Sparsity of rational points on curves

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Motivation

It is a fundamental question in mathematics to solve equations.

For example:
$f(X, Y)$= polynomial in X and Y with coefficients in $\mathbb{Q}$.
What can we say about the $\mathbb{Q}$-solutions to $f(X, Y) = 0$?

$\rightsquigarrow$

Diophantine problem. Rational points on algebraic curves.

Some examples:

$f(X, Y)$	$X^2 + Y^2 - 1$	$Y^2 - X^3 - X$	$Y^2 - X^3 - 2$	$Y^2 - X^6 - X^2 - 1$
$\mathbb{Q}$-solutions	$(3/5, 4/5)$, $(5/13, 12/13)$, $(8/17, 15/17)$, *etc.* infinitely many	$(0, 0)$, $(\pm 1, 0)$. finitely many	$(-1, 1)$, $(34/8, 71/8)$, $(2667/9261, 13175/9261)$, *etc.* infinitely many	$(0, \pm 1)$, $(\pm 1/2, \pm 9/8)$. finitely many
genus of the associated curve	0	1	1	2

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Setup and Genus 0

In what follows,

- $g \geq 0$ and $d \geq 1$ integers;
- K= number field of degree d;
- $C =$ irreducible smooth projective curve of genus g defined over K.

As usual, we use $C(K)$ to denote the set of K-points on C.

✎ If $g = 0$, then either $C(K) = \varnothing$ or $C \cong \mathbb{P}^1$ over K.

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Genus 1

Assume $g = 1$.
If $C(K) \neq \emptyset$, then $C(K)$ has a structure of abelian groups with an identity element $O \in C(K)$. $\rightsquigarrow$ Elliptic curve $E/K := (C, O)$.

Theorem (Mordell–Weil)

$E(K)$ is a finitely generated abelian group. Namely,

$$E(K) \cong \mathbb{Z}^{\rho} \oplus E(K)_{\mathrm{tor}}$$

with $\rho < \infty$ and $E(K)_{\mathrm{tor}}$ finite.

Genus 1: finite part

Theorem (Mazur '77 for $K = \mathbb{Q}$, Merel '96)

$\#E(K)_{\mathrm{tor}}$ is uniformly bounded above in terms of $[K : \mathbb{Q}]$.

Mazur proved this result by establishing the following theorem:

Theorem (Mazur '77)

If $N = 11$ or $N \geq 13$, then the only $\mathbb{Q}$-points of the modular curve $X_1(N)$ are the rational cusps.

The genus of $X_1(N)$ is ≥ 2 if $N = 13$ or $N \geq 16$.

$\rightsquigarrow$ results of rational points on curves of genus ≥ 2.

Genus ≥ 2: Mordell Conjecture

Mordell made the following conjecture about 100 years ago (1922), known as the Mordell Conjecture. It became a theorem in 1983, proved by Faltings.

Theorem (Faltings '83; known as Mordell Conjecture)

If $g \geq 2$, then the set $C(K)$ is finite.

Feature of this theorem	When applied to Mazur's result on $X_1(N)$
➢ weak topological hypothesis, very strong arithmetic conclusion!	✎ $X_1(N)$ has only finitely many $\mathbb{Q}$-points if $N \geq 16$.
➢ not constructive yet.	✎ $X_1(N)(\mathbb{Q})$ cannot be determined by Faltings's Theorem.

Genus ≥ 2: Fermat's Last Theorem

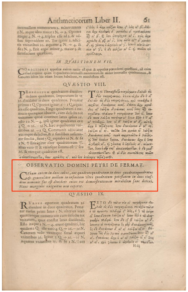
Fix $n \geq 4$ integer.

$$F_n : X^n + Y^n - 1 = 0.$$

Then $g(F_n) \geq 2$.

$\Downarrow$ Faltings

$\exists$ only finitely many $(x, y) \in \mathbb{Q}^2$ with $x^n + y^n = 1$.



For this example, more is expected.

Theorem (Wiles, Taylor–Wiles, '95; known as Fermat's Last Theorem)

If x and y are rational numbers such that $x^n + y^n = 1$, then $(x, y) = (0, \pm 1)$ or $(x, y) = (\pm 1, 0)$.

Of course if n is furthermore assumed to be odd, then -1 cannot be attained.

Genus ≥ 2

From now on, we always assume that $g \geq 2$.
The example of Fermat's Last Theorem suggests that it can be extremely hard to compute $C(\mathbb{Q})$ for an arbitrary C!
Instead, here is a more achievable but still fundamental question.

Question (Mordell, Weil, Manin, Mumford, Faltings, *etc.*)

Is there an "easy" upper bound for $\#C(K)$? How does $C(K)$ "distribute"?

Different grades of the question:

- Finiteness of $C(K)$
- Upper bound of $\#C(K)$
- Uniformity of bounds of $\#C(K)$
- Effective Mordell

Heights

Use height to measure the “size” of the rational and algebraic points.

- On $\mathbb{Q}$: $h(a/b) = \log\max\{|a|, |b|\}$, for $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$.
- On $\mathbb{P}^n(\mathbb{Q})$: $h([x_0 : \cdots : x_n]) = \log\max\{|x_0|, \ldots, |x_n|\}$, for $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$.
- Arbitrary number field K: For $[x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$ with each $x_j \in K$, $h([x_0 : \cdots : x_n]) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \Sigma_K} \log\max\{\|x_0\|_v, \ldots, \|x_n\|_v\}$.

⇝ (logarithmic) Weil height on $\mathbb{P}^n(\overline{\mathbb{Q}})$, and on any subvariety $X \subseteq \mathbb{P}^n$.

Two important properties →
↓

Northcott Property

For all B and $d \geq 1$,

$$\{\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}}) : h(\mathbf{x}) \leq B, [\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leq d\}$$

is finite.

Bounded from below

$h(\mathbf{x}) \geq 0$ *for all* $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$.

Heights

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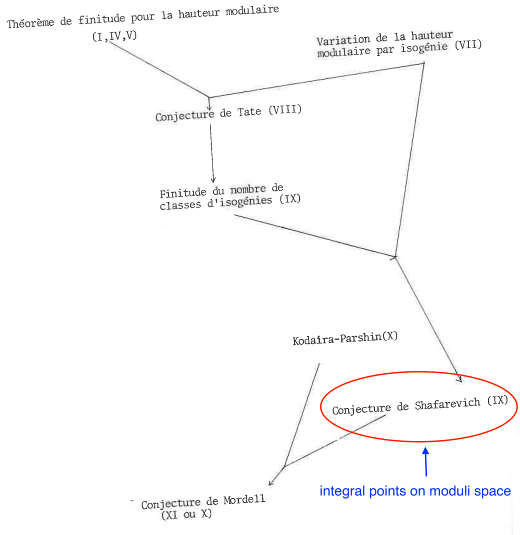
Genus ≥ 2: Faltings's proof of the Mordell Conjecture



Extracted from « Séminaire sur les pinceaux arithmétiques, La conjecture de Mordell » (Astérisque 127), Lucien Szpiro.

- $\mathbb{A}_g$ = moduli space of pp abelian varieties

New approach to treat integral points on moduli spaces: Lawrence–Venkatesh.

Faltings height

- $A/\overline{\mathbb{Q}}$ = pp abelian variety.

Faltings defined an intrinsic number $h_{\mathrm{Fal}}(A)$ associated with A (cf. Astérisque 127, or Cornell–Silverman).

$\rightsquigarrow h_{\mathrm{Fal}}\colon \mathbb{A}_g(\overline{\mathbb{Q}}) \to \mathbb{R}$.

Why is it called a height?

Fix an embedding $\mathbb{A}_g \subseteq \mathbb{P}^N$ over $\overline{\mathbb{Q}}$. $\rightsquigarrow$ Weil height $h\colon \mathbb{A}_g(\overline{\mathbb{Q}}) \to \mathbb{R}$.

Theorem (Faltings, improved constants by Bost, David, Pazuki)

$|\frac{1}{2}h_{\mathrm{Fal}}(A) - h([A])| \le c_g \log(h([A]) + 2)$.

Upshots:

- $h_{\mathrm{Fal}}(A)$ bounded from below solely in terms of g.
- Northcott property for h_{Fal}.

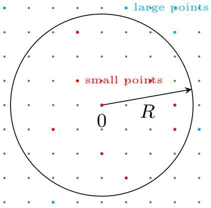
Genus ≥ 2: a new proof by Vojta

In early 90s, Vojta gave a second proof to Faltings's Theorem with Diophantine method.

- Closer to A. Weil's hope.
- Does not prove the other big conjectures (Tate, Shafarevich) as in Faltings's first proof.
- In this proof, one sees some descriptions of distribution of algebraic points on C. They lead to an upper bound on $\#C(K)$.
- The proof was simplified by Bombieri. And generalized by Faltings to some high dimensional cases.

Starting Point: Take $P_0 \in C(K)$, and see C as a curve in $J = \mathrm{Jac}(C)$ via the Abel–Jacobi embedding $C \to J$ based at P_0. Then $C(K) \subseteq J(K)$.

Vojta's proof of the Mordell Conjecture: Setup



Normalized height function $\hat{h}\colon J(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ vanishing precisely on $J(\overline{\mathbb{Q}})_{\mathrm{tor}}$.

$\rightsquigarrow$ $\hat{h}\colon J(K) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}_{\geq 0}$ quadratic, positive definite.

$\rightsquigarrow$ Normed Euclidean space $(J(K) \otimes_{\mathbb{Z}} \mathbb{R}, |\cdot| := \hat{h}^{1/2})$, with $J(K)$ a lattice.

$\rightsquigarrow$ Inner product $\langle \cdot, \cdot \rangle$ on $J(K) \otimes_{\mathbb{Z}} \mathbb{R}$, and the angle of each two points in $J(K) \otimes_{\mathbb{Z}} \mathbb{R}$.

Vojta's proof of Mordell Conjecture: Mumford's work

A starting point is the following (consequence of) Mumford's Formula: For $P, Q \in C(\overline{\mathbb{Q}})$ with $P \neq Q$, we have

$$\frac{1}{g}\left(|P|^2 + |Q|^2 - 2g\langle P, Q\rangle\right) + O(|P| + |Q| + 1) \geq 0$$

As $g \geq 2$, the leading term is an indefinite quadratic form, which a priori could take any value. This gives a strong constraint on the pair (P, Q)!
$\rightsquigarrow$ Algebraic points are "sparse" in C!

Vojta's proof of Mordell Conjecture: Both inequalities

Theorem

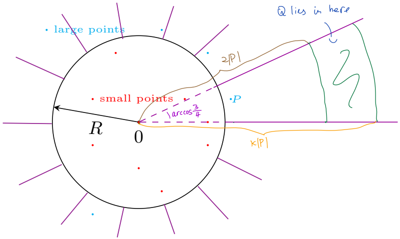
There exist $R = R(C)$ and $\kappa = \kappa(g)$ satisfying the following property. If two distinct points $P, Q \in C(\overline{\mathbb{Q}})$ satisfy $|Q| \geq |P| \geq R$ and

$$\langle P, Q\rangle \geq (3/4)|P||Q|,$$

then

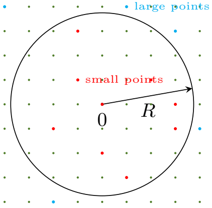
- *(Mumford, '65)* $|Q| \geq 2|P|$;
- *(Vojta, '91)* $|Q| \leq \kappa|P|$.

This finishes the proof of the Mordell Conjecture, with #large points$\leq (\log_2 \kappa + 1)7^{\mathrm{rk}J(K)}$.



If $P_1, \ldots, P_n$ are in the cone where P lies, then $\kappa|P| \geq |P_n| \geq 2|P_{n-1}| \geq \cdots \geq 2^n|P|$.
So in each cone there are $\leq \log_2 \kappa + 1$ large points!
$7^{\mathrm{rk}J(K)}$ such cones, according to the angle condition.

Genus ≥ 2: Classical bound



Theorem (Bombieri '91, de Diego '97, Alpoge 2018)

- *One can take $R^2 = c_0(g) h_{\mathrm{Fal}}(J)$.*
- *$\#$large points $\leq c(g) 1.872^{\mathrm{rk}_{\mathbb{Z}} J(K)}$. $\rightsquigarrow$ A nice bound for #large points!*

For a bound of $\#C(K)$, we have:

Theorem (David–Philippon, Rémond 2000)

$$\#C(K) \leq c(g, [K : \mathbb{Q}], h_{\mathrm{Fal}}(J))^{1+\mathrm{rk}_{\mathbb{Z}} J(K)}.$$

Genus ≥ 2

Different grades of the question:

- Finiteness of $C(K)$ ✓
- Upper bound of $\#C(K)$ ✓
- Uniformity of bounds of $\#C(K)$
- Effective Mordell

Sparsity of algebraic points:

"sparsity" of large points

- Mumford's Inequality '65
- Vojta's Inequality '91
- ?✎
- ???

And about the distribution / sparsity of points:

✎ Are there other descriptions of the "sparsity" of algebraic points on C? Or at least can we say something about "small" points?

Genus ≥ 2: Towards uniform bounds on $\#C(K)$

The cardinality $\#C(K)$ must depend on g.

Example

The hyperelliptic curve defined by

$$y^2 = x(x-1)\cdots(x-2024)$$

has genus 1012 *and has at least* 2026 *different rational points.*

The cardinality $\#C(K)$ must depend on $[K : \mathbb{Q}]$.

Example

The hyperelliptic curve

$$y^2 = x^6 - 1$$

has points $(1,0)$, $(2, \pm\sqrt{63})$, $(3, \pm\sqrt{728})$, *etc.*

Genus ≥ 2: Towards uniform bounds on $\#C(K)$

Here is a very ambitious bound.

Question

Is it possible to find a number $B(g, [K : \mathbb{Q}]) > 0$ such that

$$\#C(K) \leq B?$$

This question has an affirmative answer if one assumes a widely open conjecture of Bombieri–Lang on rational points on varieties of general type (Caporaso–Harris–Mazur, Pacelli, '97).

- ✎ Two divergent opinions towards this conditional result: either this ambitious bound is true, or one could use this to disprove this conjecture of Bombieri–Lang.

Genus ≥ 2: Mazur's Conjecture B

Theorem (Dimitrov-G'-Habegger, 2021; Mazur's Conjecture B ('86, 2000))

If $g \geq 2$, then

$$\#C(K) \leq c(g, [K:\mathbb{Q}])^{1+\mathrm{rk}_{\mathbb{Z}} J(K)}$$

where J is the Jacobian of C. Moreover, $c(g, [K:\mathbb{Q}])$ grows at most polynomially in $[K:\mathbb{Q}]$.

- Compared to the classical result, the *height of C* is no longer involved.
- We showed that c does not depend on $[K:\mathbb{Q}]$ assuming the relative Bogomolov conjecture. Kühne (2021) removed this dependence on $[K:\mathbb{Q}]$ unconditionally.
- Previous results:
  - When $J \subseteq E^n$ and some particular family of curves (David, Philippon, Nakamaye 2007). Average number of $\#C(\mathbb{Q})$ when $g = 2$ (Alpoge 2018).
  - When $\mathrm{rk} J(K) \leq g - 3$ (hyperelliptic by Stoll 2015, then Katz–Rabinoff–Zureick-Brown 2016).

Example of a 1-parameter family

Example (DGH 2019)

Let $s \geq 5$ be an integer and let C_s be the genus 2 hyperelliptic curve defined by

$$C_s : y^2 = x(x-1)(x-2)(x-3)(x-4)(x-s).$$

Then

$$\begin{aligned}\mathrm{rk}(J_s)(\mathbb{Q}) &\leq 2g\#\{p :\ p = 2 \text{ or } C_s \text{ has bad reduction at } p\} \\ &\leq 2g\#\{p : p \mid 2\cdot 3\cdot 5\cdot s(s-1)(s-2)(s-3)(s-4)\} \\ &\ll_g \frac{\log s}{\log\log s}.\end{aligned}$$

This yields, for any $\epsilon > 0$,

$$\#C_s(\mathbb{Q}) \ll_\epsilon s^\epsilon.$$

Genus ≥ 2: New Gap Principle

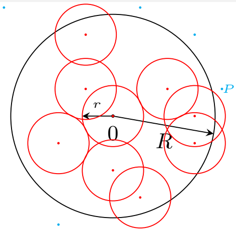
Our new contribution is a New Gap Principle.

Theorem (New Gap Principle, Dimitrov–G'–Habegger + Kühne, 2021)

Assume $g \geq 2$. Each $P \in C(\overline{\mathbb{Q}})$ satisfies

$$\#\{Q \in C(\overline{\mathbb{Q}}) : \hat{h}_L(Q-P) \leq c_1 h_{\mathrm{Fal}}(J)\} \leq c_2$$

for some positive constants c_1 and c_2 depending only on g.



$R^2 = c_0(g)h_{\mathrm{Fal}}(C)$
$r^2 = c_1(g)h_{\mathrm{Fal}}(C)$
small balls to cover all small points $\leq (R/r)^{\mathrm{rk}J(K)}$
of points in each ball $\leq c_2$

- The Bogomolov Conjecture, proved by Ullmo and S.Zhang ('98), gives this result with c_1 and c_2 depending on C (but don't know how).
- The New Gap Principle is another phenomenon of the "sparsity" of algebraic points in C of genus ≥ 2. It says that algebraic points in $C(\overline{\mathbb{Q}})$ are in general far from each other in a quantitative way.
- It implies that #small rational points $\leq c'(g)^{1+\mathrm{rk}J(K)}$ by a simple packing argument.
- Second proof by Yuan; uses Yuan–Zhang's adelic line bundle over quasi-proj var.

Genus ≥ 2

Different grades of the question:

- ➢ Finiteness of $C(K)$ ✓
- ➢ Upper bound of $\#C(K)$ ✓
- ➢ Uniformity of bounds of $\#C(K)$ ✓"subject" to the Mordell–Weil rank
- ➢ Effective Mordell

Sparsity of algebraic points:

- ➢ Mumford's Inequality -'65
- ➢ Vojta's Inequality -'91
- ➢ New Gap Principle -2021 (Dimitrov–G'–Habegger + Kühne)
- ➢ ???✎

And:

- ✎ Mumford's and Vojta's Inequalities to describe that large algebraic points are "sparse" in C.
- ✎ New Gap Principle gives another description on how all algebraic points are "sparse" in C.
- ✎ Effective Mordell is a conjectural statement which describes where to find the rational points ("no large rational points").

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✎ Effective Mordell is a conjectural statement which describes where to find the rational points ("no large rational points").

Genus ≥ 2: Effective Mordell

Conjecture (Effective Mordell, made by Szpiro)

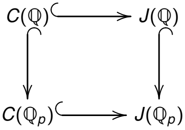
There exists an effectively computable $c = c(g, [K:\mathbb{Q}], \mathrm{disc}(K/\mathbb{Q})) > 0$ such that $\hat{h}(P) \leq c h_{\mathrm{Fal}}(J)$ for all C/K and $P \in C(K)$.

- Effective Mordell tells us where to find all the rational points on C ("no large rational points")!
- Little is known about Effective Mordell.
- Checcoli, Veneziano, and Viada proved results in this direction when $C \subseteq E^n$ for some elliptic curve E with $\mathrm{rk} E(K) < n$ (modification if E has CM) and C is *transverse*, following the method of Manin–Demjanenko.

Genus ≥ 2: Chabauty–Coleman–Kim method

✎ Another approach to compute $C(K)$ is the Chabauty–Coleman–Kim method, by obtaining sharp bounds on $\#C(K)$ when $\mathrm{rk}J(K)$ is small. Currently:

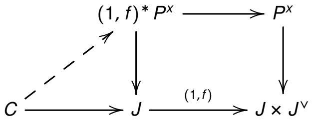
- ➢ Chabauty–Coleman: $K = \mathbb{Q}$, $\mathrm{rk}J(\mathbb{Q}) < g$.



$\dim \overline{J(\mathbb{Q})} \leq \mathrm{rk}J(\mathbb{Q}) < g \Rightarrow C(\mathbb{Q}) \subseteq C(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})}$ finite.

- ➢ Quadratic Chabauty: $\mathrm{rk}J(\mathbb{Q}) = g$, in various publications of Jennifer Balakrishnan in collaboration with Besser, Müller, Dogra *et al*.

A geometric point of view by Edixhoven–Lido:



$\Rightarrow C \hookrightarrow T$ with $T \to J$ a $\mathbb{G}_{\mathrm{m}}^{\rho-1}$-torsor, with $\rho = \mathrm{rkNS}(J)$.

Hence need $\mathrm{rk}J(\mathbb{Q}) < g + \rho - 1$.

the lifting exists $\Leftrightarrow \deg(1, f)^* P^x = 0$.

Proof of DGH: a tale of two heights

Theorem (New Gap Principle, Dimitrov–G'–Habegger + Kühne, 2021)

Assume $g \geq 2$. Each $P \in C(\overline{\mathbb{Q}})$ satisfies

$$\#\{Q \in C(\overline{\mathbb{Q}}) : \hat{h}_L(Q-P) \leq c_1 h_{\mathrm{Fal}}(J)\} \leq c_2$$

for some positive constants c_1 and c_2 depending only on g.

- $Q - P \in C - C \subseteq J$
- We are comparing:
  - $\hat{h}_L|_{C-C}$ height on J, and
  - $h_{\mathrm{Fal}}(J)$ height of J

Put all curves "together":

$\mathcal{C}_g$ universal curve

$\downarrow$

$\mathbb{M}_g$ moduli space of curves of genus g with level-4-structure

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$$\begin{array}{ccc} \mathcal{C}_g \times_{\mathbb{M}_g} \mathcal{C}_g & \xrightarrow{\mathcal{D}_1} & \mathrm{Jac}(\mathcal{C}_g/\mathbb{M}_g) \\ & \searrow & \downarrow \pi \\ & & \mathbb{M}_g \end{array} \qquad \begin{array}{c} X \subseteq \mathcal{A}_g \\ \downarrow \pi \\ \mathbb{A}_g \end{array}$$

- $\hat{h}$ fiberwise, and
- $h_{\mathrm{Fal}}(J)$ height on the base $\mathbb{M}_g$.
- Want to find the correct condition for X such that $\hat{h} \geq c h_{\mathrm{Fal}}$ when restricted on X for some constant c.

Proof of DGH: a tale of two heights

Theorem (GH 2019, DGH 2021)

The followings are equivalent:

(i) *There exists a Zariski open dense subset U of X, and a constant $c = c(X) > 0$ such that for all $x \in U(\overline{\mathbb{Q}})$,*

$$\hat{h}(x) \geq c h_{\mathrm{Fal}}(A_x) - c.$$

(ii) *X satisfies a linear algebra property, called non-degenerate.*

Non-degeneracy: Habegger 2013, GH 2019, DGH 2021. The definition uses Betti map (Masser–Zannier, Bertrand).

Proof of DGH: Non-degeneracy

- $\pi\colon \mathcal{A} \to S$ an abelian scheme

  $\downarrow$ taking Betti realization / forgetting complex structures of the fibers
- $\mathcal{T} \to S$ a local system of real torus ($\mathcal{T}_s = H_1(\mathcal{A}_s, \mathbb{R})/H_1(\mathcal{A}_s, \mathbb{Z})$)

  $\downarrow$ Betti foliation $\mathcal{F}$ on $\mathcal{A}$
- $T_x\mathcal{A} = T_x\mathcal{F} \bigoplus T_x\mathcal{A}_{\pi(x)}$ for each $x \in \mathcal{A}(\mathbb{C})$.

Definition

$X \subseteq \mathcal{A}$ is called non-degenerate if $T_xX \subseteq T_x\mathcal{A} \to T_x\mathcal{A}_{\pi(x)}$ has dimension $\dim X$ at some point $x \in X(\mathbb{C})$.

In the terminology of Yuan–Zhang 2021, non-degeneracy is equivalent to: the tautological adelic line bundle $\widetilde{\mathcal{L}}_g$ is big when restricted to X (DGH + YZ).

An immediate observation by definition: If $\dim X > g$, then X is degenerate! $\rightsquigarrow$ naive degenerate.
For example, $\mathcal{C}_g - \mathcal{C}_g = \mathcal{D}_1(\mathcal{C}_g \times_{\mathbb{M}_g} \mathcal{C}_g)$ is degenerate!

Proof of DGH: a tool (degeneracy loci) and bigness

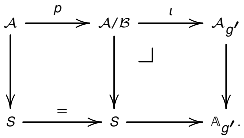
✎ (G' 2020) For each $t \in \mathbb{Z}$, one can define the *t-th degeneracy locus* $X^{\deg}(t)$ of X. ⇝Important tool to study these uniformity results.

As an application of mixed Ax–Schanuel (G') and $X^{\deg}(0)$, one proves:

Theorem (G' 2020, Betti rank)

TFAE:

- *X is degenerate, i.e. $\widetilde{\mathcal{L}}_g|_X$ is NOT big.*
- *$\exists$ abelian subscheme $\mathcal{B}$ of $\mathcal{A} \to S$ such that "a generic fiber of $\iota \circ p|_X$ is naive degenerate", i.e.* $\dim X - \dim(\iota \circ p)(X) > \dim \mathcal{B} - \dim S$.



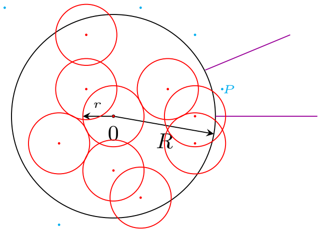
✎ Applications of this theorem and beyond:

- $X := \mathcal{D}_M(\mathcal{C}_g^{[M+1]})$ is non-degenerate if $M \geq 3g - 2$ (for DGH and K).
- the full Uniform Mordell–Lang Conjecture (G'–Ge–Kühne 2021).
- $X^{\deg}(1)$ for the Relative Manin–Mumford Conjecture (G'–Habegger 2023).

Genus ≥ 2: Some further questions related to the rather uniform bound of DGH+K

$\#C(K) \leq c_2(g)c(g)^{\mathrm{rk}J(K)}$

- ✎ How does $c_2(g)$ grow as $g \to \infty$ (Manin–Mumford constant)?
  - ➢ $c_2(g) \to \infty$ ($y^2 = x(x-1)\cdots(x-2024)$).
  - ➢ Over function fields: $\sim g^2$ by Looper–Silverman–Wilms 2022.
  - ➢ Over number fields: no explicit formula.



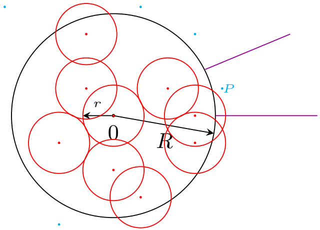
$R^2 = c_0(g)h_{\mathrm{Fal}}(C)$
$r^2 = c_1(g)h_{\mathrm{Fal}}(C)$
$\#$ small balls to cover all small points $\leq (R/r)^{\mathrm{rk}J(K)}$
$\#$ of points in each ball $\leq c_2$

- ✎ What if we confine ourselves to rational torsion points $\mathrm{TP}(C,P) := (C-P)(K) \cap J_{\mathrm{tor}}$?
  - ➢ Baker–Poonen 2001: $\#\mathrm{TP}(C,P) \leq 2$ for all but $B = B(C)$ points $P \in C(K)$.
  - ➢ Is it possible to make $B(C)$ uniform in g up to replacing 2 by 6?

Genus ≥ 2: Some further questions related to the rather uniform bound of DGH+K

$\#C(K) \leq c_2(g)c(g)^{\mathrm{rk}J(K)}$

- Is it true that $c(g) \to 1$ when $g \to \infty$, or at least give an absolute upper bound of $c(g)$ (Vojta constant)?
  - In view of Mumford's Formula
    $$\frac{1}{g}\left(|P|^2 + |Q|^2 - 2g\langle P, Q\rangle\right) + O(|P|+|Q|+1) \geq 0.$$
  - The angle condition in both inequalities can be improved.
  - A more precise version of Mumford's formula.



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small balls to cover all small points $\leq (R/r)^{\mathrm{rk}J(K)}$

of points in each ball $\leq c_2$

- Arithmetic Statistics: Average number of rational points.
  - Alpoge ('18): $K = \mathbb{Q}$ and $g = 2$, before the result of DGH.
  - Bhargava–Gross ('13): $K = \mathbb{Q}$, the average of $2^{\mathrm{rk}J(\mathbb{Q})}$ is a finite number for hyperelliptic curves having a rational Weierstrass point.

Beilinsin–Bloch height for Gross–Schoen / Ceresa cycles

- C smooth projective curve of genus $g \geq 3$;
- $J = \mathrm{Jac}(C)$;
- $\xi \in \mathrm{Pic}^1(C)$ such that $(2g-2)\xi = \omega_C$.

From these data, we obtain homologically trivial 1-cycles:

- (Gross–Schoen) $\Delta_{\mathrm{GS}}(C) \in \mathrm{Ch}_1(C^3)$ the modified diagonal;
- (Ceresa) $\mathrm{Ce}(C) := i_\xi(C) - [-1]^* i_\xi(C) \in \mathrm{Ch}_1(J)$.

Theorem (G'–S.Zhang, '24)

There exist positive constants ϵ, c and a Zariski open dense subset $\mathbb{M}_g^{\mathrm{amp}}$ of $\mathbb{M}_g$ defined over $\mathbb{Q}$ such that

$$\langle \Delta_{\mathrm{GS}}(C), \Delta_{\mathrm{GS}}(C) \rangle_{\mathrm{BB}} \geq \epsilon h_{\mathrm{Fal}}(C) - c$$
$$\langle \mathrm{Ce}(C), \mathrm{Ce}(C) \rangle_{\mathrm{BB}} \geq \epsilon h_{\mathrm{Fal}}(C) - c$$

for all $[C] \in \mathbb{M}_g^{\mathrm{amp}}(\overline{\mathbb{Q}})$.

Beilinsin–Bloch height for Gross–Schoen / Ceresa cycles

Corollary (Northcott property, G'–S.Zhang '24)

There exists a Zariski open dense subset $\mathbb{M}_g^{\mathrm{amp}}$ of $\mathbb{M}_g$ defined over $\mathbb{Q}$ such that for all $H, D \in \mathbb{R}$, we have

$$\#\{[C] \in \mathbb{M}_g^{\mathrm{amp}}(\overline{\mathbb{Q}}) : \quad \deg(\mathbb{Q}([C]) : \mathbb{Q}) < D, \quad \langle \Delta_{\mathrm{GS}}(C), \Delta_{\mathrm{GS}}(C) \rangle_{\mathrm{BB}} < H\} < \infty.$$

The definitions of the two cycles extends to any $e \in \mathrm{Pic}^1(C)$.

Corollary (Lower bound, G'–S.Zhang '24)

There exist a number c_g and a Zariski open dense subset $\mathbb{M}_g^{\mathrm{amp}}$ of $\mathbb{M}_{g,1}$ defined over $\mathbb{Q}$ such that

$$\langle \Delta_{\mathrm{GS}}(C), \Delta_{\mathrm{GS}}(C) \rangle_{\mathrm{BB}} \geq c_g$$

for all $[C] \in \mathbb{M}_g^{\mathrm{amp}}(\overline{\mathbb{Q}})$.

Same results for Ceresa cycles.

Lang–Silverman and UBC

Conjecture (Lang–Silverman)

Let $g \geq 1$ be an integer. For all number field K, there exist constants $c_1 = c_1(g, K)$, $c_2 = c_2(g, K)$, $c_3 = c_3(g, K)$ with the following property. For each abelian variety A of dimension g defined over K and each $P \in A(K)$, we have

(i) *Either P is contained in a proper abelian subvariety B of A with $\deg B \leq c_2 \deg A$ and $\mathrm{ord}(P)$ is $\leq c_3$ modulo B;*

(ii) *Or $\mathrm{End}(A) \cdot P$ is Zariski dense in A and*

$$\hat{h}(P) \geq c_1 \max\{h_{\mathrm{Fal}}(A), 1\}.$$

An immediate corollary of the Lang–Silverman Conjecture is the following widely open Uniform Boundedness Conjecture.

Conjecture (Uniform Boundedness Conjecture)

For each abelian variety A of dimension $g \geq 1$ defined over $\mathbb{Q}$, we have

$$\#A(\mathbb{Q})_{\mathrm{tor}} \leq B(g).$$

Thanks!