Seminar:
Adelic line bundles over quasi-projective varieties

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September 26, 2021

Introduction

The aim of this seminar is to read the new preprint by Yuan and Zhang [YZ21] on the theory of adelic line bundles on quasi-projective varieties. We focus on the *number field* case. Applications of this theory are for example an equidistribution theorem for quasi-projective varieties, especially for dynamical systems over quasi-projective varieties. This generalizes also the equidistribution theorem for families of abelian varieties by Kühne [Küh21]. As another application, Yuan proved in a subsequent preprint [Yua21] a uniform Bogomolov-type result. While the *Uniform Bogomolov Conjecture* has been proved by Kühne as an application of his equidistribution result, Yuan's result [Yua21] is a finer version of the more general *New Gap Principle* [Gao21, Thm.1.4] proved by Dimitrov, Gao, Habegger and Kühne. However, Yuan uses a completely different strategy.

As this theory is very technical, it is hardly possible to reach to these applications in one semester. To do not only dry theory, we will try to obtain the prove of the following height inequality.

Theorem (Theorem 5.3.5 in [YZ21]). *Let K be a number field and $\pi\colon X \to S$ a morphism of quasi-projective varieties over K. Furthermore, let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$ and $\overline{M} \in \widehat{\mathrm{Pic}}(S/\mathbb{Z})$ be adelic line bundles.*

(1) If $\overline{L}$ is big over X, then there exist $\epsilon > 0$ and a Zariski open and dense subvariety U of X such that
$$h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)), \qquad \forall x \in U(\overline{K}).$$

(2) If $\overline{L}$ is nef over $X/\mathbb{Z}$, and the image $\widetilde{L}$ of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X/K)$ is big over X/K, then for any $c > 0$ there exist $\epsilon > 0$ and a Zariski open and dense subvariety U of X such that
$$h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)) - c, \qquad \forall x \in U(\overline{K}).$$

Dimitrov, Gao and Habegger [GH19, DGH21] obtained a similar inequality for families of abelian varieties, which plays a fundamental role in the treatment of the uniform Mordell–Lang problem.

The first task in the seminar will be to define and understand the group $\widehat{\mathrm{Div}}(X/\mathbb{Z})$ of adelic divisors X as well as the group $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$ of adelic line bundles on X for any (essentially) quasi-projective variety X over $\mathbb{Z}$. The definition involves a limit over projective models and a completion with respect to the so-called boundary topology. The next step will be to establish the analytic counterparts $\widehat{\mathrm{Div}}(X^{\mathrm{an}}/\mathbb{Z})$ and $\widehat{\mathrm{Pic}}(X^{\mathrm{an}}/\mathbb{Z})$ and to study the analytification maps

$$\widehat{\mathrm{Div}}(X/\mathbb{Z}) \to \widehat{\mathrm{Div}}(X^{\mathrm{an}}/\mathbb{Z})_{\mathrm{eqv}}, \qquad \widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X^{\mathrm{an}}/\mathbb{Z})_{\mathrm{eqv}}$$

into the norm-equivariant subgroups.

Since heights are given by intersection numbers, we will introduce absolute intersection numbers on adelic line bundles as well as its relative variant given by Deligne pairings. As in classical algebraic geometry, the volume of a line bundle is an important tool. We will introduce the volume for adelic line bundles and prove some of its properties, for example the arithmetic Hilbert–Samuel formula and its continuity. Finally, we construct height functions associated to adelic line bundles and prove the theorem above.

In the paper [YZ21] the authors treat simultaneously the arithmetic and the geometric situation by using the letter k for denoting either $\mathbb{Z}$ or a field. For simplicity we will only treat the arithmetic case $k = \mathbb{Z}$ and we omit the geometric case whenever proofs or constructions are treated separately.

1 Introduction (Robert, 11.10.–15.10.)

This talk gives an overview over the seminar. It should differ from the Introduction of [YZ21] in that way, that it emphasize more on the constructions playing a role in the seminar as on the results of the paper.

2 Review of arithmetic varieties (18.10.–22.10.)

This talk should cover [YZ21, Section 2.1]. It reviews the theory of arithmetic varieties, arithmetic divisors and their arithmetic intersection numbers.

3 Mixed coefficients and essentially quasi-projective schemes (25.10.–29.10.)

In the first part of the talk we introduce the notions of $(\mathbb{Q}, \mathbb{Z})$-divisors and $(\mathbb{Q}, \mathbb{Z})$-line bundles as in [YZ21, Section 2.2]. In the second part we discuss the notion of essentially quasi-projective schemes as in [YZ21, Section 2.3]. In particular, it is good to see examples and state the properties of the lemmas. Proofs should only be treated if time allows.

4 Adelic divisors (01.11.–05.11.)

We would like to see the construction of $\widehat{\mathrm{Div}}(X/\mathbb{Z})$, one of the most important objects in this seminar. Also the class group $\widehat{\mathrm{CaCl}}(X/\mathbb{Z})$ should be introduced. The talk covers

[YZ21, Section 2.4].

5 Adelic line bundles (08.11.–12.11.)

We are now ready to define the category $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ and its group of isomorphism classes $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$ and prove that the latter one is isomorphic to $\widehat{\mathrm{CaCl}}(X/\mathbb{Z})$. The notion of nef and integrable line bundles will play a fundamental role in the rest of the seminar. Further we should shortly discuss the associated forgetful maps, functoriality and the extension to $\mathbb{Q}$-coefficients. This covers [YZ21, Section 2.5].

6 Examples (15.11.–19.11.)

For a better understanding we should see examples. In [YZ21, Section 2.6] some interested examples are collected. However, it is not necessary to cover this section in its entirety.

7 Berkovich spaces (22.11.–26.11.)

This talk should give a review of Berkovich spaces as in [YZ21, Sections 3.1.1 & 3.1.2]. One may also consult Jonsson's lecture notes [Jon16] for more details. In particular, we are interested in the Berkovich space X^{an} associated to X. We are only interested in the arithmetic case $k = \mathbb{Z}$.

8 Density result and arithmetic divisors and metrized line bundles (29.11.–03.12.)

The first part of this talk should discuss the density result on the image of the analytification of the embedding of X into a quasi-projective model [YZ21, Lemma 3.1.1]. We would like to see some aspects of its proof. In the second part we introduce arithmetic divisors and metrized line bundles over Berkovich spaces as in [YZ21, Section 3.2]. This can be considered as the analytic counterparts of the notions introduced in talks 4 and 5.

9 Analytification of adelic divisors (06.12.–10.12.)

This talk is provided to the construction of the canonical injective maps

$$\widehat{\mathrm{Div}}(X/\mathbb{Z}) \to \widehat{\mathrm{Div}}(X^{\mathrm{an}}/\mathbb{Z})_{\mathrm{eqv}}, \qquad \widehat{\mathrm{CaCl}}(X/\mathbb{Z}) \to \widehat{\mathrm{CaCl}}(X^{\mathrm{an}}/\mathbb{Z})_{\mathrm{eqv}}$$

as done in [YZ21, Section 3.3] in three steps: First for projective varieties, then for quasi-projective varieties and finally for essentially quasi-projective varieties.

10 Analytification of adelic line bundles (13.12.–17.12.)

This talk should show analogously to the previous talk the construction of the canonical injective map

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X^{\mathrm{an}}/\mathbb{Z})_{\mathrm{eqv}}$$

as covered by [YZ21, Section 3.4]. Moreover, the consequence on shrinking the underlying scheme in [YZ21, Corollary 3.4.2] should be discussed.

11 Restricted analytic spaces (20.12.–24.12.)

In this talk we would like to define the restricted analytic space $X^{\mathrm{r-an}}$ as well as the associated groups $\widehat{\mathrm{Div}}(X^{\mathrm{r-an}})$, $\widehat{\mathrm{CaCl}}(X^{\mathrm{r-an}})$ and $\widehat{\mathrm{Pic}}(X^{\mathrm{r-an}})$. We would like to see, that the canonical maps from the previous talks induce injective maps

$$\widehat{\mathrm{Div}}(X) \to \widehat{\mathrm{Div}}(X^{\mathrm{r-an}}), \qquad \widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{Pic}}(X^{\mathrm{r-an}})$$

as well as a fully faithful functor $\widehat{\mathcal{P}\mathrm{ic}}(X) \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{r-an}})$. See [YZ21, Section 3.5.1]. In a second part of the talk it would be nice to see the compatibility of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{r-an}})$ with the theory already introduced by Zhang [Zha95] in 1995. For this purpose, we would like to see (partly) the proof of [YZ21, Proposition 3.5.2].

12 Intersection theory and Deligne pairings (10.01.–14.01.)

We are interested in the existence of an absolute intersection pairing $\widehat{\mathrm{Pic}}(X/\mathbb{Z})^d_{\mathrm{int}} \to \mathbb{R}$ as proved in [YZ21, Proposition 4.1.1] by a limit process based on the arithmetic intersection pairing defined by Gillet–Soulé [GS90]. Moreover, we would like to see the construction of the Deligne pairing, which existence is guaranteed in [YZ21, Theorem 4.1.2]. For the definition of Deligne pairings, one may also consult [Zha96, Section 1.1]. We are especially interested in the construction of the metric as done in [YZ21, Section 4.2]. The proofs of the Lemmas can be left out for time reasons.

13 Effective sections of adelic line bundles (17.01.–21.01.)

In this talk we discuss the notion of effective sections of adelic line bundles as in [YZ21, Section 5.1]. The set of effective sections is denoted by $\widehat{H}^0(X, \overline{L})$ and especially, we would like to see its finiteness.

14 Volumes of adelic line bundles (24.01.–28.01.)

The goal of this talk is to introduce the notion of volume of an adelic line bundle as in [YZ21, Section 5.2]. All theorems and propositions in this section should be covered. This implies the arithmetic Hilbert–Samuel theorem [YZ21, Theorem 5.2.2] and the continuity and log-concavity of the volume. In the proofs we are more interested in the constructions than in technical calculations.

15 Heights over quasi-projective varieties (31.01.-04.02.)

In the final talk we would like to define height functions associated to adelic line bundles and study its properties, for example the Northcott property and the height inequality. This should cover [YZ21, Section 5.3]. Note, that we are only interested in the arithmetic case $k = \mathbb{Z}$. We are also more interested in number fields than general finitely generated fields. This simplifies many things in this talk. Especially, [YZ21, Section 5.3.5] can be left out, since the special case [YZ21, Section 5.3.6] is sufficient for our purpose.

References

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