Exercises course Introduction to Algebraic Topology - Singular homology, Fall 2014

In the following exercises X is a topological space.

Exercise 1. Let P be an affine independent subset of $\mathbb{R}^n$ and let $p_0, ..., p_k$ be an enumeration of the elements of P. Define

$$< p_0, ..., p_k >: \Delta^k \to [P], \quad (t_0, ..., t_k) \mapsto \sum_{i=0}^{k} t_i p_i.$$

Prove that this map is a homeomorphism. (Hint: remark that Δ^k is compact and $[P]$ is Hausdorff.)

Exercise 2. Let X be a path-connected space. Prove that $H_0(X) \cong \mathbb{Z}$.

Exercise 3. Let $X_1, \ldots, X_n$ be the path-connected components of the space X. Prove that $H_k(X) \cong \oplus_{i=1}^{n} H_k(X_i)$ for all $k \in \mathbb{Z}$.

Exercise 4. Prove that $< e_1, e_0 > + < e_0, e_1 >$ is a boundary in $\mathcal{C}_1(\Delta^1)$. Let $\sigma \colon \Delta^1 \to X$ be a singular 1-simplex given by $(1 - t)e_0 + te_1 \mapsto s(t), t \in [0, 1]$, and let $\sigma' \colon \Delta^1 \to X$ be given by $(1 - t)e_0 + te_1 \mapsto s(1 - t)$. Prove that $\sigma + \sigma'$ is a boundary in $\mathcal{C}_1(X)$.

Exercise 5. A singular 1-simplex $\sigma \colon \Delta^1 \to X$ is called a *loop* if $\sigma(e_0) = \sigma(e_1)$.
(a) Prove that a loop is a 1-cycle.
(b) Two loops σ_0 and σ_1 are called *freely homotopic* if there is a continuous map $F \colon [0, 1] \times [0, 1] \to X$ such that $F(0, t) = \sigma_0((1 - t)e_0 + te_1)$ and $F(1, t) = \sigma_1((1 - t)e_0 + te_1)$ and each $F(s, t)$ is a loop. Prove that free homotopy defines an equivalence relation on the set of loops in X.
(c) Prove that two freely homotopic loops are homologous.
(d) Choose a basepoint $x \in X$. Give a natural map $\rho \colon \pi_1(X, x) \to H_1(X)$ and prove that it is a homomorphism. So we have a natural map $\bar{\rho} \colon \pi_1(X, x)^{\mathrm{ab}} \to H_1(X)$.
(e) A 1-chain $\sigma_0 + \cdots + \sigma_{r-1}$ with $\sigma_i(e_0) = \sigma_{i-1}(e_1)$ for all $i \in \mathbb{Z}/r\mathbb{Z}$ is called an *elementary 1-cycle*. Prove that an elementary 1-cycle is a 1-cycle, homologous to a loop.
(f) Prove that the classes of loops generate $H_1(X)$.
(g) Assume that X is path-connected. Show that ρ is surjective.
Remark: it can be proved that $\bar{\rho}$ is an isomorphism.

Exercise 6. Let $\mathcal{C}'_\bullet$ be a subcomplex of a complex $\mathcal{C}_\bullet$. Let $k \in \mathbb{Z}$. Show that

$$d_{k+1}(\mathcal{C}_{k+1}) + \mathcal{C}'_k \subset d_k^{-1}(\mathcal{C}'_{k-1})$$

and that $H_k(\mathcal{C}_\bullet/\mathcal{C}'_\bullet)$ can be identified with the quotient $d_k^{-1}(\mathcal{C}'_{k-1})/(d_{k+1}(\mathcal{C}_{k+1}) + \mathcal{C}'_k)$.

Exercise 7. Let A be a subspace of X.
(a) Assume there exists a map $r \colon X \to A$ which is the identity on A (in that case we call r a *retraction map* and A a *retract* of X). Let $k \in \mathbb{Z}$. Show that $H_k(X) \cong H_k(A) \oplus \operatorname{Ker} r_k$.
(b) Assume there exists a map $R \colon X \times [0, 1] \to X$ such that $R(a, t) = a$ for all $a \in A$ and all t, and $R(x, 0) = x$ and $R(x, 1) \in A$ for all x in X (in that case we call R a *deformation retraction map* and A a *deformation retract* of X). Show that for each subspace $B \subset A$ the inclusion $(A, B) \subset (X, B)$ induces isomorphisms on homology.

Exercise 8. (a) Let $\phi\colon \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ be a chain map of exact complexes. Suppose there exist two distinct residue classes modulo 3 such that ϕ_k is an isomorphism whenever k belongs to one of these two residue classes. Prove that ϕ_k is an isomorphism for all $k \in \mathbb{Z}$.
(b) Let $\phi\colon \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ be a chain map of complexes. Assume $\mathcal{C}'_\bullet \subset \mathcal{C}_\bullet$ and $\mathcal{D}'_\bullet \subset \mathcal{D}_\bullet$ are sbucomplexes such that $\phi_k(\mathcal{C}'_k) \subset \mathcal{D}'_k$ for all $k \in \mathbb{Z}$. So we have chain maps $\phi'\colon \mathcal{C}'_\bullet \to \mathcal{D}'_\bullet$ and $\bar{\phi}\colon \mathcal{C}_\bullet/\mathcal{C}'_\bullet \to \mathcal{D}_\bullet/\mathcal{D}'_\bullet$. Prove that if two of ϕ, ϕ' and $\bar{\phi}$ induce an isomorphism on homology, then so does the third.
(c) Let $f\colon (X, Y, Z) \to (X', Y', Z')$ be a map of triads. In particular we have three maps of topological pairs $(X, Y) \to (X', Y')$, $(X, Z) \to (X', Z')$ and $(Y, Z) \to (Y', Z')$. Prove that if two of these inclusions induce isomorphisms on homology, then so does the third.

Exercise 9. Let $\phi, \phi'\colon \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ be chain maps. A *chain homotopy* from ϕ to ϕ' is a collection homomorphisms $(P_k\colon \mathcal{C}_k \to \mathcal{D}_{k+1})_{k\in\mathbb{Z}}$ such that $\phi'_k - \phi_k = P_{k-1}d_k + d_{k+1}P_k$ for all $k \in \mathbb{Z}$.
(a) Prove that chain homotopy defines an equivalence relation on the set of chain maps from $\mathcal{C}_\bullet$ to $\mathcal{D}_\bullet$.
(b) Let $\phi, \phi'\colon \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ and $\psi, \psi'\colon \mathcal{D}_\bullet \to \mathcal{E}_\bullet$ be chain homotopic. Prove that $\psi\phi, \psi'\phi'\colon \mathcal{C}_\bullet \to \mathcal{E}_\bullet$ are chain homotopic.
(c) Prove that chain homotopic maps induce the same maps on homology.

Exercise 10. The *cone CX* over a non-empty space X is obtained from $[0,1] \times X$ by identifying the subspace $\{0\} \times X$ to one point v, the *vertex* of CX.
(a) Show that CX is contractible.
Let $\{x\}$ be a one point space and let $\epsilon\colon X \to \{x\}$ be the unique map. Let $k \in \mathbb{Z}$. We define the k-th *reduced homology group* $\tilde{H}_k(X)$ to be the kernel of the map $\epsilon_k\colon H_k(X) \to H_k(\{x\})$.
(b) Prove that $H_k(CX, CX - \{v\}) \cong \tilde{H}_{k-1}(X)$.

Exercise 11. Visualize the first barycentric subdivision of Δ^3 and count the number of 3-simplices in it.

Exercise 12. The *suspension ΣX* of a non-empty space X is obtained from $[0,1] \times X$ by identifying each of the subsets $\{0\} \times X$ and $\{1\} \times X$ to a point.
(a) Prove that the projection $[0,1] \times X \to [0,1]$ defines a continuous map $h\colon \Sigma X \to [0,1]$.
(b) Compute the homology of ΣX by applying Mayer-Vietoris to the open sets $h^{-1}(0,1]$ and $h^{-1}[0,1)$.
(c) Let S^n for $n \in \mathbb{Z}_{\geq 0}$ be the n-sphere. Prove that ΣS^n and S^{n+1} are homeomorphic and compute the homology groups of S^n from this.

Exercise 13. Let $p_1, \ldots, p_n$ be distinct points in the plane $\mathbb{R}^2$. Compute the homology of $\mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$.

Exercise 14. Let S^2 be the 2-sphere and let $D_1, \ldots, D_n$ be n small open discs on S^2 with disjoint boundaries. Let X^+, X^- be two copies of $S^2 \setminus (D_1 \cup \ldots \cup D_n)$ and let X be the space obtained by identifying, for each $i = 1, \ldots, n$, the boundary of D_i on X^+ with the boundary of D_i on X^-, using the identity map. Thus, X is a "sphere with $n-1$ handles". Compute the homology of X.

Exercise 15. Each graph has the homotopy type of a bouquet of circles. Suppose that X is a graph, with the homotopy type of a bouquet of n circles. Prove that n is a homotopy-invariant of X. We call n the *Betti number* of X.

Exercise 16. Suppose that X is the union of open sets $U_0, \ldots, U_n$ such that all homology groups $H_k(Y)$ vanish for any intersection $Y = U_{i_0} \cap \ldots \cap U_{i_r}$ of these open sets and all $k > 0$ (we call the open cover $\{U_0, \ldots, U_n\}$ an *acyclic* cover in this case).
(a) Show that $H_k(X) = 0$ for $k > n$.
(b) If, in addition, each intersection Y is path-connected or empty, and $n \geq 1$, show that $H_n(X) = 0$.

Exercise 17. Let $f \colon X \to Y$ be a map between non-empty spaces.
(a) Prove that f induces a natural map $\Sigma f \colon \Sigma X \to \Sigma Y$ between the suspensions of X and Y (see Exercise 12).
(b) Let $f \colon S^n \to S^n$ be a map and let $\Sigma f \colon S^{n+1} \to S^{n+1}$ be the map induced from a homeomorphism $\Sigma S^n \cong S^{n+1}$. Prove that f and Σf have the same degree.
(c) In particular, for each $n > 0$ there exist maps $S^n \to S^n$ of arbitrary degree.

Exercise 18. In class we have seen that for any $n \geq 1$ and any $k \in \mathbb{Z}$ we have natural isomorphisms
$$H_k(\Delta^n, \hat{\Delta}^n) \cong H_{k-1}(\Delta^{n-1}, \hat{\Delta}^{n-1}).$$
Let Y be a non-empty space. By sticking in Y as a 'dummy' variable, we have natural isomorphisms
$$H_k((\Delta^n, \hat{\Delta}^n) \times Y) \cong H_{k-1}((\Delta^{n-1}, \hat{\Delta}^{n-1}) \times Y)$$
as well.
(a) Prove, by iteration, that $H_k((\Delta^n, \hat{\Delta}^n) \times Y) \cong H_{k-n}(Y)$.
(b) Hence we have $H_k(B^n \times Y, S^{n-1} \times Y) \cong H_{k-n}(Y)$.
(c) Let x be a point on S^n. Prove that $H_k(S^n \times Y, \{x\} \times Y) \cong H_{k-n}(Y)$.
(d) Prove that there is a natural isomorphism
$$H_k(S^n \times Y) \cong H_{k-n}(Y) \oplus H_k(Y).$$
Hint: the projection $S^n \times Y \to Y \cong \{x\} \times Y$ is a retraction.
(e) Compute the homology groups of $S^1 \times \ldots \times S^1$ (n factors).

Exercise 19. If $m, n \geq 0$ then every point z of $S^{m+n+1} \subset \mathbb{R}^{m+n+2} = \mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$ can be represented in the form $z = \cos(t) \cdot x + \sin(t) \cdot y$ with $x \in S^m$, $y \in S^n$, and $t \in [0, \pi/2]$, and this representation is unique except that x resp. y is undetermined when $t = \pi/2$ resp. $t = 0$. Given $f \colon S^m \to S^m$ and $g \colon S^n \to S^n$ we define their *join* $f * g \colon S^{m+n+1} \to S^{m+n+1}$ by $(f * g)(z) = \cos(t) \cdot f(x) + \sin(t) \cdot g(y)$.
(a) Prove that $\deg(f * g) = \deg(f) \cdot \deg(g)$. Hint: first prove that $f * g = (f * \mathrm{id})(\mathrm{id} * g)$ and prove $\deg(f * \mathrm{id}) = \deg(f)$ by induction on n. You may want to use the results of Exercise 17.
(b) Show that if both f and g are homotopic to the identity, then so is $f * g$.
(c) Exhibit a homotopy from id to $-$id on S^1.
(d) Prove that the antipodal map on an odd-dimensional sphere is homotopic to the identity.

Exercise 20. In this exercise we prove the Main Theorem of Algebra. Let $p(z) = z^k + c_1 z^{k-1} + \cdots + c_k$ with $k > 0$ be a non-constant polynomial with complex coefficients. We view

S^1 as the unit circle in $\mathbb{C}$. Assume p has no zeroes. We can then define a map $\hat{p}\colon S^1 \to S^1$ via

$$\hat{p}(z) = \frac{p(z)}{|p(z)|}.$$

(a) Exhibit a homotopy from $\hat{p}$ to a constant map. Hint: use that p has no zero z with $|z| \leq 1$.
(b) Exhibit a homotopy from $\hat{p}$ to the map $z \mapsto z^k$. Hint: use the identity

$$t^k p(\frac{z}{t}) = z^k + t(c_1 z^{k-1} + t c_2 z^{k-2} + \cdots + t^{k-1} c_k)$$

and the fact that $\hat{p}$ has no zero z with $|z| \geq 1$.
(c) Finish the proof of the Main Theorem of Algebra.

Exercise 21. In this exercise we prove that a sphere of positive even dimension cannot be given the structure of a topological group. Given a group G acting as a group of homeomorphisms of a space X, we say that G acts *freely* if the only element from G which has any fixed points is the identity element. Let g, h be two elements, unequal to the identity element, from a group G acting freely on S^n, where $n > 0$ is even.
(a) Prove that both g and h have degree -1.
(b) Prove that gh is the identity element.
(c) Conclude that G is either $\mathbb{Z}/2\mathbb{Z}$ or the trivial group.
(d) Prove that S^n is not a topological group.

Exercise 22. Prove that S^3 is a topological group. Hint: identify $\mathbb{R}^4$ with the Hamilton quaternions.

Exercise 23. Let $\mathbb{P}^n(\mathbb{R}) = \mathbb{P}(\mathbb{R}^{n+1})$ be the n-dimensional real projective space. Prove that any map $\mathbb{P}^n(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ has a fixed point if n is even. Describe a map $\mathbb{P}^n(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ without fixed points for each odd n.

Exercise 24. Assume the following fact: $S^n \to S^n$, $(x_1, ..., x_{n+1}) \mapsto (-x_1, x_2, ..., x_{n+1})$ is of degree -1. For each $n \in \mathbb{Z}_{>0}$ construct a surjective map $S^n \to S^n$ that has degree 0.

Exercise 25. Prove that every map $S^n \to S^n$ $(n > 0)$ is homotopic to one with a fixed point.

Exercise 26. (More about Snake Lemma) For Snake Lemma, prove that if φ is injective (resp. ψ' is surjective), then φ_0 is injective (resp. $\bar{\psi}'$ is surjective).

Exercise 27. (Snake Lemma implies long exact sequence) Let $0 \to (\mathcal{C}'_\bullet, d') \to (\mathcal{C}_\bullet, d) \to (\mathcal{C}''_\bullet, d'') \to 0$ be an exact sequence of complexes. Let $Z_k(\mathcal{C}_\bullet) := \ker(\mathcal{C}_k \to \mathcal{C}_{k-1})$ and $B_k(\mathcal{C}_\bullet) := \mathrm{im}(\mathcal{C}_{k+1} \to \mathcal{C}_k)$.
(a) Use Ex. 26 to prove the exactness of $0 \to Z_k(\mathcal{C}'_\bullet) \to Z_k(\mathcal{C}_\bullet) \to Z_k(\mathcal{C}''_\bullet)$ and of $\mathcal{C}'_k/B_k(\mathcal{C}'_\bullet) \to \mathcal{C}_k/B_k(\mathcal{C}_\bullet) \to \mathcal{C}''_k/B_k(\mathcal{C}''_\bullet) \to 0$.
(b) Show that for each k, there is a commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{C}'_k/B_k(\mathcal{C}'_\bullet) & \longrightarrow & \mathcal{C}_k/B_k(\mathcal{C}_\bullet) & \longrightarrow & \mathcal{C}''_k/B_k(\mathcal{C}''_\bullet) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow Z_{k-1}(\mathcal{C}'_\bullet) & \longrightarrow & Z_{k-1}(\mathcal{C}_\bullet) & \longrightarrow & Z_{k-1}(\mathcal{C}''_\bullet) & &
\end{array}
$$

where the vertical maps are induced by d', d and d'' respectively.
(c) Apply Snake Lemma to the diagram in (b) to get the long exact sequence of homology.