Exercises IAT - Sheaves and sheaf cohomology, Fall 2014

Exercise 1. Consider the following operations on groups G:

- forming the commutator subgroup $[G, G]$ of G;
- forming the abelianization $G^{\mathrm{ab}} = G/[G, G]$ of G;
- forming the center $Z(G)$ of G;
- taking $\mathrm{Hom}(H, G)$ where H is a fixed group;
- taking $\mathrm{Hom}(G, H)$ where H is a fixed group;
- forming the set of conjugacy classes of G;
- forming the automorphism group $\mathrm{Aut}(G)$.

Which of the above operations naturally give rise to a functor on the category of groups? Motivate your answer.

Exercise 2. Consider the following property (*) for an abelian group A:

> for every inclusion $I \subset \mathbb{Z}$ of a subgroup I in $\mathbb{Z}$, and every homomorphism $f\colon I \to A$, there exists a homomorphism $g\colon \mathbb{Z} \to A$ such that $g|_I = f$.

(i) Verify that saying that A satisfies (*) is equivalent to saying that A is divisible.
(ii) Prove that if A satisfies (*), then A is injective.
Hint: let $M \subset N$ be an inclusion of abelian groups, and let $k\colon M \to A$ be a homomorphism. Consider the set of pairs (H, h) where H is a subgroup of N with $M \subset H$ and where $h\colon H \to A$ is a homomorphism with $h|_M = k$. This set has a natural partial ordering. Prove that a maximal element of this set is of the form (N, h), and verify that such a maximal element exists by Zorn's Lemma.
(iii) Conclude that an abelian group A is injective if and only if A is divisible.

Exercise 3. Prove that the category of abelian groups has enough injectives.
Hint: for each abelian group A there exists a free abelian group F and a surjective morphism $F \to A$. As F is a direct sum of copies of $\mathbb{Z}$, the group F can be embedded in a divisible group. Furthermore, a quotient of a divisible group is divisible.

Exercise 4. Let *Ab* be the category of abelian groups, and let A be a fixed abelian group. Recall that the rule $M \mapsto \mathrm{Hom}(A, M)$ defines a left exact functor from *Ab* to itself. Let B be a cyclic group (finite or infinite).
(i) Construct an injective resolution of B.
The right derived functors of the functor $\mathrm{Hom}(A, -)$ are denoted by $\mathrm{Ext}^i(A, -)$ for $i \geq 0$. Now let A and B be cyclic groups (finite or infinite).
(ii) Compute $\mathrm{Ext}^i(A, B)$ for all $i \geq 0$.

In the following exercises X denotes a topological space.

Exercise 5. (i) Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if for every open set U of X, and every section $s \in \mathcal{G}(U)$, there exists an open cover $(U_i)_{i\in I}$ of U and sections $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ in $\mathcal{G}(U_i)$ for all $i \in I$.
(ii) Give an example of a surjective morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$, and an open set U such that $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.

Exercise 6. Let A be an abelian group. Denote by A_X the sheaf associated to the presheaf $U \mapsto A$ on X. We call A_X the *constant sheaf* on X with values in A.
(i) Let U be an open set of X. Show that $A_X(U)$ can be identified with the set of continuous maps $U \to A$. Here A is endowed with the discrete topology.
(ii) Assume that X is locally connected (i.e. for each open set U of X the connected components of U are open). Show that $A_X(U)$ can be identified with the set of locally constant functions $U \to A$. A function $U \to A$ is called locally constant if it is constant on each connected component of U.

Exercise 7. Let K be a closed subset of X, and denote by $i\colon K \to X$ the inclusion of K in X. Let $\mathcal{F}$ be a sheaf on K. Denote by $i_*\mathcal{F}$ the presheaf on X given by

$$U \mapsto \mathcal{F}(K \cap U)\,.$$

(i) Show that $i_*\mathcal{F}$ is in fact a sheaf.
(ii) Show that

$$(i_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in K \\ 0 & x \notin K\,. \end{cases}$$

We call $i_*\mathcal{F}$ the *extension of* $\mathcal{F}$ *by zero outside* K.
(iii) Show that $\mathcal{F} \mapsto i_*\mathcal{F}$ is an exact functor from Sh_K to Sh_X, i.e. sends exact sequences to exact sequences.
(iv) Show that $\mathcal{F} \mapsto i_*\mathcal{F}$ sends flasque sheaves to flasque sheaves. A sheaf $\mathcal{F}$ is called *flasque* if for each inclusion $U \subset V$ of open sets in X, the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.
(v) Recall that a sheaf always admits a flasque resolution, and that this resolution can be used to compute the cohomology groups of that sheaf. Use this to show that $\mathrm{H}^k(X, i_*\mathcal{F}) \cong \mathrm{H}^k(K, \mathcal{F})$ for all $k \geq 0$.
Let K_1, K_2 be closed subsets of X with $X = K_1 \cup K_2$. Denote by $\mathbb{Z}_i$ $(i = 1, 2)$ the constant sheaf with values in $\mathbb{Z}$ on K_i $(i = 1, 2)$. Denote by $\mathbb{Z}_{12}$ the constant sheaf with values in $\mathbb{Z}$ on $K_1 \cap K_2$. Let $i_{1,2}$ be the inclusions of $K_{1,2}$ into X, and let i_{12} be the inclusion of $K_1 \cap K_2$ into X.
(vi) Show that we have a natural exact sequence of sheaves

$$0 \to \mathbb{Z}_X \to i_{1*}\mathbb{Z}_1 \oplus i_{2*}\mathbb{Z}_2 \to i_{12*}\mathbb{Z}_{12} \to 0$$

on X.
(vii) Show that by taking the long exact sequence of cohomology of this short exact sequence we get the *Mayer-Vietoris exact sequence*

$$\cdots \to \mathrm{H}^i(X, \mathbb{Z}) \to \mathrm{H}^i(K_1, \mathbb{Z}) \oplus \mathrm{H}^i(K_2, \mathbb{Z}) \to \mathrm{H}^i(K_1 \cap K_2, \mathbb{Z}) \to \mathrm{H}^{i+1}(X, \mathbb{Z}) \to \cdots$$

(note that for readability we removed the subscripts from the $\mathbb{Z}$'s). As an intermediate step, show that sheaf cohomology commutes with taking direct sums of sheaves.