exp. VIII].

3. Shafarevich Conjecture:

    - 04/29: Reformulation of Tate conjecture; see [FW84, Chapter IV, Thm. 1.1, Cor. 1.2, Cor. 1.3]. Proof of the finiteness of isogeny classes *using Tate conjecture*; see [FW84, Chapter V, §2] and/or [Ast127, Exp. IX].

4. Kodaira–Parshin construction and conclusion.

    - 05/06: [Ast127, Exp. X].

References

[AMRT10] A. Ash, D. Mumford, M. Rapoport, Y. Tai: *Smooth Compactifications of Locally Symmetric Varieties (2nd ed)*. Cambridge University Press, 2010.

[FW84] G. Faltings, G. Wüstholz et al.: *Rational Points: Seminar Bonn/Wuppertal 1983/1984*. Aspects of Mathematics, **E6**, 1984.

[Gao22] Z. Gao: *An Introduction to Diophantine Geometry*. Lecture notes online `https://ziyangjeremygao.github.io/teaching/LectureNotes/DiophantineGeometry2022.pdf`.

[HS00] M. Hindry, J. Silverman: *Diophantine Geometry, An Introduction*. GTM, **201**, 2000.

[Nam] Y. Namikawa: *Toroidal Compactification of Siegel Spaces*. LNM, **812**, 1980.

[Ast127] L. Szpiro: *Séminaire sur les pinceaux arithmétiques : la conjecture de Mordell*. Astérique, **127**, 1985. Available at `http://www.numdam.org/item/AST_1985__127_/`.