DEGENERACY LOCI IN THE UNIVERSAL FAMILY OF ABELIAN VARIETIES

ZIYANG GAO AND PHILIPP HABEGGER

Abstract. Recent developments on the uniformity of the number of rational points on curves and subvarieties in a moving abelian variety rely on the geometric concept of the degeneracy locus. The first-named author investigated the degeneracy locus in certain mixed Shimura varieties. In this expository note we revisit some of these results while minimizing the use of mixed Shimura varieties while working in a family of principally polarized abelian varieties. We also explain their relevance for applications in diophantine geometry.

Contents

1. Introduction

The goal of this expository note is to reprove some arguments in [Gao17a, Gao20a], especially regarding the degeneracy loci, *with a minimal use of the language of mixed Shimura varieties.*

With a view towards application, we will work in the following setup. Let $\mathfrak{A}_g \to \mathbb{A}_g$ be the universal family of principally polarized g-dimensional abelian varieties with level-ℓ-structure for some $\ell \geq 3$. Then $\mathfrak{A}_g$ carries the structure of a geometrically irreducible quasi-projective variety defined over a number field.

Let X be an irreducible closed subvariety of $\mathfrak{A}_g$. In [Gao20a], the first-named author defined the *t-th degeneracy locus* $X^{\deg}(t)$ for each $t \in \mathbb{Z}$; we refer to §4 for a definition in our setting. By definition, $X^{\deg}(t)$ is an at most countably infinite union of Zariski closed subsets of X. Yet $X^{\deg}(t)$ is Zariski closed in X, see [Gao20a, Theorem 1.8].

Date: February 13, 2023.

The definition of $X^{\deg}(t)$ involves *bi-algebraic subvarieties* of $\mathfrak{A}_g$ and $\mathbb{A}_g$; bi-algebraic subvarieties are explained in the beginning of §3.2. Ullmo and Yafaev characterized [UY11] bi-algebraic subvarieties of $\mathbb{A}_g$ as the *weakly special subvarieties* of $\mathbb{A}_g$, when we view $\mathbb{A}_g$ as a Shimura variety. The first-named author [Gao17b, Corollary 8.3] showed that the bi-algebraic subvarieties of $\mathfrak{A}_g$ are precisely the *weakly special* subvarieties, when we view $\mathfrak{A}_g$ as a mixed Shimura variety, see [Pin05, Definition 4.1(b)]. Then by some computation involving mixed Shimura varieties, a geometric characterization of bi-algebraic subvarieties of $\mathfrak{A}_g$ is given by [Gao17b, Proposition 1.1].

In the current paper we revisit the geometric description of a class of bi-algebraic subvarieties of $\mathfrak{A}_g$. This is done in Proposition 3.2. Instead of obtaining a full characterization, as in work of the first-named author, we prove a slightly weaker result which is sufficient for several applications in diophantine geometry.

A key tool in the proof of this proposition is already present in the first-named author's work [Gao17b, §8] as well as Bertrand's overview of Manin's Theorem of the Kernel [Ber20]. This tool is André's normality theorem for a variation of mixed Hodge structures [And92].

We follow the ideas of [Gao20a], in particular Theorem 8.1 *loc.cit.* and derive a necessary condition for $X^{\deg}(t)$ to be sufficiently large in X. The corresponding result is stated here in Proposition 4.3 and relies on the geometric structure result Proposition 3.2.

Two values of t are of particular interest for recent applications to diophantine geometry.

In §5 we emphasize the case $t = 0$. The zeroth degeneracy locus $X^{\deg}(0)$ is of crucial importance in the recent proof of the Uniform Mordell–Lang Conjecture [DGH21, Küh21, GGK21]. The mixed Ax–Schanuel Theorem [Gao20b] for the universal family of abelian varieties links the concept of non-degeneracy, in the sense of [DGH21, Definition 1.5], with the size of $X^{\deg}(0)$ in X. We refer to recent work of Blázquez-Sanz–Casale–Freitag–Nagloo [BSCFN23] for a differential algebraic approach to the Ax–Schanuel Theorem. More precisely, we replace $\mathfrak{A}_g$ by its m-fold fiber power $\mathfrak{A}_g^{[m]}$ over $\mathbb{A}_g$ for some $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and consider $\mathfrak{A}_g^{[m]} \subseteq \mathfrak{A}_{gm}$. For the Uniform Mordell–Lang Conjecture [DGH21] for curves of genus $g \geq 2$, the subvariety $X \subseteq \mathfrak{A}_g^{[m]}$ is the image under the *Faltings–Zhang morphism* of the $(m+1)$-fold fiber power of a suitable family of smooth projective curves of genus g. Corollary 5.3 yields a sufficiently strong statement to ensure that the relevant X arising in [DGH21] is non-degenerate. This corollary is a special case of [Gao20a, Theorem 1.3]. But the theorem on which it is based, Theorem 5.1, applies to more general subvarieties of $\mathfrak{A}_g^{[m]}$ and is new.

As explained in Remark 4.4, the mixed Ax–Schanuel Theorem for the universal family implies that if X fails to be non-degenerate, then $X^{\deg}(0)$ contains a non-empty open subset of X^{an}; here X^{an} means the analytification of X and the topology in consideration is the Euclidean topology. (This and a converse claim is contained in [Gao20a, Theorem 1.7].) Zariski closedness of $X^{\deg}(0)$ is not logically required in the context of [DGH21, Küh21, GGK21].

As observed in [Gao20a, §11], $X^{\deg}(1)$ is linked to the relative Manin–Mumford Conjecture; see §6.1 for a formulation of this conjecture and a brief history. Our Corollary 6.3 contains an application of the structure result, Proposition 3.2, to reduce the relative Manin–Mumford Conjecture for $\mathfrak{A}_g$ to Conjecture 6.1. This is in the spirit of [Gao20a, Proposition 11.2]. We plan to address Conjecture 6.1 in future work. In the current paper, we restrict ourselves to the family of principally polarized abelian varieties. Again we avoid the language of mixed Shimura varieties. However, our proof of Theorem 6.2, and so ultimately Corollary 6.3, requires the Zariski closedness of $X^{\deg}(1)$ in X, which was proved using the theory of mixed Shimura varieties [Gao20a, Theorem 1.8]. We do not reprove Zariski closedness in the current paper.

Section 2 below contains some preliminaries on abelian schemes in characteristic 0.

Acknowledgements. The authors would like to thank Gabriel Dill for discussions on abelian schemes and Lemma 2.1. ZG has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement n° 945714). PH has received funding from the Swiss National Science Foundation project n° 200020_184623.

2. Preliminaries on Abelian Schemes

Let S be a smooth irreducible quasi-projective variety defined over an algebraically closed subfield of $\mathbb{C}$. By abuse of notation we often consider our varieties as defined over $\mathbb{C}$. Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$. For $s \in S$ we write $\mathcal{A}_s$ for the abelian variety over $\mathbb{C}(s)$. More generally, for a morphism $T \to S$ of schemes we let $\mathcal{A}_T$ denote the fiber power $\mathcal{A} \times_S T$. Let $\eta \in S$ denote the generic point.

Let $\mathrm{End}(\mathcal{A}/S)$ denote the group of endomorphisms of the abelian scheme $\mathcal{A} \to S$. It is a finitely generated free abelian group. For all $s \in S$ let φ_s denote the restriction of $\varphi \in \mathrm{End}(\mathcal{A}/S)$ to $\mathcal{A}_s$. The associated group homomorphism homomorphism

$$\begin{aligned} \mathrm{End}(\mathcal{A}/S) &\to \mathrm{End}(\mathcal{A}_s/\mathbb{C}(s)), \\ \varphi &\mapsto \varphi_s \end{aligned} \tag{2.1}$$

is injective. As S is smooth, an endomorphism of the generic fiber $\mathcal{A}_\eta$ extends to an endomorphism of $\mathcal{A}$ over S by [FC90, Proposition I.2.7]. Therefore, (2.1) is bijective for $s = \eta$.

Observe that any $\varphi \in \mathrm{End}(\mathcal{A}/S)$ is a proper morphism. So the image $\varphi(\mathcal{A})$ is Zariski closed in $\mathcal{A}$. We will consider $\varphi(\mathcal{A})$ as a closed subscheme of $\mathcal{A}$ with the reduced induced structure. As $\mathcal{A}$ is reduced, $\varphi(\mathcal{A})$ is the schematic image of φ.

The following lemma on endomorphisms of $\mathcal{A}/S$ relies on a result of Barroero–Dill and ultimately on the theory of group schemes.

Lemma 2.1. *Let $\varphi \in \mathrm{End}(\mathcal{A}/S)$. Then $\varphi(\mathcal{A})$ is an abelian subscheme of $\mathcal{A}$. For all $s \in S(\mathbb{C})$ the restriction $\varphi_s\colon \mathcal{A}_s \to \mathcal{B}_s$ is surjective and its kernel has dimension $g - \dim \mathcal{B} + \dim S$.*

Proof. Let B be $\varphi(\mathcal{A}_\eta)$, this is an abelian subvariety of $\mathcal{A}_\eta$ defined over the function field $\mathbb{C}(\eta)$. By [BD22, Lemma 2.9] the abelian variety B is the generic fiber of an abelian subscheme $\mathcal{B} \subseteq \mathcal{A}$.

Then $\mathcal{A}_\eta$ is contain in the closed subset $\varphi^{-1}(\mathcal{B})$ of $\mathcal{A}$. As $\mathcal{A}_\eta$ lies dense in $\mathcal{A}$ we have $\varphi(\mathcal{A}) \subseteq \mathcal{B}$, set-theoretically. Furthermore, φ is proper and its image contains the dense subset B of $\mathcal{B}$. So $\varphi(\mathcal{A}) = \mathcal{B}$ as sets. But $\mathcal{A}$ and $\mathcal{B}$ are reduced, so $\mathcal{B}$ is the schematic image of φ. In particular, $\varphi(\mathcal{A})$ is an abelian subscheme of $\mathcal{A}$.

For all $s \in S(\mathbb{C})$ we have $\varphi(\mathcal{A}_s) = \mathcal{B}_s$ and this image has dimension $\dim \mathcal{B} - \dim S$ since $\mathcal{B} \to S$ is smooth. The lemma follows as the kernel of φ_s has dimension $\dim \mathcal{A}_s - \dim \mathcal{B}_s$. □

We will often treat $\varphi(\mathcal{A})$ as an abelian scheme over S and φ as the homomorphism $\mathcal{A} \to \varphi(\mathcal{A})$.

Let V be an irreducible variety defined over $\mathbb{C}$. A subset of $V(\mathbb{C})$ is called *meager* in V if it is contained in an at most countably infinite union of Zariski closed proper subsets of V.

We assume that all geometric endomorphisms of the generic fiber $\mathcal{A}_\eta$ are defined over the function field $\mathbb{C}(\eta)$ of S. This condition is met, for example, if there is an integer $\ell \geq 3$ such that all ℓ-torsion points of $\mathcal{A}_\eta$ are $\mathbb{C}(S)$-rational [Sil92, Theorem 2.4].

A point $s \in S(\mathbb{C})$ is called *endomorphism generic* for $\mathcal{A}/S$ if the homomorphism

$$\mathrm{End}(\mathcal{A}/S) \otimes \mathbb{Q} \to \mathrm{End}(\mathcal{A}_s/\mathbb{C}) \otimes \mathbb{Q} \tag{2.2}$$

induced by (2.1) is surjective. Note that (2.2) is always injective. We define

$$S^{\mathrm{exc}} := \left\{ s \in S(\mathbb{C}) : s \text{ is not endomorphism generic for } \mathcal{A}/S \right\}. \tag{2.3}$$

Proposition 2.2. *The set S^{exc} is meager in S.*

Proof. This proposition can be proved using Hodge theory. Masser [Mas96, Proposition] gave a proof using an effective Nullstellensatz. In this reference one must assume, as we do above, that all geometric endomorphisms of $\mathcal{A}_\eta$ are already defined over $\mathbb{C}(S)$. As a consequence any endomorphism of $\mathcal{A}_s$ for s outside a meager subset is the specialization of an endomorphism of the generic fiber. Then we use that (2.1) is surjective for $s = \eta$. □

A *coset* in an abelian variety is the translate of an abelian subvariety by an arbitrary point.

Lemma 2.3. *Let Y be an irreducible closed subvariety of $\mathcal{A}$ with $\pi(Y) = S$. Assume that there is a Zariski open and dense subset $U \subseteq \pi(Y)$ such that for all $s \in U(\mathbb{C})$, some irreducible component of Y_s is a coset in $\mathcal{A}_s$. There exists $\varphi \in \mathrm{End}(\mathcal{A}/S)$ with the following properties:*

(i) *We have* $\dim \varphi(Y) = \dim \pi(Y)$.

(ii) *For all $s \in S(\mathbb{C})$ we have* $\dim \ker \varphi_s = \dim Y - \dim \pi(Y)$.

Moreover, if $\dim Y = \dim \pi(Y)$, then φ is the identity.

Proof. If $\dim Y = \dim \pi(Y)$ we take φ to be the identity, the conclusions are all true. Otherwise by generic flatness, we may and do replace U by a Zariski open and dense subset such that Y_s is equidimensional of dimension $\dim Y - \dim \pi(Y)$ for all $s \in U(\mathbb{C})$.

Let $s \in U(\mathbb{C}) \setminus S^{\mathrm{exc}}$. We fix an irreducible component Z_s of Y_s that is a coset in $\mathcal{A}_s$, necessarily of dimension $\dim Y - \dim \pi(Y)$. Next we pick $\varphi_s \in \mathrm{End}(\mathcal{A}_s/\mathbb{C})$ whose kernel contains a translate of the said coset as an irreducible component. After multiplying φ_s by a positive multiple it extends to an endomorphism φ of $\mathcal{A}$ as (2.2) is bijective.

For each $\varphi \in \mathrm{End}(\mathcal{A}/S)$ we define Σ_φ to be the set of points $s \in U(\mathbb{C}) \setminus S^{\mathrm{exc}}$ with $\varphi_s = \varphi$. We have $S(\mathbb{C}) = (S \setminus U)(\mathbb{C}) \cup S^{\mathrm{exc}} \cup \bigcup_{\varphi \in \mathrm{End}(\mathcal{A}/S)} \Sigma_\varphi$

The set S^{exc} is meager in S by Proposition 2.2 and so is $(S \setminus U)(\mathbb{C}) \cup S^{\mathrm{exc}}$. As $\mathrm{End}(\mathcal{A}/S)$ is at most countably infinite, the Baire Category Theorem implies that there exists $\varphi \in \mathrm{End}(\mathcal{A}/S)$ such that the closure of Σ_φ in S^{an} has non-empty interior. In particular, Σ_φ is Zariski dense in S.

For all $s \in \Sigma_\varphi$ we have $\dim Z_s = \dim Y - \dim \pi(Y)$. So $Z = \bigcup_{s\in\Sigma_\varphi} Z_s$ lies Zariski dense in the irreducible Y.

For all $s \in \Sigma_\varphi$, each Z_s is contained in a fiber of $\varphi|_Y$ by our choice of φ_s. So Z_s, being an irreducible component of Y_s, is an irreducible component of a fiber of $\varphi|_Y$.

Generically, fibers of $\varphi|_Y$ are equidimensional of dimension $\dim Y - \dim \varphi(Y)$. So there exists $s_0 \in \Sigma_\varphi$ such that Z_{s_0} meets such a (generic) fiber. Then $\dim Y - \dim \varphi(Y) = \dim Z_{s_0} = \dim \ker \varphi_{s_0}$. Recall that Y_{s_0} is equidimensional of dimension $\dim Y - \dim \pi(Y)$ and has Z_{s_0} as an irreducible component. We conclude $\dim Y - \dim \varphi(Y) = \dim Y - \dim \pi(Y)$ and so $\dim \varphi(Y) = \dim \pi(Y)$. This implies (i). By Lemma 2.1 all $\ker \varphi_s$ have the same dimension, here equal to $\dim Y - \dim \pi(Y)$. This concludes (ii). $\square$

The exceptional set of an irreducible closed subvariety Y of $\mathcal{A}$ is defined to be

$$Y^{\mathrm{exc}} := \{P \in Y(\mathbb{C}) : P \text{ is contained in a proper algebraic subgroup of } \mathcal{A}_{\pi(P)}\}. \tag{2.4}$$

If $N \in \mathbb{Z}$ then $[N]$ denotes the multiplication-by-N morphism $\mathcal{A} \to \mathcal{A}$.

Lemma 2.4. *Let Y be an irreducible closed subvariety of $\mathcal{A}$ and let $S' = \pi(Y)^{\mathrm{reg}}$ denote the regular locus of $\pi(Y)$. We have one of the following two alternatives.*

(i) *Either Y^{exc} is meager in Y,*
(ii) *or every $P \in \pi|_Y^{-1}(S')(\mathbb{C})$ lies in a proper algebraic subgroup of $\mathcal{A}_{\pi(P)}$. In this case, $\bigcup_{N\in\mathbb{N}}[N](Y)$ is not Zariski dense in $\pi^{-1}(\pi(Y))$ and if η is the generic point of $\pi(Y)$, then Y_η lies in a proper algebraic subgroup of $\mathcal{A}_\eta$.*

Proof. Let $Y' = Y \cap \mathcal{A}_{S'}$. Suppose $P \in Y'(\mathbb{C})$ is in a proper algebraic subgroup of $\mathcal{A}_s$ with $s = \pi(P)$. Then there exists $\varphi_s \in \mathrm{End}(\mathcal{A}_s/\mathbb{C}) \setminus \{0\}$ with $\varphi_s(P) = 0$. If $s \notin S'^{\mathrm{exc}}$, then by definition some positive multiple of φ_s extends to an element of $\mathrm{End}(\mathcal{A}_{S'}/S') \setminus \{0\}$. Therefore,

$$Y^{\mathrm{exc}} \subseteq \pi|_Y^{-1}(\pi(Y) \setminus S') \cup \pi^{-1}(S'^{\mathrm{exc}}) \cup \bigcup_{\varphi\in\mathrm{End}(\mathcal{A}_{S'}/S')\setminus\{0\}} \ker\varphi. \tag{2.5}$$

By Proposition 2.2 the set $\pi|_Y^{-1}(S'^{\mathrm{exc}})$ is meager in Y. Moreover, $\pi|_Y^{-1}(\pi(Y) \setminus S')$ is Zariski closed and proper in Y, and hence its complex points form a meager subset of Y. Moreover, the last union in (2.5) is over an at most countably infinite union of proper algebraic subsets of $\mathcal{A}_{S'}$.

So if we are not in alternative (i), then there exists $\varphi \in \mathrm{End}(\mathcal{A}_{S'}/S') \setminus \{0\}$ with $Y \subseteq \overline{\ker\varphi}$, the Zariski closure of $\ker\varphi$ in $\mathcal{A}$. Note that $Y' = Y \cap \mathcal{A}_{S'} \subseteq \overline{\ker\varphi} \cap \mathcal{A}_{S'} = \ker\varphi$. Say $P \in Y'(\mathbb{C})$, then $P \in \ker\varphi_{\pi(P)}$. By Lemma 2.1, $\ker\varphi_{\pi(P)}$ is a proper algebraic subgroup of $\mathcal{A}_{\pi(P)}$. Finally, $[N](Y') \subseteq \ker\varphi$ for all $N \in \mathbb{N}$. So $\bigcup_{N\in\mathbb{N}}[N](Y)$ lies in $\pi|_Y^{-1}(\pi(Y) \setminus S') \cup \ker\varphi$ and is thus not Zariski dense in $\pi^{-1}(\pi(Y))$. Finally, the generic fiber of $Y \to S$ lies in the generic fiber of $\ker\varphi \to S$, the latter is a proper algebraic subgroup of $\mathcal{A}_\eta$. $\square$

Here is a useful consequence of the previous lemma.

Lemma 2.5. *Let $Y \subseteq \mathcal{A}$ and $X \subseteq \mathcal{A}$ be irreducible closed subvarieties with $Y \subseteq X$ and $\pi(Y) \cap \pi(X)^{\mathrm{reg}} \neq \emptyset$. If Y^{exc} is meager in Y, then X^{exc} is meager in X.*

Proof. If X^{exc} is not meager in X, then by Lemma 2.4 every point $P \in X(\mathbb{C})$ with $\pi(P) \in \pi(X)^{\mathrm{reg}}$ lies in a proper algebraic subgroup of $\mathcal{A}_{\pi(P)}$. In particular, the complex points of $Y \cap \pi^{-1}(\pi(X)^{\mathrm{reg}})$ lie in Y^{exc}. The hypothesis implies that Y^{exc} contains a non-empty open subset of Y^{an}. So Y^{exc} is not meager in Y by the Baire Category Theorem. $\square$

3. Bi-algebraic Subvarieties and the University Family of Abelian Varieties

Ullmo and Yafaev [UY11] characterized the bi-algebraic subvarieties of (pure) Shimura varieties: they are precisely the weakly special subvarieties, *i.e.*, the geodesic subvarieties studied by Moonen [Moo98]. For a definition of bi-algebraic subgroup we refer to §3.2 below. The first-named author [Gao17a, §3] gave a complete characterization of the bi-algebraic subvarieties of $\mathfrak{A}_g$, based on [Gao17b, §8]. Below in Proposition 3.2 we follow the approach presented in these references but minimize the language of mixed Shimura varieties. Our main tool is André's normality theorem [And92] for variations of mixed Hodge structures.

3.1. **The Mumford–Tate Group.** Let $g \geq 1$ and let $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$ be the universal family of principally polarized g-dimensional abelian varieties with level-ℓ-structure for some $\ell \geq 3$. Then $\mathfrak{A}_g$ and $\mathbb{A}_g$ are geometrically irreducible, smooth quasi-projective varieties defined over a number field which we assume is a subfield $\mathbb{C}$. We consider all varieties as defined over a subfield of $\mathbb{C}$, sometimes executing a base change to $\mathbb{C}$ without mention.

Let $\mathfrak{H}_g$ denote Siegel's upper half space, *i.e.*, the symmetric matrices in $\mathrm{Mat}_{g\times g}(\mathbb{C})$ with positive definite imaginary part. By abuse of notation we write

$$\mathrm{unif}\colon \mathfrak{H}_g \to \mathbb{A}_g^{\mathrm{an}} \quad \text{and} \quad \mathrm{unif}\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{A}_g^{\mathrm{an}}$$

for both holomorphic uniformizing maps. Recall that $\mathrm{Sp}_{2g}(\mathbb{R})$, the group of real points of the symplectic group, acts on $\mathfrak{H}_g$.

We identify $\mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{H}_g$ with $\mathbb{C}^g \times \mathfrak{H}_g$ via the natural semi-algebraic bijection

$$(\tau, u, v) \leftrightarrow (\tau, z) \quad \text{where } z = \tau u + v. \tag{3.1}$$

In the former coordinates, the corresponding uniformizing map $\mathfrak{H}_g \times \mathbb{R}^{2g} \to \mathfrak{A}_g^{\mathrm{an}}$ is real analytic.

Let $s \in \mathfrak{A}_g(\mathbb{C})$ and fix $\tau \in \mathfrak{H}_g$ in its preimage under the uniformizing map, *i.e*, $s = \mathrm{unif}(\tau)$. Let 1_g denote the $g \times g$ unit matrix, then the columns of $(\tau, 1_g)$ are an $\mathbb{R}$-basis of $\mathbb{C}^g$ and $\mathfrak{A}_{g,s}^{\mathrm{an}} \cong \mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g)$. The period lattice basis $(\tau, 1_g)$ allows us to identify $H_1(\mathfrak{A}_{g,s}^{\mathrm{an}}, \mathbb{Z})$ with $\mathbb{Z}^g \times \mathbb{Z}^g$ and $H_1(\mathfrak{A}_{g,s}^{\mathrm{an}}, \mathbb{R})$ with $\mathbb{R}^g \times \mathbb{R}^g$.

We briefly recall the monodromy action of $\pi_1(\mathbb{A}_g^{\mathrm{an}}, s)$, the (topological) fundamental group of $\mathbb{A}_g^{\mathrm{an}}$ based at s, on singular homology $H_1(\mathfrak{A}_{g,s}^{\mathrm{an}}, \mathbb{Z})$.

Suppose $[\gamma] \in \pi_1(\mathbb{A}_g^{\mathrm{an}}, s)$ is represented by a loop γ in $\mathbb{A}_g^{\mathrm{an}}$ based at s. Then a lift $\tilde{\gamma}$ of γ to $\mathfrak{H}_g$ starting at τ ends at $M\tau \in \mathfrak{H}_g$ for some $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$. Then $M\tau$ is the period matrix of the abelian variety $\mathbb{C}^g/(M\tau\mathbb{Z}^g + \mathbb{Z}^g)$ which is isomorphic to $\mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g)$. To describe this isomorphism we need the identity

$$I(M,\tau)^{\top}(M\tau, 1_g) = (\tau, 1_g)M^{\top}, \tag{3.2}$$

where $I(M,\tau) = c\tau + d$, note the transpose and see [BL04, §8.1 and Remark 8.1.4] for a discussion. We rearrange this equation. The map

$$\tau u + v \mapsto (I(M,\tau)^\top)^{-1}(\tau u + v) = (M\tau, 1_g)(M^\top)^{-1}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.3}$$

here $u, v \in \mathbb{R}^g$ are column vectors, induces the isomorphism $\mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g) \to \mathbb{C}^g/(M\tau\mathbb{Z}^g + \mathbb{Z}^g)$.

By (3.3), the monodromy representation expressed in these coordinates is given by

$$\begin{aligned} \rho\colon \pi_1(\mathbb{A}_g^{\mathrm{an}}, s) &\to \mathrm{Sp}_{2g}(\mathbb{Z}) \\ [\gamma] &\mapsto (M^\top)^{-1}. \end{aligned} \tag{3.4}$$

Next we recall the definition of the Mumford–Tate group in our context.

We continue to assume $\tau \in \mathfrak{H}_g$. Choose any $M \in \mathrm{Sp}_{2g}(\mathbb{R})$ with $\tau = M(\sqrt{-1}\cdot 1_g)$; such an M exists as $\mathrm{Sp}_{2g}(\mathbb{R})$ acts transitively on $\mathfrak{H}_g$. We set

$$J_\tau = (M^\top)^{-1}\Omega M^\top \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}. \tag{3.5}$$

We claim that J_τ is independent of the choice of M; it depends only on τ. Indeed, if M' is a further element of $\mathrm{Sp}_{2g}(\mathbb{R})$ with $\tau = M'(\sqrt{-1}\cdot 1_g)$, then $M = M'N$ where $N \in \mathrm{Sp}_{2g}(\mathbb{R})$ stabilizes $\sqrt{-1}\cdot 1_g$. So N is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where $a, b \in \mathrm{Mat}_g(\mathbb{R})$. This implies $(N^\top)^{-1}\Omega N^\top = \Omega$ and so $(M'^\top)^{-1}\Omega M'^\top = J_\tau$ on substituting $M' = MN^{-1}$.

Say $x, y \in \mathbb{R}$, then

$$(x1_{2g} + yJ_\tau)^\top\Omega(x1_{2g} + yJ_\tau) = x^2\Omega + y^2 J_\tau^\top\Omega J_\tau + xy(\Omega J_\tau + J_\tau^\top\Omega).$$

The group $\mathrm{Sp}_{2g}(\mathbb{R})$ contains Ω and is mapped to itself by matrix transposition. Hence $J_\tau \in \mathrm{Sp}_{2g}(\mathbb{R})$. Moreover, $J_\tau^2 = -1_{2g}$. So $J_\tau^\top\Omega J_\tau = \Omega$ and $J_\tau^\top\Omega = \Omega J_\tau^{-1} = -\Omega J_\tau$. We conclude $h_\tau(z) = x1_{2g} + yJ_\tau \in \mathrm{GSp}_{2g}(\mathbb{R})$ for all $z = x + \sqrt{-1}y \in \mathbb{C}^\times = \mathbb{C}\setminus\{0\}$ where $x, y \in \mathbb{R}$. Moreover,

$$h_\tau\colon \mathbb{C}^\times \to \mathrm{GSp}_{2g}(\mathbb{R})$$

is a group homomorphism.

By (3.5) we have $J_\tau^\top\tau = \tau$. Below we use the well-known identity $I(MM',\tau) = I(M, M'\tau)I(M',\tau)$ for all $M, M' \in \mathrm{Sp}_{2g}(\mathbb{R})$ and all $\tau \in \mathfrak{H}_g$. We apply (3.2) to $J_\tau^\top$ where $J_\tau = (M^\top)^{-1}\Omega M^\top$ and $\tau = M(\sqrt{-1}\cdot 1_g)$ and compute

$$\begin{aligned} (\tau, 1_g)J_\tau &= I(J_\tau^\top, \tau)^\top(\tau, 1_g) \\ &= I(-M\Omega M^{-1}, \tau)^\top(\tau, 1_g) \\ &= \left(I(-M\Omega, M^{-1}\tau)I(M^{-1},\tau)\right)^\top(\tau, 1_g) \\ &= -\left(I(M\Omega, \sqrt{-1}\cdot 1_g)I(M^{-1},\tau)\right)^\top(\tau, 1_g) \\ &= -\left(I(M, \Omega(\sqrt{-1}\cdot 1_g))I(\Omega, \sqrt{-1}\cdot 1_g)I(M^{-1},\tau)\right)^\top(\tau, 1_g). \end{aligned}$$

Next we use $\Omega(\sqrt{-1}\cdot 1_g)=\sqrt{-1}\cdot 1_g$. Hence

$$
\begin{aligned}
(\tau,1_g)J_\tau &= \sqrt{-1}\left(I(M,\sqrt{-1}\cdot 1_g)I(M^{-1},\tau)\right)^\top(\tau,1_g)\\
&= \sqrt{-1}\left(I(M,M^{-1}\tau)I(M^{-1},\tau)\right)^\top(\tau,1_g)\\
&= \sqrt{-1}I(1_{2g},\tau)^\top(\tau,1_g)\\
&= \sqrt{-1}(\tau,1_g).
\end{aligned}
\tag{3.6}
$$

So J_τ represents multiplication by $\sqrt{-1}$ in the real coordinates determined by the $\mathbb{R}$-basis $(\tau,1_g)$ of $\mathbb{C}^g$.

Let $s\in\mathbb{A}_g(\mathbb{C})$ lie below $\tau\in\mathfrak{H}_g$. The *Mumford–Tate group* $\mathrm{MT}(\mathfrak{A}_{g,s})$ of $\mathfrak{A}_{g,s}$ is the smallest algebraic subgroup of $\mathrm{GSp}_{2g,\mathbb{Q}}$ whose group of $\mathbb{R}$-points contains $h_\tau(\mathbb{C}^\times)$. As $J_\tau=h_\tau(\sqrt{-1})$ we certainly have $J_\tau\in\mathrm{MT}(\mathfrak{A}_{g,s})(\mathbb{R})$.

3.2. **Bi-algebraic Subvarieties.** We keep the conventions introduced in the beginning of §3.1. An irreducible closed subvariety $Y\subseteq\mathfrak{A}_g$ is called *bi-algebraic*, if some (or equivalently any) complex analytic irreducible component of $\mathrm{unif}^{-1}(Y^{\mathrm{an}})$ equals an irreducible component of $\tilde{Y}(\mathbb{C})\cap(\mathbb{C}^g\times\mathfrak{H}_g)$ for an algebraic subset $\tilde{Y}\subseteq\mathbb{G}_{\mathrm{a},\mathbb{C}}^g\times\mathrm{Mat}_{g\times g,\mathbb{C}}$. All irreducible components of the intersection of 2 bi-algebraic subvarieties of $\mathfrak{A}_g$ are bi-algebraic. So any irreducible closed subvariety Y of $\mathfrak{A}_g$ is contained in a bi-algebraic subvariety Y^{biZar} of $\mathfrak{A}_g$ that is minimal with respect to inclusion.

Bi-algebraic subvarieties of $\mathbb{A}_g$ are defined in a similar manner. By a theorem of Ullmo–Yafaev [UY11, Theorem 1.2], the bi-algebraic subvarieties of $\mathbb{A}_g$ are precisely the weakly special subvarieties of $\mathbb{A}_g$; here we consider $\mathbb{A}_g$ as a Shimura variety. For any irreducible closed subvariety Y of $\mathbb{A}_g$, we use Y^{biZar} to denote the minimal bi-algebraic subvariety containing Y.

Lemma 3.1. *Let $Y\subseteq\mathfrak{A}_g$ be an irreducible closed subvariety that is bi-algebraic and let $\eta\in\pi(Y)$ be the generic point. For all $P\in Y(\mathbb{C})$, each irreducible component of $Y_{\pi(P)}$ is a coset in $\mathfrak{A}_{g,\pi(P)}$ of dimension at least $\dim Y-\dim\pi(Y)$.*

Proof. Let $P\in Y^{\mathrm{biZar}}(\mathbb{C})$ and let C be an irreducible component of $Y^{\mathrm{biZar}}_{\pi(P)}$. By [GW10, Corollary 14.116 and Remark 14.117] we have $\dim C\geq\dim Y-\dim\pi(Y)$.

The irreducible component C of $Y_{\pi(P)}$ is a bi-algebraic subset of the abelian variety $\mathfrak{A}_{g,\pi(P)}$. The lemma follows as by [UY11, Proposition 5.1], C is a coset in the ambient abelian variety. □

We now come to a structural result of bi-algebraic subsets. We refer to the first-named author's more comprehensive result in [Gao17a] (the statement of [Gao17a, Proposition 3.3] contains a mistake; for a correct version see [Gao20a, Proposition 5.3]) using the language of mixed Shimura varieties.

Proposition 3.2. *Let $Y\subseteq\mathfrak{A}_g$ be a bi-algebraic subvariety with Y^{exc} meager in Y.*

(i) *There is a vector space $W\subseteq\mathbb{R}^{2g}$ defined over $\mathbb{Q}$ with $\dim W=2(\dim Y-\dim\pi(Y))$ with the following property. For all $s=\mathrm{unif}(\tau)\in\pi(Y)(\mathbb{C})$, with $\tau\in\mathfrak{H}_g$, the fiber Y_s is a finite union of translates of $\mathrm{unif}(\{\tau\}\times W)\subseteq\mathfrak{A}_{g,s}^{\mathrm{an}}$ which is an abelian variety C_s*

(ii) *The quotient abelian varieties* $\mathfrak{A}_{g,s}/C_s$ *are pairwise isomorphic abelian varieties for all* $s \in \pi(Y)(\mathbb{C})$.

Each $\tau \in \mathfrak{H}_g$ endows $\mathbb{R}^{2g}$ with the structure of a $\mathbb{C}$-vector space, multiplication by $\sqrt{-1}$ is represented by J_τ from (3.5). The subspace $W \subseteq \mathbb{R}^{2g}$ from part (i) is a $\mathbb{C}$-vector space for all τ in question. The image $\mathrm{unif}(\{\tau\} \times W)$ is an abelian subvariety of $\mathfrak{A}_{g,s}$ of dimension $\dim Y - \dim \pi(Y)$. In particular, $Y \to \pi(Y)$ is equidimensional.

Proof. If $\pi(Y)$ is a point, say $s \in \mathbb{A}_g(\mathbb{C})$, then Y_s is a coset in $\mathfrak{A}_{g,s}$ by Lemma 3.1. The proposition holds in this case.

We will now assume $\dim \pi(Y) \geq 1$. We identify $\mathbb{R}^{2g} \times \mathfrak{H}_g$ with the universal covering of $\mathfrak{A}_g(\mathbb{C})$; sometimes alluding to the complex structure induced by (3.1). The fundamental group of $\mathfrak{A}_g^{\mathrm{an}}$ based at some point P is a subgroup of $\mathbb{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbb{Z})$. The element (M, ω) acts by

$$(M\tau, M * u + \omega)$$

on (τ, u); where $M * u = (M^\top)^{-1}u$.

Recall that the ambient variety $\mathfrak{A}_g$ is quasi-projective and so is Y. By Bertini's Theorem a general linear space of codimension $\dim Y - 1$ intersected with Y^{reg} is a smooth, irreducible curve $\mathbf{x}$ that is quasi-finite over $\pi(\mathbf{x})$. A suitable version of Lefschetz's Theorem for the topological fundamental group we may also assume that the homomorphism

$$\pi_1(\mathbf{x}^{\mathrm{an}}, P) \to \pi_1(Y^{\mathrm{reg},\mathrm{an}}, P) \tag{3.7}$$

induced by the inclusion $\mathbf{x} \to Y^{\mathrm{reg,an}}$ is surjective for all $P \in \mathbf{x}(\mathbb{C})$; see [Del81, Lemme 1.4]. We may fix P in very general position. For example, P is not contained in a proper algebraic subgroup of $\mathfrak{A}_{g,s}$ for $s = \pi(P)$. If we replace $\mathbf{x}$ by a Zariski open and dense subset, the image of the induced homomorphism has finite index in $\pi_1(Y^{\mathrm{reg,an}}, P)$, [Del71, Lemme 4.4.17]. This suffices for us. So we may assume that $\pi|_{\mathbf{x}} \colon \mathbf{x} \to \pi(\mathbf{x})$ is finite and étale.

Let Γ denote the image of $\pi_1(\mathbf{x}^{\mathrm{an}}, P)$ in $\mathbb{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbb{Z})$. Let $\mathrm{Mon}(\mathbf{x})$ be the neutral component of the Zariski closure of Γ in $\mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g} \rtimes \mathrm{Sp}_{2g,\mathbb{Q}}$ and let $\mathrm{Mon}(Y^{\mathrm{reg}})$ be the neutral component of the Zariski closure of the image of $\pi_1(Y^{\mathrm{reg,an}}, P)$. We call $\mathrm{Mon}(\mathbf{x})$ the *connected algebraic monodromy group* of $\mathbf{x}$. By the surjectivity of (3.7) and the discussion below we have

$$\mathrm{Mon}(\mathbf{x}) = \mathrm{Mon}(Y^{\mathrm{reg}}). \tag{3.8}$$

By Lemma 3.1 we have $P \in C(\mathbb{C})$ where C is an irreducible component of Y_s and a coset in $\mathfrak{A}_{g,s}$ with $\dim C \geq \dim Y - \dim \pi(Y)$. Now $C \cap Y^{\mathrm{reg}}$ is Zariski dense and open in C. So the image of $\pi_1(C^{\mathrm{an}} \cap Y^{\mathrm{reg,an}}, P)$ in $\pi_1(C^{\mathrm{an}}, P)$, induced by inclusion, has finite index. But $\pi_1(C^{\mathrm{an}}, P)$ can be identified with a subgroup of $\mathbb{Z}^{2g} \cong H_1(\mathfrak{A}_{g,s}^{\mathrm{an}}, \mathbb{Z})$ of rank $2\dim C \geq 2(\dim Y - \dim \pi(Y))$.

The kernel of the projection $\mathrm{pr}\colon \mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g} \rtimes \mathrm{Sp}_{2g,\mathbb{Q}} \to \mathrm{Sp}_{2g,\mathbb{Q}}$ restricted to $\mathrm{Mon}(Y^{\mathrm{reg}})$ is an algebraic subgroup of $\mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g} \times \{1_{2g}\}$. So it is $\{1_{2g}\} \times W$ with W a linear subspace of $\mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g}$. By the previous paragraph and by (3.8) we have

$$\dim W \geq 2(\dim Y - \dim \pi(Y)). \tag{3.9}$$

Let $Z = \pi(Y)^{\mathrm{reg}}$. Then there is a natural representation $\pi_1(Z^{\mathrm{an}}, s) \to \mathrm{Sp}_{2g}(\mathbb{Z})$. The connected algebraic monodromy group $\mathrm{Mon}(Z) \subseteq \mathrm{Sp}_{2g,\mathbb{Q}}$ is the neutral component of the Zariski closure of the image of $\pi_1(Z^{\mathrm{an}}, s)$.

Note that $\mathrm{pr}(\mathrm{Mon}(Y^{\mathrm{reg}})) = \mathrm{Mon}(Z)$ by [Del71, Lemme 4.4.17].

Let $M \in \mathrm{Mon}(Z)(\mathbb{C})$. The preimage $\mathrm{pr}|_{\mathrm{Mon}(Y^{\mathrm{reg}})}^{-1}(M)$ is $(M, \psi(M) + W)$ where $\psi(M)$ is a unique complex point of $\mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g}/W$. For all $M, M' \in \mathrm{Mon}(Z)(\mathbb{C})$ we have

$$(M, \psi(M)+W)(M', \psi(M')+W) = (MM', \psi(M)+M*\psi(M')+W) = (MM', \psi(MM')+W).$$

So $\psi\colon \mathrm{Mon}(Z) \to \mathbb{C}^{2g}/W(\mathbb{C})$ is a cocycle. It must be a coboundary as $\mathrm{Mon}(Z)$ is semisimple or trivial, by work of Deligne [Del71, Corollaire 4.2.9(a)]. Hence there exists $v_0 \in \mathbb{C}^{2g}$ with

$$\psi(M) = M * v_0 - v_0 + W \quad \text{for all} \quad M \in \mathrm{Mon}(Z)(\mathbb{C}).$$

Let $\tilde{Y}$ be an algebraic subset of $\mathbb{G}_{\mathrm{a},\mathbb{C}}^{g} \times \mathrm{Mat}_{g\times g,\mathbb{C}}$ such that the preimage of $Y(\mathbb{C})$ in $\mathbb{C}^g \times \mathfrak{H}_g$ and $\tilde{Y}(\mathbb{C}) \cap (\mathbb{C}^g \times \mathfrak{H}_g)$ have a common complex analytic irreducible component, say $\tilde{Y}_0$. We have $\dim \tilde{Y}_0 = \dim Y$.

Suppose $\tau_0 \in \mathfrak{H}_g$ lies above s and $(\tau_0, u_0) \in \tilde{Y}_0$ lies above P. We return to real coordinates, so $u_0 \in \mathbb{R}^{2g}$.

An element $[\gamma] \in \Gamma$ is represented by a loop γ in $\mathbf{x}^{\mathrm{an}}$ based at P. We lift γ to an arc in $\mathbb{R}^{2g} \times \mathfrak{H}_g$ starting at $(\tau_0, u_0) \in \tilde{Y}(\mathbb{C})$. In particular, the end point of the lift lies in $\tilde{Y}_0$ and equals $(M, u)(\tau_0, u_0)$ with $(M, u) = [\gamma] \in \Gamma$. For the orbit of (τ_0, u_0) under Γ we have

$$\Gamma \cdot (\tau_0, u_0) \subseteq \tilde{Y}_0 \subseteq \tilde{Y}(\mathbb{C}).$$

By definition $\tilde{Y}(\mathbb{C})$ is an algebraic subset of $\mathrm{Mat}_{g\times g}(\mathbb{C}) \times \mathbb{C}^g$. As P is a regular point of Y we have that (τ_0, u_0) is a regular point of $\tilde{Y}_0$. So

$$\mathrm{Mon}(Y^{\mathrm{reg}})(\mathbb{R})^+ \cdot (\tau_0, u_0) = \mathrm{Mon}(\mathbf{x})(\mathbb{R})^+ \cdot (\tau_0, u_0) = \{(M, u)(\tau_0, u_0) : (M, u) \in \mathrm{Mon}(\mathbf{x})(\mathbb{R})^+\} \subseteq \tilde{Y}_0; \tag{3.10}$$

the superscript $+$ signals taking the neutral component in the Euclidean topology.

In addition to the connected algebraic monodromy group, we have the corresponding Mumford–Tate group.

First, recall that $s \in \pi(Y)(\mathbb{C})$ determines a principally polarized abelian variety $\mathfrak{A}_{g,s}$ defined over $\mathbb{C}$. We consider its Mumford–Tate group $\mathrm{MT}(\mathfrak{A}_{g,s})$ coming from the corresponding weight -1 pure Hodge structure; it is a reductive algebraic group. Moreover, $\mathrm{MT}(\mathfrak{A}_{g,s})$ is naturally an algebraic subgroup of $\mathrm{GSp}_{2g,\mathbb{Q}}$.

Second, after a finite and étale base change, which is harmless for our investigations, $\mathbf{x}$ becomes a section of an abelian scheme. It is a good, smooth one-motive (of rank ≤ 1) in the sense of Deligne; see [And92, §4 and Lemma 5]. Attached to $\mathbf{x}$ is a variation of mixed Hodge structures. Restricted to each point of $\mathbf{x}$ we obtain a mixed Hodge structure. By [And92, §4 and Lemma 5] the mixed Hodge structure thus obtained has the same the Mumford–Tate group for all sufficiently general points in $\mathbf{x}$. We denote this group by $\mathrm{MT}(\mathbf{x})$. We may assume P to be such a very general point. Then $\mathrm{MT}(\mathbf{x})$ is naturally an algebraic subgroup of $\mathbb{G}_{\mathrm{a}}^{2g} \rtimes \mathrm{GSp}_{2g,\mathbb{Q}}$.

We also write pr for the projection $\mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g} \rtimes \mathrm{GSp}_{2g,\mathbb{Q}} \to \mathrm{GSp}_{2g,\mathbb{Q}}$. By [And92, Lemma 2(c)] we have surjectivity $\mathrm{pr}(\mathrm{MT}(\mathbf{x})) = \mathrm{MT}(\mathfrak{A}_{g,s})$.

André [And92, Theorem 1] proves that $\mathrm{Mon}(\mathbf{x})$ is a normal subgroup of $\mathrm{MT}(\mathbf{x})$ as P is in very general position. We do not require the statement that $\mathrm{Mon}(\mathbf{x})$ is in fact a normal subgroup of the derived Mumford–Tate group.

Before moving on, we make the following remark. The second remark on [And92, page 11] suggests that we could work directly with Y^{reg}, *i.e.*, without bypassing to $\mathbf{x}$.

Let $M \in \mathrm{Mon}(Z)(\mathbb{C})$ be arbitrary. Then $(M, M*v_0 - v_0 + W) \subseteq \mathrm{Mon}(\mathbf{x})(\mathbb{C})$. Suppose $(h, v) \in \mathrm{MT}(\mathbf{x})(\mathbb{C})$. By André's Theorem, we have

$$(h,v)(M, M*v_0 - v_0 + W)(h,v)^{-1} \subseteq \mathrm{Mon}(\mathbf{x})(\mathbb{C}),$$

so

$$(hMh^{-1}, v + hM*v_0 - h*v_0 - hMh^{-1}*v + h*W) \subseteq \mathrm{Mon}(\mathbf{x})(\mathbb{C}).$$

Recall (3.8). The second coordinate equals $\psi(M) = hMh^{-1}*v_0 - v_0 + W$, so

$$v + hM*v_0 - h*v_0 - hMh^{-1}*v + h*W = hMh^{-1}*v_0 - v_0 + W.$$

We draw two conclusions.

First, we have $h*W = W$. As the projection $\mathrm{MT}(\mathbf{x}) \to \mathrm{MT}(\mathfrak{A}_{g,s})$ is surjective, it follows that $\mathrm{MT}(\mathfrak{A}_{g,s})$ acts on W. The reductive group $\mathrm{MT}(\mathfrak{A}_{g,s})$ also acts on a linear subspace $W^\perp \subseteq \mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g}$ with $\mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g} = W \oplus W^\perp$.

Second, and putting $h=1$, we get

$$M*v - v \in W(\mathbb{C}) \quad \text{for all} \quad M \in \mathrm{Mon}(Z)(\mathbb{C}) \text{ and all } (1,v) \in \mathrm{MT}(\mathbf{x})(\mathbb{C}). \tag{3.11}$$

Now let us compute the kernel of $\mathrm{MT}(\mathbf{x}) \to \mathrm{MT}(\mathfrak{A}_{g,s})$ using [And92, Proposition 1]. In its notation we set $H = G = \mathrm{MT}(\mathbf{x})$ and claim $E' = \mathfrak{A}_{g,s}$. Indeed, P is not contained in a proper algebraic subgroup of $\mathfrak{A}_{g,s}$ by hypothesis. So the cyclic subgroup it generates is Zariski dense in $\mathfrak{A}_{g,s}$. The said proposition then implies that the kernel equals $\mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g}$. So (3.11) holds for all $v \in \mathbb{C}^{2g}$.

In particular,

$M*v - v \in W(\mathbb{C})$ for all $M \in \mathrm{Mon}(Z)(\mathbb{C})$ and all $v \in W^\perp(\mathbb{C})$. As $\mathrm{Mon}(Z)$ is an algebraic subgroup of $\mathrm{MT}(\mathfrak{A}_{g,s})$ it also acts on $W^\perp$. As $W(\mathbb{C}) \cap W^\perp(\mathbb{C}) = 0$ we conclude that $\mathrm{Mon}(Z)$ acts trivially on $W^\perp$. So $W^\perp(\mathbb{Q})$ is contained in the fixed part of the monodromy action on $H_1(\mathfrak{A}_{g,s}, \mathbb{Q})$.

Moreover, we write $v_0 = v_0' + v_0''$ such that $v_0' \in W(\mathbb{R})$ and $v_0'' \in W^\perp(\mathbb{R})$. Then $M*v_0 = M*v_0' + M*v_0'' \in v_0'' + W = v_0 + W$ and

$$\psi(M) = M*v_0' + M*v_0'' - v_0' - v_0'' + W = M*v_0' - v_0' + W = W$$

for all $M \in \mathrm{Mon}(Z)(\mathbb{C})$.

We summarize these last arguments and (3.8) by stating that the connected algebraic monodromy group satisfies

$$\mathrm{Mon}(Y^{\mathrm{reg}}) = \mathrm{Mon}(\mathbf{x}) = W \rtimes \mathrm{Mon}(Z).$$

By (3.10) we have

$$(\mathrm{Mon}(Z)(\mathbb{R})^+ \cdot \tau_0) \times (v_0 + W(\mathbb{R})) = \{(M\tau_0, v_0 + w) : M \in \mathrm{Mon}(Z)(\mathbb{R})^+, w \in W(\mathbb{R})\} \subseteq \tilde{Y}_0$$

By Moonen's work on weakly special subvarieties the orbit $\mathrm{Mon}(Z)(\mathbb{R})^+ \cdot \tau_0$ maps onto $\pi(Y)$ under the uniformizing map $\mathfrak{H}_g \to \mathbb{A}_g^{\mathrm{an}}$, see [Moo98, §3 and Proposition 3.7]. Generically, the fiber of $Y \to \pi(Y)$ has dimension $\dim Y - \dim \pi(Y)$, which is $\leq \frac{1}{2}\dim_{\mathbb{R}} W(\mathbb{R})$ by (3.9). Hence $\dim_{\mathbb{R}} \tilde{Y}_0 \leq \dim_{\mathbb{R}}(\mathrm{Mon}(Z)(\mathbb{R})^+ \cdot \tau_0) \times (v_0 + W(\mathbb{R}))$.

We now show $(\mathrm{Mon}(Z)(\mathbb{R})^+ \cdot \tau_0) \times (v_0 + W(\mathbb{R})) = \tilde{Y}_0$. Indeed, note that the left-hand side is closed in $\tilde{Y}_0$. Let T denote the singular points of the complex analytic space $\tilde{Y}_0$. As $\tilde{Y}_0$ is irreducible, $\tilde{Y}_0 \setminus T$ is a connected complex manifold. Moreover, $(\mathrm{Mon}(Z)(\mathbb{R})^+ \cdot \tau_0) \times (v_0 + W(\mathbb{R})) \setminus T$ is a topological (real) manifold of dimension $2 \dim \tilde{Y}_0$ contained in $\tilde{Y}_0 \setminus T$. So it is open in $\tilde{Y}_0 \setminus T$ by invariance of dimension. But it is also closed in $\tilde{Y}_0 \setminus T$. So $(\mathrm{Mon}(Z)(\mathbb{R})^+ \cdot \tau_0) \times (v_0 + W(\mathbb{R})) \setminus T = \tilde{Y}_0 \setminus T$. The claim follows as $\tilde{Y}_0 \setminus T$ is dense in $\tilde{Y}_0$.

In particular, $(\mathrm{Mon}(Z)(\mathbb{R})^+ \cdot \tau_0) \times (v_0 + W(\mathbb{R}))$ is complex analytic. Thus for all $\tau \in \mathfrak{H}_g$ with $\mathrm{unif}(\tau) \in \pi(Y)(\mathbb{C})$, $W(\mathbb{R})$ is a complex subspace for the complex structure on $\mathbb{R}^{2g}$ endowed by τ. Moreover, (3.9) is an equality.

This concludes (i) since W is an algebraic subgroup of $\mathbb{G}_{\mathrm{a},\mathbb{Q}}^{2g}$. Part (ii) follows from [Del71, Corollaire 4.1.2] because $W^\perp(\mathbb{Q})$ is contained in the fixed part of the monodromy action on $H_1(\mathfrak{A}_{g,s}, \mathbb{Q})$. □

We end this section with a sufficient criterion for the meagerness of the bi-algebraic closure of a variety.

Lemma 3.3. *Let Z be an irreducible closed subvariety of $\mathbb{A}_g$, then $Z \cap Z^{\mathrm{biZar,reg}} \neq \emptyset$. Let Y be an irreducible closed subvariety of $\mathfrak{A}_g$. Then $\pi(Y) \cap \pi(Y^{\mathrm{biZar}})^{\mathrm{reg}} \neq \emptyset$. If Y^{exc} is meager in Y, then $Y^{\mathrm{biZar,exc}}$ is meager in Y^{biZar}.*

Proof. First we show that Z is not contained in the singular locus of Z^{biZar}. Indeed, being a singular point of Z^{biZar} is an algebraic condition in $\mathbb{A}_g$. A component of the preimage of $Z^{\mathrm{biZar}}(\mathbb{C})$ under $\mathfrak{H}_g \to \mathbb{A}_g^{\mathrm{an}}$ is algebraic. So being a singular point is also an algebraic condition in $\mathfrak{H}_g$. Therefore, each irreducible component of $Z \setminus Z^{\mathrm{biZar,reg}}$ is bi-algebraic. As Z^{biZar} is the minimal bi-algebraic subvariety containing Z we have $Z \cap Z^{\mathrm{biZar,reg}} \neq \emptyset$. The first part of the follows.

The second claim follows from the first one and since $\pi(Y^{\mathrm{biZar}}) = \pi(Y)^{\mathrm{biZar}}$.

The third claim follows from the second one and from Lemma 2.5 with $X = Y^{\mathrm{biZar}}$. □

4. A Criterion for Non-degeneracy

Recall that $\mathfrak{A}_g$ is a geometrically irreducible quasi-projective variety defined over a number field. Again we take this number field to be a subfield of $\mathbb{C}$. For the rest of this section we consider all subvarieties as defined over $\mathbb{C}$.

Let $X \subseteq \mathfrak{A}_g$ be an irreducible closed subvariety. We set

$$\delta(X) = \dim X^{\mathrm{biZar}} - \dim \pi(X^{\mathrm{biZar}}) \geq 0,$$

and with $t \in \mathbb{Z}$, also

$$X^{\mathrm{deg}}(t) = \bigcup_{\substack{Y \subseteq X \\ \delta(Y) < \dim Y + t \\ \dim Y > 0}} Y \tag{4.1}$$

where Y ranges over positive dimensional *irreducible* closed subvarieties of X. Thus

$$X^{\mathrm{deg}}(t) \subseteq X^{\mathrm{deg}}(t+1).$$

By [Gao20a, Theorem 7.1], $X^{\deg}(t)$ is Zariski closed in X. Moreover if X is defined over some algebraically closed field $L \subseteq \mathbb{C}$ of characteristic 0, then $X^{\deg}(t)$ is also defined over L; see [Gao21, Proposition 4.2.6].

Remark 4.1. *Before moving on, let us take a look at $X^{\deg}(t)$ when $\pi(X)$ is a point. In this case, X is contained in a fiber of $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$, which is an abelian variety. Call this abelian variety A. For each irreducible subvariety Y of X, we have $\delta(Y) = \dim Y^{\mathrm{biZar}} \geq \dim Y$. In particular, $X^{\deg}(t) = \emptyset$ if $t \leq 0$.*

By [UY11, Proposition 5.1], any bi-algebraic subvariety of A is a coset in A, i.e., a translate of an abelian subvariety of A. Conversely, any coset in A is bi-algebraic. Thus Y^{biZar} is the smallest coset of A containing Y. Now if $\delta(Y) < \dim Y + 1$, then $Y^{\mathrm{biZar}} = Y$. Thus $X^{\deg}(1)$ is the union of all positive-dimensional cosets in A that are contained in X. This is precisely the Ueno locus or Kawamata locus.

For general t and X still in A, the union $X^{\deg}(t)$ was studied by Rémond [Rém09, §3] and by Bombieri, Masser, and Zannier in the multiplicative case [BMZ07, BMZ08] under the name (b-)anomalous.

We investigate necessary conditions for when $X = X^{\deg}(t)$. As a general result we mention [Gao20a, Theorem 8.1] and the exposition here is heavily motivated by this reference. Our approach works under the assumption that $X^{\deg}(t)$ contains a non-empty open subset of X^{an}.

We keep the same setup as introduced in the beginning of §3.1.

Lemma 4.2. *Let $Y \subseteq \mathfrak{A}_g$ be an irreducible closed subvariety such that Y^{exc} is meager in Y. Let S denote the regular locus of $\pi(Y^{\mathrm{biZar}})$. Then $\pi(Y) \cap S \neq \emptyset$ and there exists $\varphi \in \mathrm{End}(\mathfrak{A}_{g,S}/S)$ with the following properties hold:*

(i) *We have $\dim \ker \varphi_s = \delta(Y)$ for all $s \in S(\mathbb{C})$.*
(ii) *The fiber Y_s^{biZar} is a finite union of translates of $(\ker \varphi_s)^0$ for all $s \in S(\mathbb{C})$.*
(iii) *The abelian varieties $\varphi(\mathfrak{A}_g)_s$ are pairwise isomorphic for all $s \in S(\mathbb{C})$.*
(iv) *If $\delta(Y) = 0$, then Y is a point.*

Proof. Recall that $\pi(Y)^{\mathrm{biZar}}$ is the smallest bi-algebraic subvariety of $\mathbb{A}_g$ that contains $\pi(Y)$ and that it equals $\pi(Y^{\mathrm{biZar}})$.

By Lemma 3.3, $Y^{\mathrm{biZar,exc}}$ is meager in Y^{biZar} and $\pi(Y) \cap S \neq \emptyset$.

By Proposition 3.2 applied to Y^{biZar} each fiber of Y^{biZar} above a complex point of $\pi(Y^{\mathrm{biZar}})$ is a finite union of cosets of dimension $\delta(Y)$.

We abbreviate $\mathcal{A} = \pi^{-1}(S)$. We apply Lemma 2.3 to the abelian scheme $\mathcal{A}/S$ and the subvariety $Y^{\mathrm{biZar}} \cap \mathcal{A}$. Let φ be the endomorphism in the said lemma.

By the conclusion of Lemma 2.3(ii) we have $\dim \ker \varphi_s = \dim Y^{\mathrm{biZar}} \cap \mathcal{A} - \dim \pi(Y^{\mathrm{biZar}} \cap \mathcal{A}) = \delta(Y)$ for all $s \in S(\mathbb{C})$. Part (i) now follows. For later reference we remark that φ is the identity map if $\delta(Y) = 0$; see Lemma 2.3.

By Lemma 2.3(i) we have $\dim \varphi(Y^{\mathrm{biZar}} \cap \mathcal{A}) = \dim \pi(Y^{\mathrm{biZar}} \cap \mathcal{A})$. By the fiber dimension theorem, the general fiber of $\pi|_{\varphi(Y^{\mathrm{biZar}} \cap \mathcal{A})} \colon \varphi(Y^{\mathrm{biZar}} \cap \mathcal{A}) \to \pi(Y^{\mathrm{biZar}} \cap \mathcal{A}) = S$ is finite. For s in a Zariski open and non-empty subset of S we have that $\varphi(Y^{\mathrm{biZar}} \cap \mathcal{A})_s$ is finite. Therefore, Y_s^{biZar} is contained in a finite union of $(\ker \varphi_s)^0$ for such s. By dimension reasons, these Y_s^{biZar} are a finite union of $(\ker \varphi_s)^0$ and $(\ker \varphi_s)^0 + Y_s^{\mathrm{biZar}} = Y_s^{\mathrm{biZar}}$.

Note that $(\ker \varphi)^0$ is smooth over S with geometrically irreducible generic fiber, as it is an abelian scheme. Moreover, $Y^{\mathrm{biZar}} \cap \mathcal{A}$ is Zariski open in Y^{biZar} and thus irreducible. It follows from a purely topological consideration that $(\ker\varphi)^0\times_S(Y^{\mathrm{biZar}}\cap\mathcal{A})$ is irreducible. A Zariski open and non-empty subset is mapped to $Y^{\mathrm{biZar}}\cap\mathcal{A}$ under addition. This continues to hold on all of $(\ker\varphi)^0\times_S(Y^{\mathrm{biZar}}\cap\mathcal{A})$. Thus $(\ker\varphi_s)^0+Y_s^{\mathrm{biZar}}=Y_s^{\mathrm{biZar}}$. By dimension reasons Y_s^{biZar} is a finite union of $(\ker\varphi_s)^0$ for all $s\in S(\mathbb{C})$. Part (ii) follows.

For all $s\in S(\mathbb{C})$, the image $\varphi(\mathfrak{A}_{g,s})=\varphi(\mathfrak{A}_g)_s$ is isogenous to $\mathfrak{A}_{g,s}/(\ker\varphi_s)^0$. The latter are pairwise isomorphic abelian varieties for all s by Proposition 3.2. By consider the morphism to a suitable moduli space we conclude that the $\varphi(\mathfrak{A}_g)_s$ are indeed pairwise isomorphic. We conclude (iii).

For the proof of part (iv) we assume that $\delta(Y)=0$. As remarked above, φ is the identity. Therefore, $\mathfrak{A}_{g,s}$ are pairwise isomorphic abelian varieties for $s\in S(\mathbb{C})$. This implies that S is a point and so is $\pi(Y)$. But then $\pi(Y^{\mathrm{biZar}})$ is a point. As $0=\delta(Y)=\dim Y^{\mathrm{biZar}}-\dim\pi(Y^{\mathrm{biZar}})$ we have that Y^{biZar} is a point. The same holds for Y and this completes the proof of (iv). □

Now we are ready to prove a necessary condition for $X^{\deg}(t)$ being sufficiently large. The next proposition relies on the previous lemma and the Baire Category Theorem; recall that group of endomorphisms of an abelian scheme is at most countably infinite.

Proposition 4.3. *Let $t\in\mathbb{Z}$ and let X be an irreducible closed subvariety of $\mathfrak{A}_g$. Let η denote the generic point of $\pi(X)$ and let S denote the regular locus of $\pi(X)$. We suppose*

(a) *$X^{\deg}(t)$ contains an open and non-empty subset of X^{an}*
(b) *and X_η is not contained in a proper algebraic subgroup of $\mathfrak{A}_{g,\eta}$,*

There exists a set $\mathcal{Y}$ of irreducible closed positive dimensional subvarieties of X and $\varphi\in\mathrm{End}(\mathfrak{A}_{g,S}/S)$ with the following properties for all $Y\in\mathcal{Y}$.

(i) *We have $\dim\ker\varphi_s=\delta(Y)$ for all $s\in S(\mathbb{C})$.*
(ii) *The fiber Y_s^{biZar} is a finite union of translates of $(\ker\varphi_s)^0$ for all complex points s of a Zariski open and dense subset $\pi(Y)$.*
(iii) *The abelian varieties $\varphi(\mathfrak{A}_g)_s$ are pairwise isomorphic for all complex points s of a Zariski open and dense subset of $\pi(Y)$.*
(iv) *We have $\delta(Y)<\dim Y+t$ and $\pi(Y)\cap S\neq\emptyset$.*
(v) *The set Y^{exc} is meager in Y.*

Finally, the closure of $\bigcup_{Y\in\mathcal{Y}}Y(\mathbb{C})$ in X^{an} has non-empty interior.

Proof. By hypothesis (b) and Lemma 2.4 applied to $X\subseteq\mathfrak{A}_g$, we have that X^{exc} is meager in X. Thus $X^{\mathrm{exc}}\subseteq\bigcup_{i=1}^\infty X_i(\mathbb{C})$ such that all $X_i\subsetneq X$ are Zariski closed. For a similar reason and using Proposition 2.2 there exist Zariski closed $S_1,S_2,\ldots\subsetneq\pi(X)$, among them is $\pi(X)\setminus S$, with $S^{\mathrm{exc}}\subseteq\bigcup_{i=1}^\infty S_i(\mathbb{C})$.

By hypothesis the union of all $Y\subseteq X$ with $\dim Y>0$ and

$$\delta(Y)=\dim Y^{\mathrm{biZar}}-\dim\pi(Y^{\mathrm{biZar}})<\dim Y+t \tag{4.2}$$

contains a non-empty open subset of X^{an}. Let $\mathcal{Y}$ be the collection of those Y with $Y\not\subseteq\pi^{-1}(S_i)$ and $Y\not\subseteq X_i$ for all i. There is a set $N\subseteq X(\mathbb{C})$, meager in X, such that $N\cup\bigcup_{Y\in\mathcal{Y}}Y(\mathbb{C})$ contains a non-empty open subset of X^{an}.

Let $Y\in\mathcal{Y}$ be arbitrary. In particular, $\pi(Y)\cap S\neq\emptyset$. Set $U_Y=\pi(Y)\cap\pi(Y^{\mathrm{biZar}})^{\mathrm{reg}}\cap S$; it is a Zariski open and dense subset of $\pi(Y)$ by Lemma 3.3. Therefore, $U_Y\not\subseteq S_i$ for all i by the choice of $\mathcal{Y}$. The Baire Category Theorem implies $U_Y(\mathbb{C})\not\subseteq\bigcup_{i=1}^\infty S_i(\mathbb{C})$, so $U_Y(\mathbb{C})\not\subseteq S^{\mathrm{exc}}$.

By definition we have $Y^{\mathrm{exc}} \subseteq X^{\mathrm{exc}}$ and so $Y^{\mathrm{exc}} \subseteq \bigcup_{i=1}^{\infty}(Y \cap X_i)(\mathbb{C})$. By the choice of $\mathcal{Y}$ we conclude that Y^{exc} is meager in Y.

Apply Lemma 4.2 to Y and obtain φ_Y, and restrict φ_Y to an endomorphism of the abelian scheme $\pi^{-1}(U_Y)$. Choose $s \in U_Y(\mathbb{C}) \setminus S^{\mathrm{exc}}$. Then $\varphi_Y|_{\pi^{-1}(s)} \in \mathrm{End}(\mathfrak{A}_{g,s})$ extends to an endomorphism of $\mathfrak{A}_{g,S}/S$. This extension is unique and it coincides with φ_Y on $\pi^{-1}(U_Y)$. We use φ_Y to denote this endomorphism of $\mathfrak{A}_{g,S}/S$. Note that $\delta(Y) = \dim\ker(\varphi_Y)_s$ for all $s \in S(\mathbb{C})$.

Recall that $N \cup \bigcup_{Y\in\mathcal{Y}} Y^{\mathrm{an}}$ contains a non-empty open subset of X^{an}. We rearrange this union and conclude that the said open subset lies in $N \cup \bigcup_{\varphi\in\mathrm{End}(\mathfrak{A}_{g,S}/S)} \overline{D_\varphi}$ where $D_\varphi = \bigcup_{Y\in\mathcal{Y}:\varphi_Y=\varphi} Y(\mathbb{C})$ and $\overline{D_\varphi}$ denotes the topological closure in X^{an}.

By the Baire Category Theorem there is $\varphi \in \mathrm{End}(\mathfrak{A}_{g,S}/S)$ such that $\overline{D_\varphi}$ has non-empty interior in X^{an}. In particular, D_φ is Zariski dense in X.

We claim that the proposition follows with $\mathcal{Y}$ replaced by $\{Y \in \mathcal{Y} : \varphi_Y = \varphi\}$. Indeed, properties (i), (ii), and (iii) follow from the corresponding properties of Lemma 4.2 and (iv) and (v) follow from the choice of $\mathcal{Y}$ $\square$

Remark 4.4. *The case $t = 0$ is closely linked to large fibers of the Betti map; see [Gao20a, §3] for a definition of the Betti map. The Betti map is real analytic and defined locally on $X^{\mathrm{reg,an}}$. Suppose that the generic rank of the differential is strictly less than $2\dim X$. This is the case if X fails to be non-degenerate in the sense of [DGH21, Definition 1.5]. Then there is a non-empty open subset of X^{an} on which the rank is pointwise strictly less than $2\dim X$. Using the first-named author's Ax–Schanuel Theorem [Gao20b] for $\mathfrak{A}_g$ one can recover that $X^{\deg}(0)$ contains a non-empty open subset of X^{an}. So the hypothesis (a) for $t = 0$ in Proposition 4.3 is satisfied. See also [Gao20a, Theorem 1.7] for an equivalence.*

5. The Zeroth Degeneracy Locus in a Fiber Power

We keep the notation from §4 and consider all subvarieties as defined over $\mathbb{C}$. We study ramifications of Proposition 4.3 in the case $t = 0$ for the m-fold fiber power $\mathfrak{A}_g^{[m]}$ of $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$, here $m \in \mathbb{N}$. There is a natural morphism $\mathfrak{A}_g^{[m]} \to \mathfrak{A}_{mg}$ which is the base change of the modular map $\mathbb{A}_g \to \mathbb{A}_{mg}$ that attaches to an abelian variety its m-th power compatible with the principal polarization and level structure. It can be shown that $\mathbb{A}_g \to \mathbb{A}_{mg}$ is a closed immersion. So $\mathfrak{A}_g^{[m]} \to \mathfrak{A}_{mg}$ is a closed immersion. We will treat $\mathfrak{A}_g^{[m]}$ as a closed subvariety of $\mathfrak{A}_{mg}$.

By abuse of notation let $\pi\colon \mathfrak{A}_{mg} \to \mathbb{A}_{mg}$ denote the structure morphism.

Let X be a Zariski closed subset of an abelian variety A defined over $\mathbb{C}$. The stabilizer $\mathrm{Stab}(X)$ of X is the algebraic group determined by $\{P \in A(\mathbb{C}) : P + X = X\}$.

Theorem 5.1. *Let X be an irreducible closed subvariety of $\mathfrak{A}_g^{[m]}$. Consider $X \subseteq \mathfrak{A}_{mg}$ and let η denote the generic point of $\pi(X) \subseteq \mathbb{A}_{mg}$. We suppose*

(a) *$X^{\deg}(0)$ contains an open and non-empty subset of X^{an},*
(b) *X_η is not contained in a proper algebraic subgroup of $\mathfrak{A}_{mg,\eta}$,*
(c) *and*

$$\dim X \leq 2m. \tag{5.1}$$

Then the following hold true.

(i) *There exists a Zariski open and dense subset* $U \subseteq \pi(X)$ *such that for all* $s \in U(\mathbb{C})$ *the stabilizer* $\mathrm{Stab}(X_s)$ *has dimension at least* m.
(ii) *There is a Zariski dense subset* $D \subseteq \pi(X)(\mathbb{C})$ *such that for all* $s \in D$ *the stabilizer* $\mathrm{Stab}(X_s)$ *contains* E^m *where* $E \subseteq \mathfrak{A}_{g,s}$ *is an elliptic curve.*

Proof. We apply Proposition 4.3 to $X \subseteq \mathfrak{A}_{mg}$ in the case $t = 0$ and obtain $\mathcal{Y}$ and φ. We write S for the regular locus of $\pi(X) \subseteq \mathbb{A}_{mg}$ and $\mathcal{B}$ for the abelian scheme $\varphi(\mathfrak{A}_{g,S}^{[m]})$ over S, see Lemma 2.1.

Let $Y \in \mathcal{Y}$. Note that $\delta = \delta(Y) \geq 0$ is independent of Y by Proposition 4.3(i).

The generic fiber of $\mathcal{B}_{\pi^{-1}(Y)} \to \pi(Y)$ is an abelian variety B defined over the function field of $\pi(Y)$. By Proposition 4.3(iii) there is a finite extension L of the function field of $\pi(Y)$, such that the base change B_L is a constant abelian variety over L. We have $\dim B_L = mg - \delta$. Let A_L denote the base change of the generic fiber of $\mathfrak{A}_{g,\pi^{-1}(Y)} \to \pi(Y)$. Then B_L is a quotient of A_L^m. Thus $A_L^m \to B_L$ factors through $\mathrm{Im}_{L/\mathbb{C}}(A_L^m)_L$ where $\mathrm{Im}_{L/\mathbb{C}}(\cdot)$ denotes the $L/\mathbb{C}$-image of an abelian variety defined over L, see [Con06] for a definition and properties. Since $A_L^m \to B_L$ is surjective we have $\dim B_L \leq \dim \mathrm{Im}_{L/\mathbb{C}}(A_L^m) = m \dim \mathrm{Im}_{L/\mathbb{C}}(A_L)$. By (4.2) with $t = 0$ we find

$$mg - \dim Y < mg - \delta = \dim B_L \leq m \dim \mathrm{Im}_{L/\mathbb{C}}(A_L), \tag{5.2}$$

As $Y \subseteq X$ we have $\dim Y \leq \dim X$. The hypothesis $\dim X \leq 2m$ combined with (5.2) yields $m(g-2) < m \dim \mathrm{Im}_{L/\mathbb{C}}(A_L)$. We cancel m and obtain

$$\dim \mathrm{Im}_{L/\mathbb{C}}(A_L) \geq g - 1.$$

If $\pi(Y)$ is a point, then so is $\pi(Y^{\mathrm{biZar}}) = \pi(Y)^{\mathrm{biZar}}$. Again (4.2) with $t = 0$ implies $\dim Y^{\mathrm{biZar}} < \dim Y$ which contradicts $Y \subseteq Y^{\mathrm{biZar}}$. So

$$\dim \pi(Y) \geq 1.$$

From this we conclude $\dim \mathrm{Im}_{L/\mathbb{C}}(A_L) < g$ as otherwise general fibers of $\mathfrak{A}_g$ above $\pi(Y)$ would be pairwise isomorphic abelian varieties. Thus

$$\dim \mathrm{Im}_{L/\mathbb{C}}(A_L) = g - 1.$$

The canonical morphism $A_L \to \mathrm{Im}_{L/\mathbb{C}}(A_L)_L$ is surjective with connected kernel E as we are in characteristic 0. Here E is an elliptic curve and $\mathrm{Im}_{L/\mathbb{C}}(E) = 0$. Recall that φ induces a homomorphism $\varphi_L \colon A_L^m \to B_L$ and B_L is a constant abelian variety. The composition $E^m \to A_L^m \xrightarrow{\varphi_L} B_L$ factors through $E^m \to \mathrm{Im}_{L/\mathbb{C}}(E^m)_L = \mathrm{Im}_{L/\mathbb{C}}(E)_L^m = 0$. Therefore, E^m lies in the kernel of φ_L. In particular,

$$\delta = \dim \ker \varphi_L \geq m. \tag{5.3}$$

We fix an irreducible variety W with function field L and a quasi-finite dominant morphism $W \to \pi(Y)$. After replacing W by a Zariski open subset we can spread A_L and E out to abelian schemes $\mathcal{A}$ and $\mathcal{E}$ over W, respectively. The j-invariant of $\mathcal{E}/W$ is a morphism $W \to \mathbb{A}^1$. If $\dim W > 1$, then there is an irreducible curve $W' \subseteq W$ on which j is constant. All elliptic curves above $W'(\mathbb{C})$ are isomorphic over $\mathbb{C}$. But then infinitely many fibers of $\mathcal{A}_g$ above points of $\pi(\mathbb{C})$ are pairwise isomorphic. This is impossible and so we have $\dim W \leq 1$. But $\dim W = \dim \pi(Y) \geq 1$, hence

$$\dim \pi(Y) = 1. \tag{5.4}$$

Recall that φ is defined above all but finitely many points of the curve $\pi(Y)$. Recall also that $\ker\varphi$ contains the m-th power of an elliptic curve on the generic fiber. So $\ker\varphi_s$ contains the m-th power of an elliptic curve in $\mathfrak{A}_{g,s}$ for all but finitely many $s\in\pi(Y)(\mathbb{C})$.

We draw the following conclusion from Proposition 4.3(i) and (ii) for a Zariski open and dense $U_Y\subseteq\pi(Y)$. If $s\in U_Y(\mathbb{C})$, then Y_s is contained in a finite union of translates of $(\ker\varphi_s)^0$. The latter is an algebraic group of dimension δ. Let $P\in\pi|_Y^{-1}(U_Y)(\mathbb{C})$ with $\pi(P)=s$. Any irreducible component C of Y_s containing P has dimension at least $\dim Y-\dim\pi(Y)$. So $\dim C\geq\dim Y-1\geq\delta$ by (5.4) and (4.2) with $t=0$. But $C\subseteq Y_s$, so C is contained in a translate of $(\ker\varphi_s)^0$. Thus $C=P+(\ker\varphi)_s^0\subseteq X$. We conclude

$$\dim_P\varphi|_{X\cap\mathfrak{A}_{g,S}^{[m]}}^{-1}(\varphi(P))\geq\delta\quad\text{for all}\quad P\in\pi|_Y^{-1}(U_Y)(\mathbb{C})\text{ and all }Y\in\mathcal{Y}.\tag{5.5}$$

By possibly removing finitely many points from U_Y we may arrange that $(\ker\varphi_{\pi(P)})^0$ contains the m-th power of an elliptic curve in $\mathfrak{A}_{g,\pi(P)}$ for all $P\in\pi|_Y^{-1}(U_Y)(\mathbb{C})$.

We write $D=\bigcup_{Y\in\mathcal{Y}}\pi|_Y^{-1}(U_Y(\mathbb{C}))$. The closure of D in X^{an} equals the closure of $\bigcup_{Y\in\mathcal{Y}}Y(\mathbb{C})$ in X^{an}. Indeed, this requires some point-set topology and the fact that $\pi|_Y^{-1}(U_Y(\mathbb{C}))$ lies dense in Y^{an}. In particular, D is Zariski dense in X by Proposition 4.3.

So (5.5) holds on the Zariski dense subset D of X. Therefore, the dimension inequality holds for all $P\in(X\cap\mathfrak{A}_{g,S}^{[m]})(\mathbb{C})$ by the semi-continuity theorem on fiber dimensions.

Each fiber of $\varphi\colon\mathfrak{A}_{g,S}^{[m]}\to\varphi(\mathfrak{A}_{g,S}^{[m]})$ is the translate of some $(\ker\varphi_{\pi(P)})^0$, which has dimension δ. We conclude that if $P\in(X\cap\mathfrak{A}_{g,S}^{[m]})(\mathbb{C})$, then $\varphi|_X^{-1}(\varphi(P))$ contains $P+(\ker\varphi_{\pi(P)})^0$ as an irreducible component. So $(\ker\varphi_{\pi(P)})^0$ lies in the stabilizer of $X_{\pi(P)}$.

The first claim of the theorem follows from (5.3) with $U=\pi(X\cap\mathfrak{A}_{g,S}^{[m]})=S$.

The second claim follows as $\ker\varphi_s$ contains the m-th power of an elliptic curve for all s in $\pi(D)$, which is Zariski dense in $\pi(X)$. $\square$

For an abelian variety A and $m\in\mathbb{N}$ we define $D_m\colon A^{m+1}\to A^m$ to be the Faltings–Zhang morphism determined by $D_m(P_0,\ldots,P_m)=(P_1-P_0,\ldots,P_m-P_0)$.

Lemma 5.2. *Let A be an abelian variety defined over $\mathbb{C}$ and let $C\subseteq A$ be an irreducible closed subvariety of dimension 1. Suppose $m\geq2$ and let $X=D_m(C^{m+1})\subseteq A^m$. If B is an abelian subvariety of A with $B^m\subseteq\mathrm{Stab}(X)$, then $B=0$ or C is a translate of B.*

Proof. Let $\varphi\colon A\to A/B$ denote the quotient homomorphism and $\varphi^m\to A^m/B^m=(A/B)^m$ its m-th power. We set $Z=\varphi^m(X)$. Then $\dim Z\leq\dim X-m\dim B$ as B^m is in the stabilizer of X. We have $\dim X\leq C^{m+1}=m+1$. So

$$0\leq\dim Z\leq\dim X-m\dim B\leq m+1-m\dim B.\tag{5.6}$$

This implies $\dim B\leq1+1/m<2$ as $m\geq2$.

Let us suppose $B\neq0$, then $\dim B=1$. Hence $\dim Z\leq1$ by (5.6). We have

$$(\varphi(P_1-P_0),\ldots,\varphi(P_m-P_0))=\varphi^m(P_1-P_0,\ldots,P_m-P_0)\in Z(\mathbb{C})$$

for all $P_0,\ldots,P_m\in C(\mathbb{C})$. We fix P_0 and let $P_1,\ldots,P_m$ vary. As $\dim Z\leq1$ and $m\geq2$ it follows that φ is constant on the curve C. For dimension reasons we conclude that C equals a translate of B. $\square$

Corollary 5.3. *Let $g\geq2,m\geq2$, and let X be an irreducible closed subvariety of $\mathfrak{A}_g^{[m]}$ with $\dim\pi(X)\leq m-1$. We suppose that for all complex points s of a Zariski open* *and dense subset of $\pi(X)$, the fiber X_s is of the form $D_m(C^{m+1})$ where $C \subseteq \mathfrak{A}_{g,s}$ is not contained in the translate of a proper algebraic subgroup of $\mathfrak{A}_{g,s}$. Then $X^{\deg}(0)$ does not contain an non-empty open subset of X^{an}. Moreover, the generic Betti rank on X is $2\dim X$ and X is non-degenerate in the sense of [DGH21, Definition 1.5].*

Proof. Let $X_s = D_m(C^{m+1})$ with C as in the hypothesis. In particular, C is not equal to the translate of an abelian subvariety of $\mathfrak{A}_{g,s}$. By Lemma 5.2, $\mathrm{Stab}(X_s)$ does not contain the m-th power of a non-zero abelian subvariety of $\mathfrak{A}_{g,s}$. So conclusion (ii) of Theorem 5.1 cannot hold.

Moreover, X_s is not contained in a proper algebraic subgroup of $\mathfrak{A}_{g,s}^m$ and this remains true for the generic point of $\pi(X)$. Moreover, $\dim X \leq \dim \pi(X) + m + 1 \leq (m-1) + m + 1 = 2m$. So hypotheses (b) and (c) of Theorem 5.1 hold. Therefore, hypothesis (a) cannot hold. This is the first claim of the corollary. The second claim follows from Remark 4.4. □

6. The First Degeneracy Locus and the Relative Manin–Mumford Conjecture

In this section we provide an exposition of the proof of Proposition 11.2 of the first-named author's work [Gao20a]. We proceed slightly differently and concentrate our efforts on subvarieties of the universal family of principally polarized abelian varieties with suitable level structure.

We keep the notation of §3.1 with an important additional restriction. Let $g \geq 1$ be an integer and equip $\mathbb{A}_g$ with suitable level structure. Let $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$ denote the universal family. In this section we consider $\mathfrak{A}_g$ and $\mathbb{A}_g$ as irreducible quasi-projective varieties defined over a number field $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

The set of torsion points $\mathfrak{A}_{g,\mathrm{tors}}$ is $\bigcup_{s\in\mathbb{A}_g(\mathbb{C})}\{P \in \mathfrak{A}_{g,s}(\mathbb{C}) : P \text{ has finite order.}\}$.

We consider here a variant of the Relative Manin–Mumford Conjecture, inspired by S. Zhang [Zha98] and formulated in work of Pink [Pin05] as well as Bombieri–Masser–Zannier [BMZ07]. We also refer to Zannier's book [Zan12] for a formulation. In contrast to the general case, we retain $\mathfrak{A}_g$ as an ambient group scheme and work only with varieties defined over $\overline{\mathbb{Q}}$.

The following conjecture depends on the dimension parameter $g \in \mathbb{N}$.

Conjecture RelMM(g). *Let X be an irreducible closed subvariety of $\mathfrak{A}_g$ defined over $\overline{\mathbb{Q}}$ and let $\eta \in \pi(X)$ denote the generic point. We assume that $\dim X < g$ and that X_η is not contained in a proper algebraic subgroup of $\mathfrak{A}_{g,\eta}$. Then $X(\overline{\mathbb{Q}}) \cap \mathfrak{A}_{g,\mathrm{tors}}$ is not Zariski dense in X.*

For curves, this conjecture is known, even over $\mathbb{C}$, thanks to work of Masser–Zannier and Corvaja–Masser–Zannier [MZ08, MZ12, MZ14, MZ15, CMZ18, MZ20]. Stoll [Sto17] proved an explicit case. For surfaces some results are due to the first-named author [Hab13] and Corvaja–Tsimerman–Zannier [CTZ23].

The goal of this section is to prove Conjecture RelMM(g) for all g conditional on the following conjecture. Below, a subscript $\mathbb{C}$ indicates base change to $\mathbb{C}$.

Conjecture 6.1. *Let $g \in \mathbb{N}$, let X be an irreducible closed subvariety of $\mathfrak{A}_g$ defined over $\overline{\mathbb{Q}}$. If $\dim X > 0$ and $X(\overline{\mathbb{Q}}) \cap \mathfrak{A}_{g,\mathrm{tors}}$ is Zariski dense in X, then $X_{\mathbb{C}}^{\deg}(1)$ is Zariski dense in X.*

The goal will be achieved by induction on g. The induction step, which is conditional, is the following theorem.

Theorem 6.2. *Suppose Conjecture 6.1 holds. Let $g \geq 2$ be an integer and suppose Conjecture RelMM(g') holds for all $g' \in \{1, \ldots, g-1\}$. Then Conjecture RelMM($g$) holds.*

Proof. Let $X \subseteq \mathfrak{A}_g$ be an irreducible closed subvariety defined over $\overline{\mathbb{Q}}$ satisfying the hypothesis of Conjecture RelMM(g). Let η denote the generic point of $\pi(X)$. We assume $X(\overline{\mathbb{Q}}) \cap \mathfrak{A}_{g,\mathrm{tors}}$ is Zariski dense in X and will derive a contradiction.

We observe $\dim X > 0$. So Conjecture 6.1 implies that $X^{\deg}_{\mathbb{C}}(1)$ is Zariski dense in X. (Note that X satisfies $\dim X < g$ and a condition on X_η, but the two are not required to invoke Conjecture 6.1). By [Gao20a, Theorem 7.1], $X^{\deg}_{\mathbb{C}}(1)$ is Zariski closed in X. Thus $X_{\mathbb{C}} = X^{\deg}_{\mathbb{C}}(1)$.

By Lemma 2.4 and since X_η is not contained in a proper algebraic subgroup of $\mathfrak{A}_{g,\eta}$ we conclude that X^{exc} is meager in X.

We may apply Proposition 4.3 to $X_{\mathbb{C}} \subseteq \mathfrak{A}_{g,\mathbb{C}}$ in the case $t = 1$, both hypotheses (a) and (b) are met as X is as in Conjecture RelMM(g). We obtain $\mathcal{Y}$ and $\varphi \in \mathrm{End}(\mathfrak{A}_{g,S}/S)$ as in the proposition with S the regular locus of $\pi(X)$. (We note that φ is defined over $\overline{\mathbb{Q}}$.) Let δ denote the common value of $\delta(Y)$ for $Y \in \mathcal{Y}$.

Observe that $\bigcup_{Y \in \mathcal{Y}} Y \cap \mathfrak{A}_{g,S,\mathbb{C}}$ lies Zariski dense in $X_{\mathbb{C}}$. It is harmless to assume that $\dim Y$ are equal for all $Y \in \mathcal{Y}$, the same can be assumed for $\dim \pi(Y)$.

Let $\mathcal{B} = \varphi(\mathfrak{A}_{g,S})$, which is an abelian scheme over S of relative dimension g', say.

For each $Y \in \mathcal{Y}$ the exceptional locus Y^{exc} is meager in Y by Proposition 4.3(v). So $Y^{\mathrm{biZar,exc}}$ is meager in Y^{biZar} by Lemma 3.3. Proposition 3.2(i) applied to Y^{biZar} implies that each Y^{biZar}_s is a finite union of $\mathrm{unif}(\{\tau\} \times W)$, for $s = \mathrm{unif}(\tau)$; here $W \subseteq \mathbb{R}^{2g}$ is a linear subspace defined over $\mathbb{Q}$ with $\dim W = 2\delta$ and independent of s. Recall that if $s \in S(\mathbb{C})$, then Y^{biZar}_s is a finite union of translates of $(\ker \varphi_s)^0$, see Proposition 4.3(ii).

For each $s \in S(\mathbb{C})$, the endomorphism φ_s lifts to a linear map $\mathbb{R}^{2g} \to \mathbb{R}^{2g}$ mapping $\mathbb{Z}^{2g}$ to itself. The lift is independent of s and necessarily vanishes on W. Therefore, $\varphi(Y^{\mathrm{an}} \cap \mathfrak{A}^{\mathrm{an}}_{g,S})$ is the image of $\mathrm{unif}(\tilde{Y} \times \{\text{finite set}\})$ where $\mathrm{unif}(\tilde{Y}) = \pi(Y) \cap S(\mathbb{C})$.

The abelian scheme $\mathcal{B}/S$ may not be principally polarized. But its geometric generic fiber is isogenous to a principally polarized abelian variety. We fix an étale morphism $S_{\mathrm{pp}} \to S$ with base change $\mathcal{B}' = \mathcal{B} \times_S S_{\mathrm{pp}}$ as in the diagram (6.1) below. After spreading out and possibly replacing S_{pp} by a Zariski open and dense subset we may arrange that $\mathcal{B}' \to \mathcal{B}_{\mathrm{pp}}$ is a fiberwise an isogeny over S_{pp} and $\mathcal{B}_{\mathrm{pp}}$ is principally polarized. But now the level structure from $\mathfrak{A}_{g,S}$ may have been lost under the isogeny. To remedy this we fix yet another étale morphism $S_{\mathrm{pp,ls}} \to S_{\mathrm{pp}}$ and do a base change to add suitable torsion sections to $\mathcal{B}_{||}$ and ultimately obtain suitable level structure. This does not affect the principal polarization. Thus we get a principally polarized abelian scheme $\mathcal{B}_{\mathrm{pp,ls}}/S_{\mathrm{pp,ls}}$ with suitable level structure. Its relative dimension equals the relative dimension of $\mathcal{B}/S$. We obtain a Cartesian diagram into the corresponding fine moduli space as in the right of the following commutative diagram

$$
\begin{tikzcd}
\mathfrak{A}_{g,S} \arrow[r, "\varphi"] \arrow[d] & \mathcal{B} \arrow[d] & \mathcal{B}' \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & \mathcal{B}_{\mathrm{pp}} \arrow[d] & \mathcal{B}_{\mathrm{pp,ls}} \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{A}_{g'} \arrow[d] \\
S \arrow[r, equal] & S & S_{\mathrm{pp}} \arrow[l] \arrow[r, equal] & S_{\mathrm{pp}} & S_{\mathrm{pp,ls}} \arrow[l] \arrow[r] & \mathbb{A}_{g'}
\end{tikzcd}
\tag{6.1}
$$

We chase $\varphi(X \cap \mathfrak{A}_{g,S}) \subseteq \mathcal{B}$ through the correspondences $\mathcal{B} \leftarrow \mathcal{B}' \to \mathcal{B}_{\mathrm{pp}}$ and $\mathcal{B}_{\mathrm{pp}} \leftarrow \mathcal{B}_{\mathrm{pp,ls}} \to \mathfrak{A}_{g'}$ by taking preimages and images and fix an irreducible component X' of the Zariski closure of the image inside $\mathfrak{A}_{g'}$. Consider $Y \in \mathcal{Y}$ and chase $\varphi(Y \cap \mathfrak{A}_{g,S,\mathbb{C}}) \subseteq \mathcal{B}$ through the diagram as just described. Recall that $\varphi(Y(\mathbb{C}) \cap \mathfrak{A}_{g,S}(\mathbb{C}))$ is the image of $\mathrm{unif}(\tilde{Y} \times \{\text{finite set}\})$. Locally in the Euclidean topology on the base, our abelian schemes are trivializable in the real analytic category. Moreover, all abelian varieties of $\mathcal{B}$ above $S(\mathbb{C}) \cap \pi(Y(\mathbb{C}))$ are isomorphic by Proposition 3.2(ii). So $\varphi(Y(\mathbb{C}) \cap \mathfrak{A}_{g,S}(\mathbb{C}))$ ends up as a finite set in $\mathfrak{A}_{g'}$. Thus applying both correspondence has fibers of dimension at least $\dim \varphi(Y \cap \mathfrak{A}_{g,S,\mathbb{C}}) = \dim \pi(Y)$. Thus $\dim X' \leq \dim \varphi(X \cap \mathfrak{A}_{g,S,\mathbb{C}}) - \dim \pi(Y)$ by analysis of the fibers of the two correspondences.

Note $\dim \varphi(X \cap \mathfrak{A}_{g,S}) \leq \dim X - (\dim Y - \dim \pi(Y))$ because all fibers of $Y \to \pi(Y)$ have finite image under φ. We find $\dim X' \leq \dim X - \dim Y < g - \dim Y$, having used $\dim X < g$.

The relative dimension of $\mathcal{B}/S$ is $g' = g - \delta$. As all elements in $\mathcal{Y}$ have positive dimension, Lemma 4.2(iv) applied to an element in $\mathcal{Y}$ implies $\delta \geq 1$. Therefore, $g' \leq g - 1$. We have further $\delta \leq \dim Y$ by Proposition 4.3(iv) with $t = 1$. We conclude $\dim X' < g - \dim Y \leq g - \delta = g'$. In particular, $g' \geq 1$.

Chasing the Zariski dense set of torsion points in $X(\overline{\mathbb{Q}}) \cap \mathfrak{A}_{g,\mathrm{tors}}$ through the diagram shows that the torsion points in $X'(\overline{\mathbb{Q}})$ are Zariski dense in X'. The generic fiber of $\varphi(X \cap \mathfrak{A}_{g,S}) \to S$ is not contained in a proper algebraic subgroup of the generic fiber of $\mathcal{B} \to S$ by the hypothesis on X in $\mathrm{RelMM}(g)$. This implies that the generic fiber of $X' \to \pi(X')$ is not contained in a proper algebraic subgroup of the generic fiber of $\mathfrak{A}_{g',\pi^{-1}(X')} \to \pi(X')$.

Recall that $g' \in \{1, \ldots, g-1\}$ and so $\mathrm{RelMM}(g')$ holds by hypothesis. But then the properties of X' contradict the conclusion of $\mathrm{RelMM}(g')$. □

Corollary 6.3. *Conjecture 6.1 implies Conjecture RelMM(g) for all $g \in \mathbb{N}$.*

Proof. By Theorem 6.2 it suffices to prove $\mathrm{RelMM}(1)$. The condition on $\dim X$ in Conjecture $\mathrm{RelMM}(1)$ implies that X is a point. The condition on X_η and $g = 1$ imply that X is not a torsion point. So Conjecture $\mathrm{RelMM}(1)$ holds true. □

References

[And92] Y. André. Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. *Compositio Math.*, 82(1):1–24, 1992.

[BD22] F. Barroero and G. Dill. On the Zilber–Pink conjecture for complex abelian varieties. *Ann. Sci. École Norm. Sup.*, 55(1):261–282, 2022.

[Ber20] D. Bertrand. Revisiting Manin's theorem of the kernel. *Ann. Fac. Sci. Toulouse Math. (6)*, 29(5):1301–1318, 2020.

[BL04] C. Birkenhake and H. Lange. *Complex Abelian Varieties.* Springer, 2004.

[BMZ07] E. Bombieri, D.W. Masser, and U. Zannier. Anomalous Subvarieties - Structure Theorems and Applications. *Int. Math. Res. Not. IMRN*, (19):1–33, 2007.

[BMZ08] E. Bombieri, D. Masser, and U. Zannier. On unlikely intersections of complex varieties with tori. *Acta Arith.*, 133(4):309–323, 2008.

[BSCFN23] D. Blázquez-Sanz, G. Casale, J Freitag, and J. Nagloo. A differential approach to the Ax-Schanuel, I. *arXiv:2102.03384*, 2023.

[CMZ18] P. Corvaja, D. Masser, and U. Zannier. Torsion hypersurfaces on abelian schemes and Betti coordinates. *Mathematische Annalen*, 371(3):1013–1045, 2018.

[Con06] B. Conrad. Chow's K/k-image and K/k-trace, and the Lang-Néron theorem. *Enseign. Math. (2)*, 52(1-2):37–108, 2006.

[CTZ23] P. Corvaja, J. Tsimerman, and U. Zannier. Finite Orbits in Surfaces with a Double Elliptic Fibration and Torsion Values of Sections. *preprint arXiv:2302.00859*, 2023.

[Del71] P. Deligne. Théorie de Hodge : II. *Publications Mathématiques de l'IHÉS*, 40:5–57, 1971.

[Del81] Pierre Deligne. Le groupe fondamental du complément d'une courbe plane n'ayant que des points doubles ordinaires est abélien (d'après W. Fulton). In *Bourbaki Seminar, Vol. 1979/80*, volume 842 of *Lecture Notes in Math.*, pages 1–10. Springer, Berlin-New York, 1981.

[DGH21] V. Dimitrov, Z. Gao, and P. Habegger. Uniformity in Mordell–Lang for curves. *Annals of Mathematics*, 194(1):237–298, 2021.

[FC90] G. Faltings and C.-L. Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.

[Gao17a] Z. Gao. A special point problem of André-Pink-Zannier in the universal family of Abelian varieties. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 17(1):231–266, 2017.

[Gao17b] Z. Gao. Towards the André-Oort conjecture for mixed Shimura varieties: the Ax-Lindemann-Weierstrass theorem and lower bounds for Galois orbits of special points. *J.Reine Angew. Math (Crelle)*, 732:85–146, 2017.

[Gao20a] Z. Gao. Generic rank of Betti map and unlikely intersections. *Compos. Math.*, 156(12):2469–2509, 2020.

[Gao20b] Z. Gao. Mixed Ax-Schanuel for the universal abelian varieties and some applications. *Compos. Math.*, 156(11):2263–2297, 2020.

[Gao21] Z. Gao. Distribution of points on varieties: various aspects and interactions. HDR (Habilitation à Diriger des Recherches), Sorbonne Université, 2021.

[GGK21] Z. Gao, T. Ge, and L. Kühne. The uniform Mordell–Lang conjecture. *arXiv: 2105.15085*, 2021.

[GW10] U. Görtz and T. Wedhorn. *Algebraic geometry I.* Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.

[Hab13] P. Habegger. Torsion points on elliptic curves in Weierstrass form. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 12(3):687–715, 2013.

[Küh21] L. Kühne. Equidistribution in families of abelian varieties and uniformity. *arXiv: 2101.10272*, 2021.

[Mas96] D. W. Masser. Specializations of endomorphism rings of abelian varieties. *Bull. Soc. Math. France*, 124(3):457–476, 1996.

[Moo98] B. Moonen. Linearity properties of Shimura varieties. I. *J. Algebraic Geom.*, 7(3):539–567, 1998.

[MZ08] D.W. Masser and U. Zannier. Torsion anomalous points and families of elliptic curves. *Comptes Rendus Mathematique*, 346(9):491–494, 2008.

[MZ12] D. Masser and U. Zannier. Torsion points on families of squares of elliptic curves. *Mathematische Annalen*, 352(2):453–484, 2012.

[MZ14] D. Masser and U. Zannier. Torsion points on families of products of elliptic curves. *Advances in Mathematics*, 259:116 – 133, 2014.

[MZ15] D. Masser and U. Zannier. Torsion points on families of simple abelian surfaces and Pell's equation over polynomial rings (with an appendix by E. V. Flynn). *Journal of the European Mathematical Society*, 17:2379–2416, 2015.

[MZ20] D. Masser and U. Zannier. Torsion points, Pell's equation, and integration in elementary terms. *Acta Mathematica*, 225(2):227–312, 2020.

[Pin05] R. Pink. A combination of the conjectures of Mordell-Lang and André-Oort. In *Geometric methods in algebra and number theory*, volume 235 of *Progr. Math.*, pages 251–282. Birkäuser, 2005.

[Rém09] G. Rémond. Intersection de sous-groupes et de sous-variétés. III. *Comment. Math. Helv.*, 84(4):835–863, 2009.

[Sil92] A. Silverberg. Fields of definition for homomorphisms of abelian varieties. *J. Pure Appl. Algebra*, 77(3):253–262, 1992.

[Sto17] M. Stoll. Simultaneous torsion in the Legendre family. *Exp. Math.*, 26(4):446–459, 2017.

[UY11] E. Ullmo and A. Yafaev. A characterisation of special subvarieties. *Mathematika*, 57(2):263–273, 2011.

[Zan12] U. Zannier. *Some problems of unlikely intersections in arithmetic and geometry*, volume 181 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012. With appendixes by David Masser.

[Zha98] S Zhang. Small points and Arakelov theory. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 217–225, 1998.

February 13, 2023

IAZD (Institute of Algebra, Number Theory and Discrete Mathematics), Leibniz University Hannover, Welfengarten 1, 30167 Hannover, Germany

Email address: ziyang.gao@math.uni-hannover.de

Department of Mathematics and Computer Science, University of Basel, Spiegelgasse 1, 4051 Basel, Switzerland

Email address: philipp.habegger@unibas.ch