HEIGHTS IN FAMILIES OF ABELIAN VARIETIES AND THE GEOMETRIC BOGOMOLOV CONJECTURE

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Abstract. On an abelian scheme over a smooth curve over $\overline{\mathbb{Q}}$ a symmetric relatively ample line bundle defines a fiberwise Néron–Tate height. If the base curve is inside a projective space, we also have a height on its $\overline{\mathbb{Q}}$-points that serves as a measure of each fiber, an abelian variety. Silverman proved an asymptotic equality between these two heights on a curve in the abelian scheme. In this paper we prove an inequality between these heights on a subvariety of any dimension of the abelian scheme. As an application we prove the Geometric Bogomolov Conjecture for the function field of a curve defined over $\overline{\mathbb{Q}}$. Using Moriwaki's height we sketch how to extend our result when the base field of the curve has characteristic 0.

Contents

1. Introduction

In 1998, Ullmo [49] and S. Zhang [66] proved the Bogomolov Conjecture over number fields. However its analog over function fields, which came to be known as the Geometric Bogomolov Conjecture, remains open in full generality.

The main goal of this paper is to prove a height inequality on a subvariety of an abelian scheme over a smooth curve over $\overline{\mathbb{Q}}$, Theorem 1.4. It is then not hard to deduce the Geometric Bogomolov Conjecture over the function field of a curve in the characteristic

2000 *Mathematics Subject Classification.* 11G10, 11G50, 14G25, 14K15.

0 case. See §11 and Appendix A. Our height inequality may be of independent interest and does not seem to follow from the Geometric Bogomolov Conjecture. It can serve as a substitute in higher dimension of Silverman's Height Limit Theorem [46], used by Masser and Zannier [37] to prove a first case of the relative Manin–Mumford Conjecture for sections of the base curve; we refer to Pink's work [42] and Zannier's book [63] on such problems.

Let k be an algebraically closed field of characteristic 0, K a field extension of k, and $\overline{K}$ a fixed algebraic closure of K. Let A be an abelian variety over K. We let $A^{\overline{K}/k}$ denote the $\overline{K}/k$-trace of $A \otimes_K \overline{K}$; it is an abelian variety over k and we have a trace map $\tau_{A,\overline{K}/k} \colon A^{\overline{K}/k} \otimes_k \overline{K} \to A \otimes_K \overline{K}$, which is a closed immersion since $\mathrm{char}(k) = 0$. By abuse of notation we consider $A^{\overline{K}/k} \otimes_k \overline{K}$ as an abelian subvariety of $A \otimes_K \overline{K}$. We refer to §2 for references and more information on the trace.

Suppose now that K is the function field of a smooth projective curve over k. In particular, we have $\mathrm{trdeg}(K/k) = 1$.

Let L be a symmetric ample line bundle on A. We can attach to A, L, and K the Néron–Tate height $\hat{h}_{K,A,L} \colon A(\overline{K}) \to [0,\infty)$, see §2.1 for additional background on the Néron–Tate height. This height satisfies: For any $P \in A(\overline{K})$ we have

$$\hat{h}_{K,A,L}(P) = 0 \text{ if and only if } P \in (A^{\overline{K}/k} \otimes_k \overline{K})(\overline{K}) + A_{\mathrm{tor}},$$

here A_{tor} denotes the subgroup of points of finite order of $A(\overline{K})$.

A *coset* in an abelian variety is the translate of an abelian subvariety, we call it a *torsion coset* if it contains a point of finite order.

Our main result towards the Geometric Bogomolov Conjecture is the following theorem. We first concentrate on the important case $k = \overline{\mathbb{Q}}$.

Theorem 1.1. *We keep the notation from above and assume $k = \overline{\mathbb{Q}}$. Let X be an irreducible, closed subvariety of A defined over K such that $X \otimes_K \overline{K}$ is irreducible and not of the form $B + (Z \otimes_k \overline{K})$ for some closed irreducible subvariety Z of $A^{\overline{K}/k}$ and some torsion coset B in $A \otimes_K \overline{K}$. Then there exists a constant $\epsilon > 0$ such that*

$$\{x \in X(\overline{K}) : \hat{h}_{K,A,L}(x) \leq \epsilon\}$$

is not Zariski dense in X.

In Appendix A we sketch a proof for when k is any algebraically closed field of characteristic 0 using Moriwaki's height.

Yamaki [59, Conjecture 0.3] proposed a general conjecture over function fields which we will call the Geometric Bogomolov Conjecture; it allows $\mathrm{trdeg}(K/k)$ to be greater than 1 and k algebraically closed of arbitrary characteristic. The reference to geometry distinguishes Yamaki's Conjecture from the arithmetic counterpart over a number field.

The Geometric Bogomolov Conjecture was proven by Gubler [26] when A is totally degenerate at some place of K. He has no restriction on the characteristic of k and does not assume that K/k has transcendence degree 1. When X is a curve embedded in its Jacobian A and when $\mathrm{trdeg}(K/k) = 1$, Yamaki dealt with nonhyperelliptic curves of genus 3 in [57] and with hyperelliptic curves of any genus in [58]. If moreover $\mathrm{char}(k) = 0$, Faber [18] proved the conjecture for X of small genus (up to 4, effective) and Cinkir [13] covered the case of arbitrary genus. Prior to these work, Moriwaki also gave some partial results in [38]. Yamaki [60] reduced the Geometric Bogomolov Conjecture to the case where A has good reduction everywhere and has trivial $\overline{K}/k$-trace. He also proved the cases (co)dim$X = 1$ [62] and $\dim(A \otimes_K \overline{K}/(A^{\overline{K}/k} \otimes_k \overline{K})) \leq 5$ [61]. As in Gubler's setup, Yamaki works in arbitrary characteristic and has no restriction on K/k. These results involve techniques ranging from analytic tropical geometry [27] to Arakelov theory; the latter method overlaps with Ullmo and S. Zhang's original approach for number fields.

Our approach differs and is based on a height inequality on a model of A to be stated below in Theorem 1.4 (see Appendix A for a version involving the Moriwaki height). In a recent collaboration with Cantat and Xie [11] we were able to resolve the Geometric Bogomolov Conjecture completely in characteristic 0. While the methods in [11] there were motivated by those presented here, they do not bypass through or recover a height inequality such as Theorem 1.4.

To prove Theorem 1.1 we must work in the relative setting. Let us setup the notation. Let S be a smooth irreducible curve over k, and let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$. Let A be the generic fiber of $\mathcal{A} \to S$; it is an abelian variety over $k(S)$, the function field of S. We will prove the Geometric Bogomolov Conjecture for A and $K = k(S)$. Let us also fix an algebraic closure $\overline{k(S)}$ of $k(S)$.

Definition 1.2. *An irreducible closed subvariety Y of $\mathcal{A}$ is called a* ***generically special*** *subvariety of $\mathcal{A}$, or just generically special, if it dominates S and if its geometric generic fiber $Y \times_S \operatorname{Spec} \overline{k(S)}$ is a finite union of $(Z \otimes_k \overline{k(S)}) + B$, where Z is a closed irreducible subvariety of $A^{\overline{k(S)}/k}$ and B is a torsion coset in $A \otimes_{k(S)} \overline{k(S)}$.*

For any irreducible closed subvariety X of $\mathcal{A}$, we set

$$X^* = X \setminus \bigcup_{\substack{Y \subseteq X \\ Y \text{ is a generically special} \\ \text{subvariety of } \mathcal{A}}} Y.$$

We start with the following proposition which clarifies the structure of X^*. Its proof relies on a uniform version of Raynaud's [44] resolution of the Manin–Mumford Conjecture in characteristic 0 as well as the Lang–Néron Theorem, the generalization of the Mordell–Weil Theorem to finitely generated fields.

Proposition 1.3. *Let X and $\mathcal{A}$ be as above. There are at most finitely many generically special subvarieties of $\mathcal{A}$ that are contained in X, maximal with respect to the inclusion for this property. In particular, X^* is Zariski open in X and it is empty if and only if X is generically special.*

Let us now assume $k = \overline{\mathbb{Q}}$ and turn to height functions. We write $h(\cdot)$ for the absolute logarithmic Weil height on projective space.

Let $\overline{S}$ be a smooth projective curve over $\overline{\mathbb{Q}}$ containing S as a Zariski open and dense subset. Let $\overline{\mathcal{M}}$ be an ample line bundle on $\overline{S}$ and let $\mathcal{M} = \overline{\mathcal{M}}|_S$. The Height Machine [7, Chapter 2.4] attaches to $(\overline{S}, \overline{\mathcal{M}})$ a function $\overline{S}(\overline{\mathbb{Q}}) \to \mathbb{R}$ that is well defined up to addition of a bounded function. Let $h_{S,\mathcal{M}}$ be the restriction to $S(\overline{\mathbb{Q}})$ of a representative of this class of functions. As $\overline{\mathcal{M}}$ is ample on $\overline{S}$, we may take such a representative that $h_{S,\mathcal{M}}(s) \geq 0$ for each $s \in S(\overline{\mathbb{Q}})$.

Let $\mathcal{L}$ be a relatively ample and symmetric line bundle on $\mathcal{A}/S$ defined over $\overline{\mathbb{Q}}$. Then for any $s \in S(\overline{\mathbb{Q}})$, the line bundle $\mathcal{L}_s$ on the abelian variety $\mathcal{A}_s = \pi^{-1}(s)$ is symmetric;

note that $\mathcal{A}_s$ is defined over $\overline{\mathbb{Q}}$. Tate's Limit Process provides a fiberwise Néron–Tate height $\hat{h}_{\mathcal{A}_s,\mathcal{L}_s}\colon \mathcal{A}_s(\overline{\mathbb{Q}}) \to [0,\infty)$. It is determined uniquely by the restriction of $\mathcal{L}$ to $\mathcal{A}_s$, there is no need to fix a representative here. Finally define $\hat{h}_{\mathcal{A},\mathcal{L}}\colon \mathcal{A}(\overline{\mathbb{Q}}) \to [0,\infty)$ to be the *total Néron–Tate height* given by $P \mapsto \hat{h}_{\mathcal{A}_{\pi(P)},\mathcal{L}_{\pi(P)}}(P)$ for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$.

These two height functions are unrelated in the following sense. It is not difficult to construct an infinite sequence of points $P_1, P_2, \ldots \in \mathcal{A}(\overline{\mathbb{Q}})$ such that $\hat{h}_{\mathcal{A},\mathcal{L}}(P_i)$ is constant and $h_{S,\mathcal{M}}(\pi(P_i))$ unbounded; just take P_i of finite order in $\mathcal{A}_{\pi(P_i)}$ and the sequence $\pi(P_i)$ of unbounded height.

The main technical result of this paper is a height inequality that relates these two heights on an irreducible subvariety X of $\mathcal{A}$. The discussion in the last paragraph suggests that we should at least remove all curves in X that dominate S and contain infinitely many points of finite order. This turns out to be insufficient and we must also remove subvarieties that are contained in constant abelian subschemes of $\mathcal{A}$. In fact, we must remove precisely generically the special subvarieties of $\mathcal{A}$ from Definition 1.2 that are contained in X.

Theorem 1.4. *Let $\pi\colon \mathcal{A} \to S$, $\mathcal{L}$ and $\mathcal{M}$ be as above with $k = \overline{\mathbb{Q}}$ and $\dim S = 1$. Let X be a closed irreducible subvariety of $\mathcal{A}$ over $\overline{\mathbb{Q}}$ and let X^* be as above Proposition 1.3. Then there exists $c > 0$ such that*

$$h_{S,\mathcal{M}}(\pi(P)) \leq c\left(1 + \hat{h}_{\mathcal{A},\mathcal{L}}(P)\right) \quad \text{for all} \quad P \in X^*(\overline{\mathbb{Q}}). \tag{1.1}$$

Suppose X dominates S, so we think of X as a family of $(\dim X - 1)$-dimensional varieties. Then our height inequality (1.1) can be interpreted as a uniform version of the Bogomolov Conjecture along the 1-dimension base S if $h_{S,\mathcal{M}}(\pi(P)) \geq 2c$. Indeed, then $\hat{h}_{\mathcal{A},\mathcal{L}}(P) \geq \frac{1}{2c} h_{S,\mathcal{M}}(\pi(P))$. From this point of view it would be interesting to have an extension of Theorem 1.4 to $\dim S > 1$. For the main obstacle to pass from $\dim S = 1$ to the general case, we refer to §1.1, Part 1 and above.

Theorem 1.4 was proven by the second-named author [28] when $\mathcal{A}$ is a fibered power of a non-isotrivial 1-parameter family of elliptic curves. This theorem had applications towards special points problems [28, Theorems 1.1 and 1.2] and towards some cases of the relative Manin–Mumford Conjecture [29].

After this work was submitted, Ben Yaacov and Hrushovski informed the authors of their similar height inequality for a 1-parameter family of genus $g \geq 2$ curves in an unpublished note [4] by reducing it to Cinkir's result [13].

In this paper we treat arbitrary abelian schemes over algebraic curves, possibly with non-trivial isotrivial part, and hope to extend the aforemention applications in future work.

Before proceeding, we point out that we shall prove Theorem 1.4 for a particular relatively ample line bundle $\mathcal{L}$ on $\mathcal{A}/S$ that is fiberwise symmetric and a particular ample line bundle $\mathcal{M}$ on $\overline{S}$. Then Theorem 1.4 holds for arbitrary such $\mathcal{L}$ and $\mathcal{M}$ by formal properties of the Height Machine. Moreover we will prove the following slightly stronger form of Theorem 1.4.

We may attach a third height function on $\mathcal{A}$ in the following way. Let $\mathcal{L}' = \mathcal{L} \otimes \pi^*\mathcal{M}$. By [43, Théorème XI 1.4] and [22, Corollaire 5.3.3 and Proposition 4.1.4], our abelian scheme admits a closed immersion $\iota\colon \mathcal{A} \to \mathbb{P}^M_{\overline{\mathbb{Q}}} \times S$ over S arising from $(\mathcal{L}')^{\otimes n}$ for some

$n \gg 1$. As we will see in §2.2 the existence of a closed immersion is more straightforward if we allow ourselves to remove finitely many points from S, a procedure that is harmless in view of our application. Define the *naive height* of P to be $h_{\mathcal{A},\mathcal{L}'}(P) = \frac{1}{n}h(P') + h_{S,\mathcal{M}}(\pi(P))$ where $\iota(P) = (P', \pi(P)) \in \mathbb{P}^M_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) \times S(\overline{\mathbb{Q}})$.

Theorem 1.4′. *Let $\pi\colon \mathcal{A} \to S$ and ι be as above with $k = \overline{\mathbb{Q}}$ and $\dim S = 1$. Let X be a closed irreducible subvariety of $\mathcal{A}$ over $\overline{\mathbb{Q}}$ and let X^* be as above Proposition 1.3. Then there exists $c > 0$ such that*

$$h_{S,\mathcal{M}}(\pi(P)) \le h_{\mathcal{A},\mathcal{L}'}(P) \le c\left(1 + \hat{h}_{\mathcal{A},\mathcal{L}}(P)\right) \quad \textit{for all} \quad P \in X^*(\overline{\mathbb{Q}}).$$

1.1. **Outline of Proof of Theorems 1.1 and 1.4 and Organization of the Paper.** We give an overview of the proof of Theorem 1.4 in three parts.

Consider an abelian scheme $\pi\colon \mathcal{A} \to S$ over a smooth algebraic curve S of relative dimension $g \ge 1$.

The Ax–Schanuel Theorem [3] is a function theoretic version of the famous and open Schanuel Conjecture in transcendence theory. Stated for algebraic groups, the case of an abelian variety deals with algebraic independence of functions defined using the uniformizing map. It has seen many applications to problems in diophantine geometry [63].

For our purpose we need an Ax-Schanuel property for families of abelian varieties which is not yet available. However the assumption $\dim S = 1$ simplifies the situation: Instead of the full power of functional transcendence, we only need to study a functional constancy property. The first part of the proof deals with this functional constancy property where the so-called Betti map plays the role of the uniformizing map. We briefly explain this map and refer to §4 for more details.

Part 1: The Betti Map and a Functional Constancy Property. Any point of $S(\mathbb{C})$ has a complex neighborhood that we can biholomorphically identify with the open unit disc $\Delta \subseteq \mathbb{C}$. The fiber of $\mathcal{A} \to S$ above a point $s \in \Delta$ is biholomorphic to a complex torus $\mathbb{C}^g/\Omega(s)\mathbb{Z}^{2g}$ where the columns of $\Omega(s) \in \mathrm{Mat}_{g,2g}(\mathbb{C})$ are a period lattice basis. Of course $\Omega(s)$ is not unique. The choice of a period lattice basis $\Omega(s)$ enables us to identify $\mathcal{A}_s$ with $\mathbb{T}^{2g}$ as real Lie groups, with $\mathbb{T}$ the unit circle in $\mathbb{C}$. As Δ is simply connected, we can arrange that the period map $s \mapsto \Omega(s)$ is holomorphic on Δ. In turn we can identify $\mathcal{A}_\Delta = \pi^{-1}(\Delta)$ with the constant family $\Delta \times \mathbb{T}^{2g}$ as families over Δ. This can be done in a way that we get group isomorphisms $\mathcal{A}_s(\mathbb{C}) \to \mathbb{T}^{2g}$ fiberwise. Note that the complex structure is lost, and that the isomorphism in play is only real analytic.

The *Betti map* $b\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ is the composite of the isomorphism $\mathcal{A}_\Delta \cong \Delta \times \mathbb{T}^{2g}$ with the projection to $\mathbb{T}^{2g}$. It depends on several choices, but two different Betti maps on $\mathcal{A}_\Delta$ differ at most by composing with a continuous automorphism of $\mathbb{T}^{2g}$.

Let X be an irreducible closed subvariety of $\mathcal{A}$ that dominates S. In §5 we study the restriction of b to X^{an}; the superscript $^{\mathrm{an}}$ denotes complex analytification. We say that X is *degenerate* if the restriction of $b\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ to $X^{\mathrm{an}} \cap \mathcal{A}_\Delta$ has positive dimensional fibers on a non-empty, open subset X^{an}; being open refers to the complex topology. The main result of §5, Theorem 5.1, states that a degenerate subvariety is generically special. This will allow us to explain the analytic notion of degeneracy in purely algebraic terms.

Let us give some ideas of what goes into the proof of Theorem 5.1.

In general, the period mapping $s \mapsto \Omega(s)$ cannot extend to the full base due to monodromy. This obstruction is a powerful tool in our context. Indeed, fix a base point

$s \in S^{\mathrm{an}}$. Monodromy induces a representation on the fundamental group $\pi_1(S^{\mathrm{an}}, s) \to \mathrm{Aut}(H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z}))$. Moreover, by transporting along the fibers of the Betti map above a loop in S^{an} we obtain a representation $\pi_1(S^{\mathrm{an}}, s) \to \mathrm{Aut}(\mathcal{A}_s^{\mathrm{an}})$ whose target is the group of real analytic automorphisms of $\mathcal{A}_s$. This new representation induces the representation on homology. Moreover, we can identify $\mathrm{Aut}(\mathcal{A}_s^{\mathrm{an}})$ with $\mathrm{GL}_{2g}(\mathbb{Z})$ because $\mathcal{A}_s$ and $\mathbb{T}^{2g}$ are isomorphic in the real analytic category. So the canonical mapping $\mathrm{Aut}(\mathcal{A}_s^{\mathrm{an}}) \to \mathrm{Aut}(H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z}))$ is an isomorphism of groups.

Now suppose that X is degenerate. The assumption $\dim S = 1$ forces that $X^{\mathrm{an}} \cap \mathcal{A}_\Delta = b^{-1}(b(X^{\mathrm{an}} \cap \mathcal{A}_\Delta))$.[1] In other words the fibers $X_s = \pi|_X^{-1}(s)$ do not depend on $s \in \Delta$ for the identification $\mathcal{A}_s^{\mathrm{an}} = \mathbb{T}^{2g}$. So the action of $\pi_1(S^{\mathrm{an}}, s)$ on $\mathcal{A}_s^{\mathrm{an}}$ leaves X_s invariant. Thus it suffices to understand subsets of $\mathcal{A}_s$ that are invariant under the action of a subgroup of $\mathrm{GL}_{2g}(\mathbb{Z})$. We use Deligne's Theorem of the Fixed Part [16] and the Tits Alternative [48] to extract information from this subgroup. Indeed, under a natural hypothesis on $\mathcal{A}$, the image of the representation in $\mathrm{GL}_{2g}(\mathbb{Z})$ contains a free subgroup on two generators. In particular, the image is a group of exponential growth. We then use a variant of the Pila–Wilkie Counting Theorem [40], due to Pila and the second-named author, and Ax's Theorem [3] for a constant abelian variety. From this we will be able to conclude that X is generically special if it is degenerate.

Let us step back and put some of these ideas into a historic perspective. In the special case where $\mathcal{A}$ is the fibered power of the Legendre family of elliptic curves, the second-named author [28] used local monodromy to investigate degenerate subvarieties. In the current work local monodromy is insufficient as S could be complete to begin with. So we need global information. Zannier introduced the point counting strategy and together with Pila gave a new proof of the Manin–Mumford Conjecture [41] using the Pila–Wilkie Theorem [40]. Masser and Zannier [37] showed the usefulness of the Betti coordinates for problems in diophantine geometry by solving a first case of the relative Manin–Mumford Conjecture. Ullmo and Yafaev [50] exploited exponential growth in groups in combination with the Pila–Wilkie Theorem to prove their hyperbolic Ax–Lindemann Theorem for projective Shimura varieties. A recent result of André, Corvaja, and Zannier [1] also deals with the rank of the Betti map on the moduli space of principally polarized abelian varieties of a given dimension. More recently, Cantat, Xie, and the authors [11] gave a different approach to the functional constancy problem that does not rely on the Pila–Wilkie Theorem but rather on results from dynamical systems.

Part 2: Eliminating the Néron–Tate Height. The second part of the proof deals with reducing the height inequality in Theorem 1.4 to one that only involves Weil heights; we refer to §2.1 for nomenclature on heights.

We will embed S into $\mathbb{P}^m$ and $\mathcal{A}$ into $\mathbb{P}^M \times \mathbb{P}^m$ such that $\pi\colon \mathcal{A} \to S$ is compatible with the projection $\mathbb{P}^M \times \mathbb{P}^m \to \mathbb{P}^m$ and other technical conditions are fulfilled. Let $h(P, Q) = h(P) + h(Q)$ for $P \in \mathbb{P}^M(\overline{\mathbb{Q}}), Q \in \mathbb{P}^m(\overline{\mathbb{Q}})$, where h denotes the absolute logarithmic Weil height on projective space. For an integer N let $[N]$ denote the multiplication-by-N morphism $\mathcal{A} \to \mathcal{A}$.

[1] This is no longer true if $\dim S > 1$, making the remaining argument in this part fail for $\dim S > 1$.

Let $N \geq 1$ be a sufficiently large integer (which we assume to be a power of 2 for convenience). If X is not generically special we show in Proposition 9.1 that

$$N^2 h(P) \leq c_1 h([N](P)) + c_2(N) \tag{1.2}$$

for all $P \in U(\overline{\mathbb{Q}})$ where U is Zariski open and dense in X and where $c_1 > 0$ and $c_2(N)$ are both independent of P. Note that U and $c_2(N)$ may depend on N.

One is tempted to divide (1.2) by N^2 and take the limit $N \to \infty$ as in Tate's Limit Process. However, this is not possible *a priori*, as U and $c_2(N)$ could both depend on N. So we mimic Masser's strategy of "killing Zimmer constants" explained in [63, Appendix C]. This step is carried out in §10 where we terminate Tate's Limit Process after finitely many steps when N is large enough in terms of c_1; for this it is crucial that c_1 is independent of N.

Part 3: Counting Lattice Points and an Inequality for the Weil Height. At this stage we have reduced proving Theorem 1.4 to (1.2) if X is not generically special. Recall that from Part 1 of the proof we know that X is not degenerate. Therefore, the restricted Betti map $b|_{X^{\mathrm{an}}} : X^{\mathrm{an}} \to \mathbb{T}^{2g}$ has discrete fibers. The image of this restriction has the same real dimension as X^{an} (the dimension is well-defined as the image is subanalytic).

Part 3a: The Hypersurface Case. To warm up let us assume for the moment that X is a hypersurface in $\mathcal{A}$, so $\dim X = g$. In this case the image $b(X^{\mathrm{an}} \cap \mathcal{A}_\Delta)$ contains a non-empty, open subset of $\mathbb{T}^{2g}$. By a simple Geometry of Numbers argument in the covering $\mathbb{R}^{2g} \to \mathbb{T}^{2g}$, the image contains $\gg N^{2g}$ points of order dividing N; the implicit constant is independent of N. As the Betti map is a group isomorphism on each fiber of $\mathcal{A}$ we find that X contains $\gg N^{2g}$ points of order dividing N.

Obtaining (1.2) requires an auxiliary rational map $\varphi\colon \mathcal{A} \dashrightarrow \mathbb{P}^g$. Suppose for simplicity that we can choose φ such that $\varphi^{-1}([1 : 0 : \cdots : 0])$ is the image of the zero section $S \to \mathcal{A}$. Then the composition $\varphi \circ [N]$ restricts to a rational map $X \dashrightarrow \mathbb{P}^g$ that maps the $\gg N^{2g}$ torsion points constructed above to $[1 : 0 : \cdots : 0]$.

If we are lucky and all these torsion points are isolated in the fiber of $\varphi \circ [N]$, then $\deg(\varphi \circ [N]) \gg N^{2g}$. A height-theoretic lemma [28], restated here as Lemma 9.4, implies (1.2) for N a power of 2. The factor N^2 on the left in (1.2) equals $N^{2\dim X}/N^{2(\dim X - 1)}$ and has the following interpretation. The numerator comes from the degree lower bound as $\dim X = g$. The denominator is a consequence of the following fact. Given a suitable embedding of an abelian variety into some projective space, the duplication morphism can be described by a collection of homogeneous polynomials of degree $2^2 = 4$. So $[N]$ can be described by homogeneous polynomials of degree $\leq N^2$.

If we are less lucky and some torsion point is not isolated in $\varphi \circ [N]$, then an irreducible component of $\ker[N] \subseteq \mathcal{A}$ is contained in X. This situation is quite harmless, as roughly speaking, it cannot happen too often for a variety that is not generically special.

The restriction $\dim X = g$ is more serious, however. The second-named author was able to reduce [28] to the hypersurface case inside a fibered power of the Legendre family of elliptic curves. This is not possible for general $\mathcal{A}$, so we must proceed differently.

Part 3b: The General Case. For general X we will still construct a suitable $\varphi : X \dashrightarrow \mathbb{P}^{\dim X}$ as above and apply Lemma 9.4. As a stepping stone we first construct in §6 an auxiliary subvariety Z of $\mathcal{A}$ in sufficiently general position such that

$$\dim X + \dim Z = \dim \mathcal{A} = g + 1. \tag{1.3}$$

The rational map φ is constructed using Z in §9, and one should think of Z as an irreducible component of $\varphi|_X^{-1}([1:0:\cdots:0])$. Being in general position and (1.3) mean that $\varphi|_X$ has finite generic fiber on its domain.

Now we want $\varphi\circ[N]\colon X \to \mathbb{P}^{\dim X}$ to have degree $\gg N^{2\dim X}$ and for this it suffices to find $\gg N^{2\dim X}$ isolated points in the preimage of $[1:0:\cdots:0]$. As $\varphi(Z) = [1:0:\cdots:0]$ we need to find $\gg N^{2\dim X}$ isolated points in $X \cap [N]^{-1}(Z)$. (Additional verifications must be made to ensure that isolated intersection points lead to isolated fibers.)

We ultimately construct these points using the Geometry of Numbers. More precisely, we need a volume estimate and Blichfeldt's Theorem. Since X is not degenerate we have a point around which the local behavior of X is similar to the local behavior of its image in $\mathbb{T}^{2g}$ under the Betti map. This allows us to linearize the problem as follows. In order to count the number of such $X \cap [N]^{-1}(Z)$, we can instead count points $x \in \mathbb{T}^{2g}$, coming from points of $X(\mathbb{C})$ under the Betti map, such that $Nx = z$ lies in the image of $Z(\mathbb{C})$. If we let x, z range over the image under the Betti map of small enough open subsets of $X(\mathbb{C})$ and $Z(\mathbb{C})$, then $Nx - z$ ranges over an open subset of $\mathbb{T}^{2g}$. This conclusion makes crucial use of the fact that X is not degenerate and that Z is in general position. Lifting via the natural map $\mathbb{R}^{2g} \to \mathbb{T}^{2g}$ we are led to the counting lattices points. Indeed, we must construct elements of $N\tilde{x} - \tilde{z} \in \mathbb{Z}^{2g}$ where $\tilde{x}, \tilde{z}$ are lifts of points x, z as before. We denote the set of all possible $N\tilde{x} - \tilde{z}$ by U_N. A careful volume estimate done in §7 leads to $\mathrm{vol}(U_N) \gg N^{2\dim X}$. So we expect to find this many lattice points. But there is no reason to believe that U_N is convex and it is not hard to imagine open subsets of $\mathbb{R}^{2g}$ of arbitrary large volume that meet $\mathbb{Z}^{2g}$ in the empty set. To solve this problem we apply Blichfeldt's Theorem, which claims that some translate $\gamma + U_N$ of U_N contains at least $\mathrm{vol}(U_N)$ lattice points in $\mathbb{Z}^{2g}$.

This approach ultimately constructs enough points to prove a suitable lower bound for the degree of $\varphi\circ[N]$ and to complete the proof. However, additional difficulties arise. For example, we must deal with non-zero γ and making sure that the points constructed are isolated in $X \cap [N]^{-1}(Z)$. These technicalities are addressed in §8.

The Remaining Results. In §3 we prove Proposition 1.3. This section is mainly self-contained and the main tool is a uniform version of the Manin–Mumford Conjecture in characteristic 0.

The proof of Theorem 1.1 in §11 follows the blueprint laid out in [28]. We need to combine our height bound, Theorem 1.4, with Silverman's Height Limit Theorem [46] used in his specialization result.

In Appendix A we sketch how to adapt our height inequality, Theorem 1.4, to more general fields in characteristic 0. This shows how to deduce Theorem 1.1 for any algebraically closed field of characteristic 0. Appendix B contains some comments on the situation when $\dim S > 1$. Finally, in Appendix C we give a self-contained and quantitative version of Brotbek's Hyperbolicity Theorem [9] in the case of an abelian variety (which is much simpler than the general case).

Acknowledgements. Both authors would like to thank the organizers of the conference "On Lang and Vojta's conjectures" in Luminy 2014, where our collaboration began.

They would like to thank Bas Edixhoven for providing the current proof of Proposition 6.7. They would like to thank Shouwu Zhang for providing the proof using the essential minimum at the end of Appendix A. The authors also thank Gabriel Dill and Walter Gubler for useful remarks. ZG would like to thank the Department of Mathematics and Computer Science at the University of Basel for the invitation and for the stimulating atmosphere and SNF grant Nr. 165525 for financial support.

2. Notation

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of positive integers. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

If X is a variety defined over $\mathbb{C}$, then we write X^{an} for $X(\mathbb{C})$ with its structure as a complex analytic space, we refer to Grauert and Remmert's book [21] for the theory of complex analytic spaces.

Given an abelian scheme $\mathcal{A}$ over any base scheme and an integer N, we let $[N]$ denote the multiplication-by-N morphism $\mathcal{A} \to \mathcal{A}$. The kernel of $[N]$ is $\mathcal{A}[N]$, it is a group scheme over the base of $\mathcal{A}$. An endomorphism of $\mathcal{A}$ is a morphism $\mathcal{A} \to \mathcal{A}$ that takes the zero section to itself.

If A is an abelian variety over field K and if $\overline{K} \supseteq K$ is a given algebraic closure of K, then A_{tor} denotes the group of points of finite order of $A(\overline{K})$.

Suppose k is a subfield of K whose algebraic closure in K equals k and $\mathrm{char}(k) = 0$. We write $A^{K/k}$ for the K/k-trace of A and let $\tau_{A,K/k}\colon A^{K/k} \otimes_k K \to A$ denote the associated trace map, we refer to [14, §6] for general facts and the universal property. Note that our notation $A^{K/k}$ is denoted by $\mathrm{Tr}_{K/k}(A)$ in *loc.cit.* By [14, Theorem 6.2 and below] $\tau_{A,K/k}$ is a closed immersion since $\mathrm{char}(k) = 0$. We sometimes consider $A^{K/k} \otimes_k K$ as an abelian subvariety of A.

By abuse of notation we sometimes abbreviate $(A \otimes_K \overline{K})^{\overline{K}/k}$ by $A^{\overline{K}/k}$ and, in this notation, consider $A^{\overline{K}/k} \otimes_k \overline{K}$ as an abelian subvariety of $A \otimes_k \overline{K}$.

2.1. Heights.

A *place* of a number field K is an absolute value $|\cdot|_v : K \to [0, \infty)$ whose restriction to $\mathbb{Q}$ is either the standard absolute value or a p-adic absolute value for some prime p with $|p| = p^{-1}$. We set $d_v = [K_v : \mathbb{R}]$ in the former and $d_v = [K_v : \mathbb{Q}_p]$ in the latter case. The *absolute, logarithmic, projective Weil height*, or just *height*, of a point $P = [p_0 : \ldots : p_n] \in \mathbb{P}^n_{\mathbb{Q}}(K)$ with $p_0, \ldots, p_n \in K$ is

$$h(P) = \frac{1}{[K : \mathbb{Q}]} \sum_v d_v \log \max\{|p_0|_v, \ldots, |p_n|_v\}$$

where the sum runs over all places v of K. The value $h(P)$ is independent of the choice of projective coordinates by the product formula. For this and other basic facts we refer to [7, Chapter 1]. Moreover, the height does not change when replacing K by another number field that contains the $p_0, \ldots, p_n$. Therefore, $h(\cdot)$ is well-defined on $\mathbb{P}^n_{\mathbb{Q}}(\overline{K})$ where $\overline{K}$ is an algebraic closure of K.

In this paper we also require heights in a function field K. With our results in mind, we restrict to the case where $K = k(S)$ and S is a smooth projective irreducible curve over an algebraically closed field k. Let $\overline{K}$ be an algebraic closure of K. In this case, we can construct a height $h_K : \mathbb{P}^n_K(\overline{K}) \to \mathbb{R}$ as follows.

The points $S(k)$ correspond to the set of places $|\cdot|_v$ of K. They extend in the usual manner to finite extensions of K. If $P = [p_0 : \cdots : p_n] \in \mathbb{P}^n_K(\overline{K})$ with $p_0, \ldots, p_n \in K'$, where K' is a finite extension of K, then we set

$$h_K(P) = \frac{1}{[K' : K]} \sum_v d_v \log \max\{|p_0|_v, \ldots, |p_n|_v\}$$

where d_v are again local degrees such that the product formula holds. We refer to [7, §1.3. and 1.4.6] for more details or [14, §8] on generalized global fields. In the function field case we keep K in the subscript of h_K to emphasize that K is our base field. Indeed, in the function field setting one must keep track of the base field "at the bottom" that plays the role of $\mathbb{Q}$ in the number field setting.

Now let K be either a number field or a function field as above. Suppose that A is an abelian variety defined over K that is embedding in some projective space $\mathbb{P}^M_K$ with a symmetric line bundle. Tate's Limit Process induces the *Néron–Tate* or *canonical height* on $A(\overline{K})$. If K is a number field, we write

$$\hat{h}_A(P) = \lim_{N\to\infty} \frac{h([2^N](P))}{4^N} \tag{2.1}$$

for the Néron–Tate height on $A(\overline{K})$, we refer to [7, Chapter 9.2] for details. The Néron–Tate height depends also on the choice of the symmetric, ample line bundle, but we do not mention it in $\hat{h}_A$.

The construction in the function field is the same. For the same reason as above we retain the symbol K and write $\hat{h}_{A,K}$ for the Néron–Tate height on $A(\overline{K})$.

2.2. **Embedding our Abelian Scheme.** In the paper we are often in the following situation. Let k be an algebraically closed subfield of $\mathbb{C}$. Let S be a smooth irreducible algebraic curve over k and let $\mathcal{A}$ be an abelian scheme of relative dimension $g \geq 1$ over S with structural morphism $\pi\colon \mathcal{A} \to S$.

Let us now see how to embed $\mathcal{A}$ into $\mathbb{P}^M_S = \mathbb{P}^M_k \times S$ for some $M > 0$ after possibly removing finitely many points from S. Note that removing finitely many points is harmless in the context of our problems. Indeed, our Theorem 1.4 is not weakened by this action and so we do it at leisure.

The generic fiber A of $\mathcal{A} \to S$ is an abelian variety defined over the function field of S. Let L be a symmetric ample line bundle on A. Then $L^{\otimes 3}$ is very ample. Replace L by $L^{\otimes 3g}$. A basis of $H^0(A, L)$ gives a projectively normal closed immersion $A \to \mathbb{P}^M_{k(S)}$ for some $M > 0$.

We take the scheme theoretic image $\mathcal{A}'$ of $A \to \mathbb{P}^M_{k(S)} \to \mathbb{P}^M_S$, hence $\mathcal{A}'$ is the Zariski closure of the image of A in $\mathbb{P}^M_S$ with the reduced induced structure. After removing finitely many points of S we obtain an abelian scheme $\mathcal{A}' \subseteq \mathbb{P}^M_S$ such that the morphism from A to the generic fiber of $\mathcal{A}' \to S$ is an isomorphism. An abelian scheme over S is the Néron model of its generic fiber, so the Néron mapping property holds. Therefore the canonical morphism $\mathcal{A} \to \mathcal{A}'$ is an isomorphism and we have thus constructed a closed immersion

$$\iota_S\colon \mathcal{A} \to \mathbb{P}^M_S = \mathbb{P}^M_k \times S.$$

Note that L is the generic fiber of the relatively very ample line bundle $\mathcal{L} = \iota_S^*\mathcal{O}_S(1)$ on $\mathcal{A}/S$. Moreover for any $s \in S(k)$, we have that $\mathcal{L}_s$ is the g-th tensor-power of a very ample line bundle on $\mathcal{A}_s$.

We may furthermore find an immersion (which need not be open or closed) of S into some $\mathbb{P}_k^m$. Composing yields the desired immersion $\mathcal{A} \to \mathbb{P}_k^M \times \mathbb{P}_k^m$. By abuse of notation we consider $A \subseteq \mathbb{P}_{k(S)}^M$ and $\mathcal{A} \subseteq \mathbb{P}_k^M \times \mathbb{P}_k^m$ from now on. Let us recapitulate.

(A1) We have an immersion $\mathcal{A} \to \mathbb{P}_k^M \times \mathbb{P}_k^m$ such that the diagram involving $\pi\colon \mathcal{A} \to S$ and the projection $\mathbb{P}_k^M \times \mathbb{P}_k^m \to \mathbb{P}_k^m$ commutes. Moreover, for all $s \in S(k)$ the closed immersion $\mathcal{A}_s \to \mathbb{P}_k^M$ is induced by a symmetric very ample line bundle.

Of course this immersion depends on the choice of the immersions of A and of S.

The image of A in $\mathbb{P}_{k(S)}^M$ is projectively normal and $[2]^*L$ is isomorphic to $L^{\otimes 4}$. Therefore, $[2]$ is represented globally by $M+1$ homogeneous polynomials of degree 4 on the image of A. Here the base field is the function field $k(S)$. But we can extend it to the model after possibly removing finitely many points of S. So we may assume the following.

(A2) The morphism $[2]$ is represented globally on $\mathcal{A} \subseteq \mathbb{P}_k^M \times \mathbb{P}_k^m$ by $M+1$ bihomogeneous polynomials, homogeneous of degree 4 in the projective coordinates of $\mathbb{P}_k^M$ and homogeneous of a certain degree in the projective coordinates of $\mathbb{P}_k^m$.

Finally, we explain why we took the additional factor g in the exponent $3g$ of $L^{\otimes 3g}$. By Proposition C.1 we have the additional and useful property.

(A3) For given $s \in S(k)$ and $P \in \mathcal{A}_s$, any generic hyperplane section of $\mathcal{A}_s$ passing through P does not contain a positive dimensional coset in $\mathcal{A}_s$.

At the cost of possibly increasing the factor g we could also refer to Brotbek's deep result [9] for more general projective varieties.

An immersion $\iota\colon \mathcal{A} \to \mathbb{P}_k^M \times \mathbb{P}_k^m$ for which (A1), (A2), and (A3) above are satisfied will be called *admissible*.

The construction above adapts easily to show the following fact. Let A be an abelian variety defined over $k(S)$. After possibly shrinking S we can realize A as the generic fiber of an abelian scheme $\mathcal{A} \to S$ with an admissible immersion $\mathcal{A} \to \mathbb{P}_k^M \times \mathbb{P}_k^m$.

If $k = \overline{\mathbb{Q}}$ we have two height functions on $\mathcal{A}(\overline{\mathbb{Q}})$.

Say $P \in \mathcal{A}(\overline{\mathbb{Q}})$, we write $P = (P', \pi(P))$ with $P' \in \mathbb{P}_{\overline{\mathbb{Q}}}^M(\overline{\mathbb{Q}})$ and $\pi(P) \in \mathbb{P}_{\overline{\mathbb{Q}}}^m(\overline{\mathbb{Q}})$. Then

$$h(P) = h(P') + h(\pi(P)) \tag{2.2}$$

defines our first height $\mathcal{A}(\overline{\mathbb{Q}}) \to [0,\infty)$ which we call the *naive height* on $\mathcal{A}$ (relative to the immersion $\mathcal{A} \subseteq \mathbb{P}_{\overline{\mathbb{Q}}}^M \times \mathbb{P}_{\overline{\mathbb{Q}}}^m$).

The second height is the *fiberwise Néron–Tate* or *canonical height*

$$\hat{h}_{\mathcal{A}}(P) = \hat{h}_{\mathcal{A}_{\pi(P)}}(P), \tag{2.3}$$

cf. (2.1). We obtain a function $\hat{h}_{\mathcal{A}} : \mathcal{A}(\overline{\mathbb{Q}}) \to [0,\infty)$. It is quadratic on each fiber as the line bundle on the generic fiber A is symmetric and this extends along the fibers of $\mathcal{A} \to S$.

In the end we explain these height functions in terms of Height Machine. Let $\overline{\mathcal{A}}$, resp. $\overline{S}$, be the Zariski closure of the image of the immersion $\mathcal{A} \subseteq \mathbb{P}_{\overline{\mathbb{Q}}}^M \times \mathbb{P}_{\overline{\mathbb{Q}}}^m$, resp. $S \subseteq \mathbb{P}_{\overline{\mathbb{Q}}}^m$. If we let $\mathcal{L}' = \mathcal{O}(1,1)|_{\overline{\mathcal{A}}}$ and $\mathcal{M} = \mathcal{O}(1)|_{\overline{S}}$, then $h(\cdot)$ represents the class of functions $h_{\mathcal{A},\mathcal{L}'}$

defined up to $O(1)$ and $h \circ \pi$ represents $h_{S,\mathcal{M}} \circ \pi$. And the fiberwise Néron–Tate height $\hat{h}_{\mathcal{A}}$ is the map $P \mapsto \hat{h}_{\mathcal{A}_{\pi(P)},\mathcal{L}_{\pi(P)}}(P)$, where $\mathcal{L} = \iota_S^*\mathcal{O}_S(1)$ as above.

3. Proof of Proposition 1.3

The goal of this section is to prove Proposition 1.3. In fact we will prove a statement of independent interest that implies Proposition 1.3.

Let k be an algebraically closed field of characteristic 0. Let S be a smooth irreducible curve over k and fix an algebraic closure $\overline{K}$ of the function field $K = k(S)$. Let A be an abelian variety over $\overline{K}$.

Furthermore, let V_0 be an irreducible variety defined over k and $V = V_0 \otimes_k \overline{K}$. We consider $V_0(k)$ as a subset of $V(\overline{K})$.

The next proposition characterizes subvarieties $V \times A$ that contain a Zariski dense set of points in $\Sigma = V_0(k) \times A_{\mathrm{tor}} \subseteq V(\overline{K}) \times A(\overline{K})$. See Yamaki's [60, Proposition 4.6] for a related statement.

Proposition 3.1. *Suppose $A^{\overline{K}/k} = 0$ and let Y be an irreducible closed subvariety of $V \times A$.*

(i) *If $Y(\overline{K}) \cap \Sigma$ is Zariski dense in Y, then $Y = (W_0 \otimes_k \overline{K}) \times (P + B)$ with $W_0 \subseteq V_0$ an irreducible closed subvariety, $P \in A_{\mathrm{tor}}$, and B an abelian subvariety of A.*

(ii) *There are at most finitely many subvarieties of the form $(W_0 \otimes_k \overline{K}) \times (P + B)$ (with W_0, P, and B as in (i)) that are contained in Y, maximal with respect to the inclusion for this property.*

Part (i) implies (ii) for the following reason. If W_0, P, and B are as in the conclusion of (i), then it suffices to observe that $((W_0 \otimes_k \overline{K}) \times (P + B))(\overline{K}) \cap \Sigma$ is Zariski dense in $(W_0 \otimes_k \overline{K}) \times (P + B)$.

The assumption $\dim S = 1$ is used only at one place in the proof. In Appendix B, we will explain how to remove it.

3.1. **Proposition 3.1 implies Proposition 1.3.** Now we go back to the setting of Proposition 1.3: S is a smooth irreducible curve over k and $\pi \colon \mathcal{A} \to S$ is an abelian scheme of relative dimension $g \geq 1$. Let A denote the geometric generic fiber of π, it is an abelian variety over $\overline{K}$.

By [14, Theorem 6.4 and below] there is a unique abelian subvariety $A' \subseteq A$ such that $(A/A')^{\overline{K}/k} = 0$ and such that we may identify A' with $A^{\overline{K}/k} \otimes_k \overline{K}$.

We fix an abelian subvariety $A'' \subseteq A$ with $A' + A'' = A$ and such that $A' \cap A''$ is finite. Then the addition morphism restricts to an isogeny $\psi \colon A' \times A'' \to A$ and $(A'')^{\overline{K}/k} = 0$.

Let W_0 be an irreducible closed subvariety of $A^{\overline{K}/k}$, B an abelian subvariety of A'', and $P \in A''(\overline{K})$. We can map $\psi((W_0 \otimes_k \overline{K}) \times (P + B)) \subseteq A$ to the generic fiber of $\mathcal{A}$; its Zariski closure is a generically special subvariety of $\mathcal{A}$. Conversely, any generically special subvariety of $\mathcal{A}$ arises this way.

Let $\mathcal{X} \subseteq \mathcal{A}$ be an irreducible closed subvariety that dominates S and $X \subseteq A$ its geometric generic fiber. We apply Proposition 3.1 where $A^{\overline{K}/k}, A''$ play the role of V_0, A respectively. There are at most finitely many subvarieties of A of the form $(W_0 \otimes_k \overline{K}) \times (P + B)$ that are contained in $\psi^{-1}(X)$, maximal for this property. This shows that there are at most finitely many generically special subvarieties of $\mathcal{A}$ that are contained in $\mathcal{X}$, maximal for this property. $\square$

3.2. **Proof of Proposition 3.1.** Now we prove Proposition 3.1. To do this we require a uniform version of the Manin–Mumford Conjecture in characteristic 0.

Theorem 3.2 (Raynaud, Hindry, Hrushovski, Scanlon)**.** *Let $\overline{K}$ be as above, let A be an abelian variety, and let V be an irreducible, quasi-projective variety, both defined over $\overline{K}$. Suppose Y is an irreducible closed subvariety of $V \times A$. For $v \in V(\overline{K})$ we let Y_v denote the projection of $V \cap (\{v\} \times A)$ to A. Then there exists a finite set M of abelian subvarieties of A and $D \in \mathbb{Z}$ with the following property. For all $v \in V(\overline{K})$ the Zariski closure of $Y_v(\overline{K}) \cap A_{\mathrm{tor}}$ in Y_v is a union of at most D translates of members of M by points of finite order in A_{tor}.*

Proof. Raynaud proved the Manin–Mumford Conjecture in characteristic zero. Automatical uniformity then follows from Scanlon's [45, Theorem 2.4]; see also work of Hrushovski [34] and Hindry's [32, Théorème 1] for $k = \overline{\mathbb{Q}}$. Indeed, the number of irreducible components is uniformly bounded in an algebraic family. Moreover, it is well-known that if an irreducible component of a fiber of an algebraic family is a coset in A, then only finitely many possible underlying abelian subvarieties arise as one varies over the fibers. □

Proof of Proposition 3.1. We have already seen that it suffices to prove (i). We keep the notation Y_v for fibers of Y above $v \in V(\overline{K})$ introduced in Theorem 3.2. Let M and D be as in this theorem.

For all $v \in V(\overline{K})$ we have

$$\overline{Y_v(\overline{K}) \cap A_{\mathrm{tor}}} = \bigcup_i (P_{v,i} + B_{v,i})$$

where $P_{v,i} \in A_{\mathrm{tor}}, B_{v,i} \in M$, and the union has at most D members. Note that for $v \in V(\overline{K})$ any torsion coset contained in Y_v is contained in some $P_{v,i} + B_{v,i}$. Moreover,

$$Y(\overline{K}) \cap \Sigma = \bigcup_{v \in V_0(k)} \bigcup_i \{v\} \times \left(P_{v,i} + (B_{v,i})_{\mathrm{tor}}\right). \tag{3.1}$$

We decompose $Y(\overline{K}) \cap \Sigma$ into a finite union of

$$\Sigma_B = \bigcup_{v \in V_0(k)} \bigcup_{\substack{i \\ B_{v,i} = B}} \{v\} \times (P_{v,i} + B_{\mathrm{tor}})$$

by collecting entries on the right of (3.1) that come from $B \in M$. The set Σ_B must be Zariski dense in Y for some $B \in M$. There is possibly more than one such B so we choose one that is maximal with respect to inclusion.

We fix a finite field extension F/K with $F \subseteq \overline{K}$, such that Y, A, and B are stable under the action of $\mathrm{Gal}(\overline{K}/F)$. For this proof we consider these three varieties and V as over F. Note that $A^{\overline{K}/k} = 0$ remains valid.

Now suppose $v \in V_0(k)$ and let $P_{v,i}$ be as in the definition of Σ_B, hence $B_{v,i} = B$ and $P_{v,i} + B \subseteq Y$.

For all $\sigma \in \mathrm{Gal}(\overline{K}/F)$ we have $\sigma(P_{v,i}) + B = \sigma(P_{v,i} + B) \subseteq \sigma(Y_v) = Y_v$, by our choice of F and since σ acts trivially on $V_0(k)$. So the torsion coset $\sigma(P_{v,i}) + B$ is contained in $P_{v,j} + B'$ for some $B' \in M$ and some j. This implies $B \subseteq B'$. If $B \subsetneq B'$, then by maximality of B the Zariski closure $\overline{\Sigma_{B'}}$ is not all of Y. After replacing F by a finite extension of itself we may assume $\sigma(\overline{\Sigma_{B'}}) = \overline{\Sigma_{B'}}$ for all $\sigma \in \mathrm{Gal}(\overline{K}/F)$. In particular, $\{v\} \times (P_{v,i} + B) \subseteq \overline{\Sigma_{B'}}$. We remove such torsion cosets from the union defining Σ_B to obtain a set $\Sigma' \subseteq Y(\overline{K}) \cap \Sigma$ that remains Zariski dense in Y.

If $P_{v,i} + B$ is in the union defining Σ', then $B = B'$ and $\sigma(P_{v,i} + B) = P_{v,j} + B$ for some j and there are at most D possibilities for $\sigma(P_{v,i} + B)$ with $\sigma \in \mathrm{Gal}(\overline{K}/F)$.

Let $\varphi\colon A \to A/B$ be the canonical map, then $\sigma(\varphi(P_{v,i})) = \varphi(\sigma(P_{v,i})) = \varphi(P_{v,j})$. We have proven that the Galois orbit of $\varphi(P_{v,i})$ has at most D elements, in particular $[F(\varphi(P_{v,i})) : F] \leq D$; recall that D is independent of v and i.

Claim: Without loss of generality we may assume that the torsion points $P_{v,i}$ contributing to Σ' have uniformly bounded order.

Indeed, we may replace each $P_{v,i}$ by an element of $P_{v,i} + B_{\mathrm{tor}}$. So by a standard argument involving a complement of B in A it is enough to show the following statement: The order of any point in

$$\{P \in (A/B)_{\mathrm{tor}} : [F(P) : F] \leq D\} \tag{3.2}$$

is bounded in terms of A/B and D only.

This is the only place in the proof of Proposition 3.1 where we use the hypothesis $\dim S = 1$. In Appendix B we will explain how to remove this hypothesis.

Let $\overline{S}'$ be an irreducible smooth projective curve with $k(\overline{S}') = F$ and P as in (3.2). The inclusion $F \subseteq F(P)$ corresponds to a finite covering $\overline{S}'' \to \overline{S}'$ of degree $[F(P) : F]$ where $\overline{S}''$ is another smooth projective curve with function field $F(P)$. Then A/B has good reduction above $S'(k)\backslash Z$ for some finite subset Z of $S'(k)$, where we have identified $\overline{S'}(k)$ with the set of places of F. Note that S' and Z are independent of P. All residue characteristics are zero, so by general reduction theory of abelian varieties we find that $F(P)/F$ is unramified above the places in $S'(k) \setminus Z$. In other words, the finite morphism $\overline{S}'' \to \overline{S}'$ is unramified above $S' \setminus Z$. So we get a finite étale covering of $S \setminus Z$ of degree $[F(P) : F]$.

Let L be the compositum in $\overline{K}$ of all extensions of F of degree at most D that are unramified above $S' \setminus Z$. Then L/F a finite field extension by [53, Corollary 7.11] if $k \subseteq \mathbb{C}$ and for general k of characteristic 0 since the étale fundamental group of $S' \setminus Z$ is topologically finitely generated by [25, Exposé XIII Corollaire 2.12]. In particular, $P \in (A/B)(L)$ for all P in (3.2).

Now $(A/B)^{\overline{K}/k} = 0$ as the same holds for A. The extension L/k is finitely generated, so the Lang–Néron Theorem, cf. [36, Theorem 1] or [14, Theorem 7.1], implies that $(A/B)(L)$ is a finitely generated group. Thus $[N](P) = 0$ for some $N \in \mathbb{N}$ that is independent of P. Our claim follows.

Define a morphism $\psi\colon V \times A \to V \times (A/B)$ by $\psi(v,t) = (v, [N] \circ \varphi(P))$. By choice of N we have $\psi(\Sigma') \subseteq V \times \{0\}$, so $\Sigma' \subseteq V \times (\Theta + B)$ where $\Theta \subseteq A_{\mathrm{tor}}$ is finite. We pass to the Zariski closure and find $Y \subseteq V \times (P + B)$ for some $P \in A_{\mathrm{tor}}$ as Y is irreducible.

Let $p\colon V \times A \to V$ be the first projection, it is proper and $p(Y)$ is Zariski closed in V. A fiber of $p|_Y$ containing a point of Σ' contains a subvariety of dimension $\dim B$. We use that Σ' is Zariski dense in Y one last time together with the Fiber Dimension Theorem [31, Exercise II.3.22] to conclude $\dim B \leq \dim Y - \dim p(Y)$. As $Y \subseteq p(Y) \times (P + B)$

we conclude

$$Y = p(Y) \times (P + B). \tag{3.3}$$

Finally, $p(\Sigma')$ is Zariski dense in $p(Y) \subseteq V$. But $p(\Sigma')$ consists of elements in $V_0(k)$, with k the base field of V_0. We conclude $p(Y) = W_0 \otimes_k K$ for some irreducible subvariety $W_0 \subseteq V_0$. We conclude the proposition from (3.3). $\square$

4. The Betti Map

In this section we describe the construction of the Betti map.

Let S be a smooth, irreducible, algebraic curve over $\mathbb{C}$ and suppose $\pi\colon \mathcal{A} \to S$ is an abelian scheme of relative dimension g. We construct:

Proposition 4.1. *Let $\mathcal{A}$ and S be as above. For all $s \in S(\mathbb{C})$ there exists an open neighborhood Δ of s in S^{an} and a real analytic mapping $b\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$, called Betti map, with the following properties.*

(i) *For each $s \in \Delta$ the restriction $b|_{\mathcal{A}_s^{\mathrm{an}}}\colon \mathcal{A}_s^{\mathrm{an}} \to \mathbb{T}^{2g}$ is a group isomorphism.*
(ii) *For each $\xi \in \mathbb{T}^{2g}$ the preimage $b^{-1}(\xi)$ is a complex analytic subset of $\mathcal{A}_\Delta^{\mathrm{an}}$.*
(iii) *The product $(b, \pi|_{\mathcal{A}_\Delta})\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g} \times \Delta$ is real bianalytic.*

Remark 4.2. *We remark that b from the proposition above is not unique as we can compose it with a continuous group endomorphism of $\mathbb{T}^{2g}$. However, if $b, b'\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ both satisfy the conclusion of the proposition and if Δ is path-connected, then using homotopy and (iii) we find $b' = \alpha \circ b$ for some $\alpha \in \mathrm{GL}_{2g}(\mathbb{Z})$.*

Before giving the concrete construction, let us explain the idea. Assume $S = \mathbb{A}_g$ is the moduli space of principally polarized abelian varieties with level-3-structure, and $\mathcal{A} = \mathfrak{A}_g$ is the universal abelian variety. The universal covering $\mathfrak{H}_g^+ \to \mathbb{A}_g$, where $\mathfrak{H}_g^+$ is the Siegel upper half space, gives a polarized family of abelian varieties $\mathcal{A}_{\mathfrak{H}_g^+} \to \mathfrak{H}_g^+$

$$\begin{array}{ccc} \mathcal{A}_{\mathfrak{H}_g^+} := \mathfrak{A}_g \times_{\mathbb{A}_g} \mathfrak{H}_g^+ & \longrightarrow & \mathfrak{A}_g \;. \\ \downarrow & & \downarrow \\ \mathfrak{H}_g^+ & \longrightarrow & \mathbb{A}_g \end{array}$$

For the universal covering $u\colon \mathbb{C}^g \times \mathfrak{H}_g^+ \to \mathcal{A}_{\mathfrak{H}_g^+}$ and for each $\tau \in \mathfrak{H}_g^+$, the kernel of $u|_{\mathbb{C}^g \times \{\tau\}}$ is $\mathbb{Z}^g + \tau\mathbb{Z}^g$. Thus the map $\mathbb{C}^g \times \mathfrak{H}_g^+ \to \mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{H}_g^+ \to \mathbb{R}^{2g}$, where the first map is the inverse of $(a, b, \tau) \mapsto (a + \tau b, \tau)$ and the second map is the natural projection, descends to a map $\mathcal{A}_{\mathfrak{H}_g^+} \to \mathbb{T}^{2g}$. Now for each $s \in S(\mathbb{C}) = \mathbb{A}_g(\mathbb{C})$, there exists an open neighborhood Δ of s in $\mathbb{A}_g^{\mathrm{an}}$ such that $\mathcal{A}_\Delta = (\mathfrak{A}_g)|_\Delta$ can be identified with $\mathcal{A}_{\mathfrak{H}_g^+}|_{\Delta'}$ for some open subset of $\mathfrak{H}_g^+$. The composite $b\colon \mathcal{A}_\Delta \cong \mathcal{A}_{\mathfrak{H}_g^+}|_{\Delta'} \to \mathbb{T}^{2g}$ is clearly real analytic and satisfies the three properties listed in Proposition 4.1. Thus b is the desired Betti map in this case. Note that for a fixed (small enough) Δ, there are infinitely choices of Δ'; but for Δ small enough, if Δ'_1 and Δ'_2 are two such choices, then $\Delta'_2 = \alpha \cdot \Delta'_1$ for some $\alpha \in \mathrm{Sp}_{2g}(\mathbb{Z})$.

Now let us give the concrete construction. Let $s_0 \in S^{\mathrm{an}}$. By Ehresmann's Theorem [51, Theorem 9.3], there is an open neigborhood Δ of s_0 in S^{an} such that $\mathcal{A}_\Delta = \pi^{-1}(\Delta)$ and

$\mathcal{A}_{s_0} \times \Delta$ are diffeomorphic as families over Δ. The map f in

(4.1)
$$\begin{tikzcd} \mathcal{A}_{s_0} \times \Delta \arrow[r, "f"] \arrow[dr] & \mathcal{A}_\Delta \arrow[d] \\ & \Delta \end{tikzcd}$$

is a diffeomorphism, the diagonal arrow is the natural projection, and the vertical arrow is the restriction of the structural morphism. After translating we may assume that $(0, s)$ maps to the unit element in $\mathcal{A}_s$ for all $s \in \Delta$. We may assume that Δ is simply connected. Fiberwise we obtain a diffeomorphism $f_s : \mathcal{A}_{s_0} \to \mathcal{A}_s$.

As $\mathcal{A}^{\mathrm{an}}$ is a complex analytic space we may assume that the fibers of f^{-1} in (4.1) are complex analytic, see [51, Proposition 9.5].

We fix a basis $\gamma_1, \ldots, \gamma_{2g}$ of the $\mathbb{Z}$-module $H_1(\mathcal{A}_{s_0}^{\mathrm{an}}, \mathbb{Z})$. Each γ_i is represented by a loop $\widetilde{\gamma}_i : [0, 1] \to \mathcal{A}_{s_0}^{\mathrm{an}}$ based at the origin of $\mathcal{A}_{s_0}^{\mathrm{an}}$.

For all $s \in \Delta$ we have a map $H^1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{R}) \to H^1(\mathcal{A}_{s_0}^{\mathrm{an}}, \mathbb{R})$ resp. $H^1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{C}) \to H^1(\mathcal{A}_{s_0}^{\mathrm{an}}, \mathbb{C})$ induced by f_s, it is an isomorphism of $\mathbb{R}$- resp. $\mathbb{C}$-vector spaces. We denote the latter by f_s^* and note that $f_s^*(\overline{v}) = \overline{f_s^*(v)}$ where complex conjugation $\bar{\cdot}$ is induced by the real structure.

The Hodge decomposition yields

$$H^1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{C}) = H^0(\mathcal{A}_s^{\mathrm{an}}, \Omega^1) \oplus \overline{H^0(\mathcal{A}_s^{\mathrm{an}}, \Omega^1)}$$

where $H^0(\mathcal{A}_s^{\mathrm{an}}, \Omega^1)$ is the g-dimensional vector space of global holomorphic 1-forms on $\mathcal{A}_s^{\mathrm{an}}$. As s varies over Δ we obtain a collection

$$f_s^* H^0(\mathcal{A}_s^{\mathrm{an}}, \Omega^1)$$

of subspaces of $H^1(\mathcal{A}_{s_0}^{\mathrm{an}}, \mathbb{C})$. As f_s^* commutes with complex conjugation we have

$$H^1(\mathcal{A}_{s_0}^{\mathrm{an}}, \mathbb{C}) = f_s^* H^0(\mathcal{A}_s^{\mathrm{an}}, \Omega^1) \oplus \overline{f_s^*(H^0(\mathcal{A}_s^{\mathrm{an}}, \Omega^1))}.$$

For $s \in \Delta$ the image $f_s^* H^0(\mathcal{A}_s^{\mathrm{an}}, \Omega^1)$ corresponds to a point in the Grassmannian variety of g-dimensional subspaces of $H^1(\mathcal{A}_{s_0}^{\mathrm{an}}, \mathbb{C})$. As a particular case of Griffith's Theorem, this association is a holomorphic function. We draw the following conclusion from Griffith's result.

Fix a basis $\omega_1^0, \ldots, \omega_g^0$ of $H^0(\mathcal{A}_{s_0}^{\mathrm{an}}, \Omega^1)$; then $\omega_1^0, \ldots, \omega_g^0, \overline{\omega_1^0}, \ldots, \overline{\omega_g^0}$ is a basis $H^1(\mathcal{A}_{s_0}^{\mathrm{an}}, \mathbb{C})$. There exist holomorphic functions

$$a_{ij} : \Delta \to \mathbb{C} \quad \text{and} \quad b_{ij} : \Delta \to \mathbb{C} \quad (1 \le i, j \le g)$$

such that

$$f_s^* \omega_i(s) = \sum_{j=1}^{g} \left(a_{ij}(s) \omega_j^0 + b_{ij}(s) \overline{\omega_j^0} \right)$$

for all $i \in \{1, \ldots, g\}$ and all $s \in \Delta$ where $\omega_1(s), \ldots, \omega_g(s)$ is a basis of $H^0(\mathcal{A}_s^{\mathrm{an}}, \Omega^1)$ with $\omega_i(s_0) = \omega_i^0$ for all i.

For $s \in \Delta$ we define the period matrix

$$\Omega(s) = \left(\int_{f_{s*}\widetilde{\gamma}_j} \omega_i(s) \right)_{\substack{1 \le i \le g \\ 1 \le j \le 2g}} \in \mathrm{Mat}_{g,2g}(\mathbb{C})$$

for all $s \in \Delta$; the integral is taken over the loop in $\mathcal{A}_{s_0}^{\mathrm{an}}$ fixed above. Note that

$$\int_{f_{s*}\widetilde{\gamma}_j} \omega_i(s) = \int_{\widetilde{\gamma}_j} f_s^*\omega_i(s) = \sum_{j=1}^{g} \left(a_{ij}(s) \int_{\widetilde{\gamma}_j} \omega_j^0 + b_{ij}(s) \int_{\widetilde{\gamma}_j} \overline{\omega_j^0} \right)$$

by a change of variables. So $\Omega(s)$ is holomorphic in s. In this notation and with $A(s) = (a_{ij}(s)) \in \mathrm{Mat}_g(\mathbb{C})$ and $B(s) = (b_{ij}(s)) \in \mathrm{Mat}_g(\mathbb{C})$ we can abbreviate the above by

$$\left(\begin{array}{c} \Omega(s) \\ \overline{\Omega(s)} \end{array} \right) = \left(\begin{array}{cc} A(s) & B(s) \\ \overline{B(s)} & \overline{A(s)} \end{array} \right) \left(\begin{array}{c} \Omega(0) \\ \overline{\Omega(0)} \end{array} \right) \tag{4.2}$$

here $\Omega(0) = \Omega(s_0)$. So the first matrix on the right of (4.2) is invertible.

Let $P \in \mathcal{A}^{\mathrm{an}}$ where $s = \pi(P) \in \Delta$ and suppose γ_P is a path in $\mathcal{A}_s^{\mathrm{an}}$ connecting 0 and P. Let $Q \in \mathcal{A}_{s_0}^{\mathrm{an}}$ with $f_s(Q) = P$ and γ_Q the path in $\mathcal{A}_{s_0}^{\mathrm{an}}$ such that $f_{s*}\gamma_Q = \gamma_P$. We define

$$\mathcal{L}(P) = \left(\begin{array}{c} \int_{\gamma_P} \omega_1(s) \\ \vdots \\ \int_{\gamma_P} \omega_g(s) \end{array} \right) = \left(\begin{array}{c} \int_{\gamma_Q} f_s^*\omega_1(s) \\ \vdots \\ \int_{\gamma_Q} f_s^*\omega_g(s) \end{array} \right) = (A(s)B(s)) \left(\begin{array}{c} \mathcal{L}^*(Q) \\ \overline{\mathcal{L}^*(Q)} \end{array} \right) \tag{4.3}$$

where

$$\mathcal{L}^*(Q) = \left(\begin{array}{c} \int_{\gamma_Q} \omega_1^0(s) \\ \vdots \\ \int_{\gamma_Q} \omega_g^0(s) \end{array} \right).$$

Replacing γ_P by another path connecting 0 and P in $\mathcal{A}_s^{\mathrm{an}}$ will translate the value of $\mathcal{L}(P)$ by a period in $\Omega(s)\mathbb{Z}^{2g}$. By passing to the quotient we obtain the Albanese map $\mathcal{A}_s^{\mathrm{an}} \to \mathbb{C}^g/\Omega(s)\mathbb{Z}^{2g}$. It is a group isomomorphism.

We set further

$$\tilde{b}(P) = \left(\begin{array}{c} \Omega(s) \\ \overline{\Omega(s)} \end{array} \right)^{-1} \left(\begin{array}{c} \mathcal{L}(P) \\ \overline{\mathcal{L}(P)} \end{array} \right).$$

and observe $\tilde{b}(P) \in \mathbb{R}^{2g}$ as these are the coordinates of $\mathcal{L}(P)$ in terms of the period lattice basis $\Omega(s)$.

By replacing γ_P by another path connecting 0 and P we find that $\tilde{b}(P)$ is translated by a vector in $\mathbb{Z}^{2g}$. Therefore, $\tilde{b}$ induces a real analytic map $b\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$, where $\mathbb{T}$ is the circle group which we identify with $\mathbb{R}/\mathbb{Z}$. We will prove that b satisfies the three properties listed in Proposition 4.1.

On a given fiber, *i.e.* for fixed s, the map b restricts to a group isomorphism $\mathcal{A}_s \to \mathbb{T}^{2g}$ as we have seen above. So part (i) of Proposition 4.1 holds.

Let us investigate such a fiber. For this we recall (4.3). By the period transformation formula (4.2) we see

$$\tilde{b}(P) = \left(\begin{array}{c} \Omega(0) \\ \overline{\Omega(0)} \end{array} \right)^{-1} \left(\begin{array}{c} \mathcal{L}^*(Q) \\ \overline{\mathcal{L}^*(Q)} \end{array} \right).$$

Fixing the value of $\tilde{b}$ amounts to fixing the value of $\mathcal{L}^*(Q)$. As $\mathcal{L}^*$ induces the Albanese map on $\mathcal{A}_s^{\mathrm{an}}$, fixing b amounts to fixing Q. Recall that Q maps to P under the trivialization (4.1). Therefore, a fiber of b equals a fiber of the trivialization. As these fibers are complex analytic we obtain part (ii) of Proposition 4.1.

Finally, the association

$$(\xi + \mathbb{Z}^{2g}, s) \mapsto \left(\begin{pmatrix} \Omega(0) \\ \overline{\Omega(0)} \end{pmatrix} (\xi + \mathbb{Z}^{2g}), s \right) \mapsto (Q, s) \mapsto f_s(Q) \in \mathcal{A}_\Delta$$

induces the inverse of the product $\mathcal{A}_\Delta \to \mathbb{T}^{2g} \times \Delta$. This is part (iii) of Proposition 4.1.

5. Degenerate Subvarieties

Let S be a smooth irreducible algebraic curve over $\mathbb{C}$ and let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$. We define and characterize the degenerate subvarieties of $\mathcal{A}$ in this section. Let Y be an irreducible closed subvariety of $\mathcal{A}$ that dominates S.

Let $s_0 \in S(\mathbb{C})$ and let $\Delta \subseteq S^{\mathrm{an}}$ be an open neighborhood of s_0 in S^{an} with the Betti map $b\colon \mathcal{A}_\Delta = \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$ as in Proposition 4.1 with $\mathbb{T} \subseteq \mathbb{C}$ the circle group. We say that a point $P \in Y^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ is *degenerate* for Y if it is not isolated in $b|_{Y^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta}^{-1}(b(P))$. We say that Y is *degenerate* if there is a non-empty and open subset of $Y^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ consisting of points that are degenerate for Y.

For technical purposes our notation of degeneracy formally depends on the choice of Δ. But this dependency is harmless as we will see.

Recall that generically special subvarieties of $\mathcal{A}$ were introduced in Definition 1.2. A generically special subvariety is degenerate. In this section we prove the converse.

Theorem 5.1. *An irreducible closed subvariety of $\mathcal{A}$ that is degenerate is a generically special subvariety of $\mathcal{A}$.*

This proposition, which has a definite Ax–Schanuel flavor, is proved using a variant of the Pila–Wilkie Counting Theorem for definable sets in an o-minimal structure. Abundantly many rational points arise from the exponential growth of a certain monodromy group.

5.1. **Invariant Subsets of the Torus.**

We write $|\cdot|_2$ for the ℓ^2-norm on $\mathbb{R}^n$.

For $n \in \mathbb{N}$ we consider the real n-dimensional torus $\mathbb{T}^n$ equipped with the standard topology. We will use the continuous left-action of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathbb{T}^n$ and use the additive notation for $\mathbb{T}^n$. Suppose X is a closed subset of $\mathbb{T}^n$ such that

$$\gamma(X) = X$$

for all γ in a subgroup Γ of $\mathrm{GL}_n(\mathbb{Z})$. What can we say about X?

To rule out subgroups that are too small we ask that Γ contains a (non-abelian) free subgroup on 2 generators. Moreover, we will assume that X is sufficiently "tame" as a set.

To formulate the last property precisely, let $\exp\colon \mathbb{R}^n \to \mathbb{T}^n$ denote the exponential map $(t_1, \ldots, t_n) \mapsto (e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})$. Let $X \subseteq \mathbb{T}^n$ be a subset and

$$\mathcal{X} = \exp|_{[0,1]^n}^{-1}(X).$$

We will work in a fixed o-minimal structure and call X definable if $\mathcal{X}$ is a definable subset of $\mathbb{R}^n$ in the given o-minimal structure. We refer to van den Dries' book [17] for the theory of o-minimal structures. We will work with $\mathbb{R}_{\mathrm{an}}$, the o-minimal structure generated by restricting real analytic functions on $\mathbb{R}^n$ to $[-1,1]^n$.

We say that $X \subseteq \mathbb{T}^n$ is of *Ax-type* if it satisfies the following property. For any continuous, semi-algebraic map $y\colon [0,1] \to \mathcal{X}$ that is real-analytic on $(0,1)$, there is a closed subgroup $G \subseteq \mathbb{T}^n$ such that $\exp \circ y([0,1]) \subseteq y(0) + G \subseteq X$.

The main example comes from a g-dimensional abelian variety A defined over $\mathbb{C}$. Indeed, then there is a real bianalytic map $A^{\mathrm{an}} \to \mathbb{T}^{2g}$. Moreover, the image of $X(\mathbb{C})$ is definable and of Ax-type for any Zariski closed subset of A; for the latter claim we refer to Ax's Theorem [2].

Lemma 5.2. *Let $X \subseteq \mathbb{T}^n$ be a closed definable set of Ax-type. Let Γ be a free subgroup of $\mathrm{GL}_n(\mathbb{Z})$ on 2 generators such that $\gamma(X) = X$ for all $\gamma \in \Gamma$. Then one of the following properties holds true:*

1. *The set X is contained in a finite union of closed and proper subgroups of $\mathbb{T}^n$.*
2. *There are a non-empty, open subset U of X and a closed, connected, infinite subgroup $G \subseteq \mathbb{T}^n$ with $U + G \subseteq X$.*

Proof. By assumption, Γ is generated by elements γ_1, γ_2 that do not satisfy any non-trivial relation. Any element $\gamma \in \Gamma$ is uniquely represented by a reduced word in $\gamma_1^{\pm 1}, \gamma_2^{\pm 1}$ whose length is $l(\gamma)$. For all real $t \geq 1$ we have

$$\#\{\gamma \in \Gamma : l(\gamma) \leq t\} \geq 2^t.$$

We define $c_1 = \max\{2, |\gamma_1|_2, |\gamma_1|_2\} \geq 2$ and observe $|\gamma|_2 \leq c_1^{l(\gamma)}$ for all $\gamma \in \Gamma$. The height $H(b)$ of any integral vector $b = (b_1, \ldots, b_m) \in \mathbb{Z}^m$ is $\max\{1, |b_1|, \ldots, |b_m|\}$. So

$$H(\gamma) \leq c_1^{l(\gamma)}.$$

Let $T \geq c_1$ and let $t = (\log T)/(\log c_1) \geq 1$. There are at least $2^t = T^{(\log 2)/\log c_1}$ elements $\gamma \in \Gamma$ with $H(\gamma) \leq T$.

Let $x \in \mathcal{X} = \exp|_{[0,1]^n}^{-1}(X)$. For all $\gamma \in \Gamma$ there is $a = a_\gamma \in \mathbb{Z}^n$ such that $y_\gamma = \gamma x - a_\gamma \in \mathcal{X}$. Then $(x, \gamma, a_\gamma, y_\gamma)$ lies in the definable set

$$\mathcal{Z} = \{(x, \gamma, a, y) \in \mathcal{X} \times \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n \times \mathcal{X} : \gamma x - a = y\}.$$

We view it as a family of definable sets parametrized by $x \in \mathcal{X}$ with fibers $\mathcal{X}_x \subseteq \mathbb{R}^{n^2+n+n}$. Moreover,

$$\begin{aligned} H(a_\gamma) &\leq \max\{1, |a_\gamma|_2\} = \max\{1, |\gamma x - y_\gamma|_2\} \\ &\leq \max\{1, |\gamma|_2 |x|_2 + |y_\gamma|_2\} \leq \sqrt{n}(|\gamma|_2 + 1) \leq 2n^2 H(\gamma). \end{aligned} \tag{5.1}$$

Let c_2 be the constant from the semi-rational variant of the Pila–Wilkie Theorem [30, Corollary 7.2] applied to the family $\mathcal{Z}$ and $\epsilon = (\log 2)/(2\log c_1)$. Here the coordinates assigned to (γ, a) are treated as rational and the coordinates assigned to y are not. We fix T large enough in terms of c_1 and c_2, more precisely we will assume that $T \geq c_1$ and

$$T^{(\log 2)/\log c_1} > c_2 (2n^2 T)^{(\log 2)/(2\log c_1)}. \tag{5.2}$$

We keep x fixed and vary γ. Let us first see how to reduce to the case that many different y_γ must arise this way if $H(\gamma) \leq T$.

Indeed, suppose $\gamma' \in \Gamma$ satisfies $H(\gamma') \leq T$. Then $y'_{\gamma'} = \gamma' x - a'_{\gamma'} \in \mathcal{X}$ for some $a' \in \mathbb{Z}$. If $y_\gamma = y'_{\gamma'}$, then $\gamma x - a_\gamma = \gamma' x - a'_{\gamma'}$, so $\gamma x - \gamma' x \in \mathbb{Z}^n$. Then $\exp(x)$ lies in the closed subgroup of $\mathbb{T}^n$ defined by the kernel of $\gamma^{-1}\gamma' - 1 \neq 0$, *i.e.* the largest subgroup of $\mathbb{T}^n$ stabilized by $\gamma^{-1}\gamma'$. So it lies in a finite union $G_1 \cup \cdots \cup G_N$ of closed proper subgroups of $\mathbb{T}^n$, each defined as the subgroup stabilized by some $\gamma^{-1}\gamma'$ as above. Here N is bounded only in terms of T and thus only in terms of c_1, c_2, and n. It is independent of x.

If $X \subseteq G_1 \cup \cdots \cup G_N$, then we are in case (1).

Otherwise $V = X \setminus (G_1 \cup \cdots \cup G_N)$ lies open in X and is non-empty.

Now suppose $x \in \mathcal{X}$ with $\exp(x) \in V$ and $\gamma \in G$ with $H(\gamma) \leq T$. Recall that $y_\gamma = \gamma x - a_\gamma \in \mathcal{X}$. By our choice of V and the arguments above the number of y_γ that arise is the number of elements in Γ of height at most T. This number is at least $T^{(\log 2)/\log c_1}$. Note that the height of (γ, a_γ) equals $\max\{H(\gamma), H(a_\gamma)\}$ and this is at most $2n^2T$ by (5.1).

By (5.2) we have enough y_γ to apply the counting result [30, Corollary 7.2]. We thus obtain continuous, definable maps $\gamma\colon [0,1] \to \mathrm{GL}_n(\mathbb{R})$, $a\colon [0,1] \to \mathbb{R}^n$, and $y\colon [0,1] \to \mathcal{X}$ such that γ and a are semi-algebraic, y is non-constant, and

$$\gamma(s)x - a(s) = y(s)$$

for all $s \in [0,1]$. So $s \mapsto y(s)$ is semi-algebraic too and $\exp \circ y([0,1]) \subseteq X$. After rescaling $[0,1]$ we may assume that y is real-analytic on $(0,1)$. By looking at the proof of [30, Corollary 7.2(iii)] we may arrange $\gamma(0) \in \Gamma$ and $a(0) \in \mathbb{Z}^n$. Recall that X is of Ax-type. So there is a closed subgroup $G'_x \subseteq \mathbb{T}^n$ with $\exp \circ y([0,1]) \subseteq \exp(y(0)) + G'_x \subseteq X$. We may assume that G'_x is connected. Observe that G'_x is infinite as $\exp \circ y$ is continuous and non-constant. We find $\exp(x) + G_x \subseteq \gamma(0)^{-1}(X) = X$ where $G_x = \gamma(0)^{-1}G'_x$.

We have proved that for any $x \in \mathcal{X}$ with $\exp(x) \in V$ we have

$$\exp(x) + G_x \subseteq X$$

for some connected, closed, infinite subgroup $G_x \subseteq \mathbb{T}^n$.

For any connected closed subgroup $G \subseteq \mathbb{T}^n$ we define

$$E(G) = \{z \in V : z + G \subseteq X\} = V \cap \bigcap_{g \in G} (X - g).$$

Then $E(G)$ is closed in V. Our conclusion from above can be restated as

$$V = \bigcup_{x \in \exp|_{\mathcal{X}}^{-1}(V)} E(G_x).$$

By Kronecker's Theorem $\mathbb{T}^n$ has countably many closed subgroups. So this union countains at most countably many different members. Now V, being non-empty, Hausdorff, and locally compact satisfies the hypothesis of Baire's Theorem. Hence there exists an connected, closed, infinite subgroup $G \subseteq \mathbb{T}^n$ such that $V \setminus E(G)$ is not dense in V. So $E(G)$ contains a non-empty and open subset of X, as claimed in (2). $\square$

Now suppose that A is an abelian variety of dimension $g \geq 1$ defined over $\mathbb{C}$.

We attach to A the associated complex manifold A^{an} whose underlying set of points is $A(\mathbb{C})$. There is a real bi-analytic map $b\colon A^{\mathrm{an}} \to \mathbb{T}^{2g}$ which is a group isomorphism, we will not need to vary A in a family here as in Proposition 4.1.

Suppose a group Γ acts faithfully and continuously on A^{an}; we do not ask for elements of Γ to act by holomorphic maps. Any continuous group automorphism of $\mathbb{T}^{2g}$ can be identified with an element of $\mathrm{GL}_{2g}(\mathbb{Z})$. So using b we may consider Γ as a subgroup of $\mathrm{GL}_{2g}(\mathbb{Z})$.

We say that the action of Γ is of *monodromy-type* if $\gamma(B(\mathbb{C})) = B(\mathbb{C})$ for all $\gamma \in \Gamma$ and all abelian subvarieties $B \subseteq A$.

Later we will study the action of the fundamental group of an abelian scheme on a fixed fiber in sufficiently general position. This action will leave the abelian subvarieties of the said fiber invariant and is thus of monodromy-type.

Proposition 5.3. *Let $A, g, b,$ and $\Gamma \subseteq \mathrm{GL}_{2g}(\mathbb{Z})$ be above, so in particular Γ acts continuously on A^{an} and is of monodromy-type. We assume in addition that Γ contains a free subgroup on 2 generators and that there are no Γ-invariant elements in $\mathbb{Z}^{2g} \setminus \{0\}$. Let Z be an irreducible closed subvariety of A with $\gamma(Z(\mathbb{C})) = Z(\mathbb{C})$ for all $\gamma \in \Gamma$. Then one of the following properties holds:*

(1) *The subvariety Z is contained in a proper torsion coset in A.*
(2) *There exists an abelian subvariety $B \subseteq A$ with $\dim B \geq 1$ and $Z + B = Z$.*

Proof. We write X for the image of $Z(\mathbb{C})$ under the real analytic isomorphism $b\colon A^{\mathrm{an}} \to \mathbb{T}^{2g}$. Then X is closed and definable in the sense as introduced before Lemma 5.2. By Ax's Theorem [2], the set X is of Ax-type. We apply Lemma 5.2 to a free subgroup of Γ on 2 generators.

If we are in case (1) of Lemma 5.2, then X is contained in a finite union of proper closed subgroups $G_1, \ldots, G_N \subsetneq \mathbb{T}^{2g}$. By the Baire Category Theorem we may assume that $X \cap G_1$ has non-empty interior in X.

The analytification Z^{an} is an irreducible complex analytic space and $Z^{\mathrm{sm,an}}$ is arc-wise connected by [21, Theorems 9.1.2 and 9.3.2]. Moreover, $Z^{\mathrm{sm,an}}$ is an open and dense subset of Z^{an}.

Let $P, Q \in Z^{\mathrm{sm,an}}$ and suppose $b(P)$ lies in the interior of $X \cap G_1$. We can connect P and Q via an arc $[0,1] \to Z^{\mathrm{sm,an}}$ whose restriction to $(0,1)$ is piece-wise real analytic on finitely many pieces. A neighborhood of $b(P)$ in X lies in G_1 and G_1 is defined globally by relations in integer coefficients. By analytic continuation we find that $b(Q) \in G_1$. In particular, $b(Z^{\mathrm{sm,an}}) \subseteq G_1$ and thus $b(Z^{\mathrm{an}}) \subseteq G_1$. So Z^{an} is contained in the proper subgroup $b^{-1}(G_1) \subsetneq A^{\mathrm{an}}$.

The sum of sufficiently many copies of $Z - Z$ is an abelian subvariety B of A. We have $B \neq A$ because $B(\mathbb{C})$ lies in $b^{-1}(G_1)$. So $Z \subseteq P + B$ for some $P \in A(\mathbb{C})$. Moreover, any coset in A containing Z must contain $P + B$.

Let B' be the complementary abelian subvariety of B in A with respect to a fixed polarization, see [5, §5.3]. So $B + B' = A$ and $B \cap B'$ is finite. By the former property we may assume $P \in B'(\mathbb{C})$.

By hypothesis we have $Z(\mathbb{C}) = \gamma(Z(\mathbb{C})) \subseteq \gamma(P) + \gamma(B(\mathbb{C})) = \gamma(P) + B(\mathbb{C})$ for all $\gamma \in \Gamma$. Thus $\gamma(P) - P \in B(\mathbb{C})$ for all $\gamma \in \Gamma$. As B' is invariant under γ we find $\gamma(P) - P \in (B \cap B')(\mathbb{C})$. So $\gamma(Q) - Q = 0$ for all $\gamma \in \Gamma$ where $Q = [\#B \cap B'](P)$.

The point $b(Q) \in \mathbb{T}^{2g}$ is the image of some $t \in \mathbb{R}^{2g}$ under the canonical map $\mathbb{R}^{2g} \to \mathbb{T}^{2g}$. Our action of Γ on A^{an} was defined using b and Γ acts on $\mathbb{T}^{2g}$ via a matrix in $\mathrm{Mat}_{2g}(\mathbb{Z})$. We find that $\gamma(t) - t \in \mathbb{Z}^{2g}$ for all $\gamma \in \Gamma$ with the standard action of $\mathrm{GL}_{2g}(\mathbb{Z})$ on $\mathbb{R}^{2g}$.

Thus $t \in \mathbb{R}^{2g}$ is the solution of a system of inhomogenous linear equations, parametrized by Γ, with integral coefficients and integral solution vector. The corresponding homogeneous equation has only the trivial solution as there are no non-trivial Γ-invariant vectors in $\mathbb{Z}^{2g}$. So t was the unique solution and we conclude $t \in \mathbb{Q}^{2g}$. Therefore Q and thus P have finite order. So $P + B$ is a torsion coset in A and we are in case (1) of the current proposition.

Now suppose we are in case (2) of Lemma 5.2 and U and G are as given in therein. Then $\bigcap_{P \in b^{-1}(G)}(Z - P)$ is Zariski closed in Z since $0 \in G$. By Lemma 5.2 its complex points contain $b^{-1}(U)$, which is Zariski dense in Z. So $Z - P = Z$ for all $P \in b^{-1}(G)$. This equality continues to hold for $\mathbb{C}$-points in the Zariski closure B of $b^{-1}(G)$ in A. As G is a connected subgroup of A^{an} we find that B is an abelian subvariety of A. Moreover, $\dim B \geq 1$ since G is infinite. So we are in case (2) of the proposition. □

5.2. **Degeneracy and Global Information.** Let S be an irreducible and smooth curve over $\mathbb{C}$ and let $\mathcal{A}$ be an abelian scheme over S of relative dimension $g \geq 1$.

Recall that Betti maps were introduced in §4. Around each point of S^{an} we fix an open neighborhood in S^{an} and a Betti map as in Proposition 4.1. This yields an open cover of S^{an} which we now refine for our application later on. After shrinking each member we may assume that each member is bounded and diffeomorphic to an open subset of $\mathbb{R}^2$. As S^{an} is paracompact we may refine this cover to obtain an open cover of S^{an} that is locally finite. Each member of this cover is relatively compact. We may refine the cover again and assume that a finite intersection of members is empty or contractible, see Weil's treatment [55, §1]. A non-empty open subset of S^{an} is naturally a Riemann surface; if it is contractible then it is homeomorphic to the open unit disc. Therefore, a finite intersection of members of our cover is empty or homeomorphic to the open unit disc.

Let $s \in S^{\mathrm{an}}$ be a base point. We describe the monodromy representation of $\pi_1(S^{\mathrm{an}}, s)$ using the Betti map.

Let $\gamma\colon [0,1] \to S^{\mathrm{an}}$ be a loop around s. We can find a Betti map in a neighborhood around each point of $\gamma([0,1])$. As this image is compact we find $0 = a_0 < a_1 < \cdots < a_n = 1$ such that $\gamma([a_{i-1}, a_i]) \subseteq \Delta_i$ where Δ_i is a member of the cover above and b_i is its associated Betti map.

We can glue the Betti maps as follows. For each $i \in \{1, \ldots, n-1\}$ we have $s_i = \gamma(a_i) \in \Delta_i \cap \Delta_{i+1}$. So $b_i|_{\mathcal{A}_{s_i}^{\mathrm{an}}} \circ (b_{i+1}|_{\mathcal{A}_{s_i}^{\mathrm{an}}})^{-1}$ is a continuous group isomorphism $M\colon \mathbb{T}^{2g} \to \mathbb{T}^{2g}$, thus represented by a matrix in $\mathrm{GL}_{2g}(\mathbb{Z})$. On replacing b_{i+1} by $M \circ b_{i+1}$ we may arrange that b_i and b_{i+1} coincide on $\mathcal{A}_{s_i}^{\mathrm{an}}$.

Now $\gamma(0) = \gamma(1) = s$ and both b_1 and b_n define homeomorphisms $\mathcal{A}_s^{\mathrm{an}} \to \mathbb{T}^{2g}$. By composing we obtain a homeomorphism $\mathcal{A}_s^{\mathrm{an}} \to \mathcal{A}_s^{\mathrm{an}}$ that is a group isomomorphism. This homeomorphism induces an automorphism of the $\mathbb{Z}$-module $H^1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})$ that depends on the loop γ. Another loop that is homotopic to γ relative $\{0,1\}$ will lead to the same automorphism of $H^1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})$. The induced mapping $\pi_1(S^{\mathrm{an}}, s) \to \mathrm{Aut}(H^1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z}))$ is the monodromy representation from [52, §3.1.2]. We denote its dual by

$$\rho\colon \pi_1(S^{\mathrm{an}}, s) \to \mathrm{Aut}(H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})). \tag{5.3}$$

Proposition 5.4. *In the notation above there is a group homomorphism*

$$\widetilde{\rho} = \widetilde{\rho}_{\mathcal{A}}\colon \pi_1(S^{\mathrm{an}}, s) \to \{\textit{homeomorphisms } \mathcal{A}_s^{\mathrm{an}} \to \mathcal{A}_s^{\mathrm{an}} \textit{ that are group homomorphisms}\} \tag{5.4}$$

that satisfies

$$\widetilde{\rho}(h)_* = \rho(h) \quad \textit{for all} \quad h \in \pi_1(S^{\mathrm{an}}, s) \tag{5.5}$$

with the following properties.

(i) *There exists a path-connected open neighborhood* $\Delta \subseteq S^{\mathrm{an}}$ *of* s *and* b *a Betti map on* $\mathcal{A}_\Delta$ *as in Proposition 4.1. Let* $Y \subseteq \mathcal{A}$ *be an irreducible closed subvariety such that* $P \in Y^{\mathrm{an}}$ *with* $\pi(P) = s$ *is not isolated in the fiber of* $b|_{Y^{\mathrm{an}} \cap \mathcal{A}_\Delta}$. *Then* $\widetilde{\rho}(h)(P) \in Y^{\mathrm{an}}$ *for all* $h \in \pi_1(S^{\mathrm{an}}, s)$. *Moreover, if* P *has finite order* N *in* $\mathcal{A}_s(\mathbb{C})$ *then* $\dim_P Y \cap \mathcal{A}[N] \geq 1$.

(ii) *Let* $\mathcal{B}$ *be a further abelian scheme over* S *and* $\alpha \colon \mathcal{A} \to \mathcal{B}$ *a morphism of abelian schemes over* S. *Then*

$$\widetilde{\rho}_{\mathcal{B}}(h)(\alpha|_{\mathcal{A}_s^{\mathrm{an}}}) = (\alpha|_{\mathcal{A}_s^{\mathrm{an}}})\widetilde{\rho}_{\mathcal{A}}(h)$$

for all $h \in \pi_1(S^{\mathrm{an}}, s)$

Although the Betti map b in Proposition 4.1 is not uniquely determined, Remark 4.2 implies that the non-isolation condition in the hypothesis above is independent of any choice of b.

Before we come to the proof we will patch together the Betti maps and extract global information.

Suppose $i \in \{1, \ldots, n-1\}$ and set $\Delta = \Delta_i \cap \Delta_{i+1} \ni \gamma(a_i)$. We consider the two real bi-analytic maps

$$b_i^*|_{\mathcal{A}_\Delta} \text{ and } b_{i+1}^*|_{\mathcal{A}_\Delta} \colon \mathcal{A}_\Delta \to \mathbb{T}^{2g} \times \Delta$$

where the star signifies passing to the product as in Proposition 4.1(iii). By composing we obtain

$$b_{i+1}^*|_{\mathcal{A}_\Delta} \circ (b_i^*|_{\mathcal{A}_\Delta})^{-1} \colon \mathbb{T}^{2g} \times \Delta \to \mathbb{T}^{2g} \times \Delta \tag{5.6}$$

which is, over each fiber of Δ, a continuous group isomorphism $\mathbb{T}^{2g} \to \mathbb{T}^{2g}$. By construction it is the identity over $\gamma(a_i) \in \Delta$. Each continuous group isomorphism of $\mathbb{T}^{2g}$ is represented by a matrix in $\mathrm{GL}_{2g}(\mathbb{Z})$. By homotopy, (5.6) is the identity above all points in the path component of Δ containing $\gamma(a_i)$. But Δ is path connected by construction, and therefore $b_i|_{\mathcal{A}_\Delta} = b_{i+1}|_{\mathcal{A}_\Delta}$ for all $i \in \{1, \ldots, n-1\}$.

Proof of Proposition 5.4. Let s, Y, and P be as in the hypothesis. We abbreviate $Y_{\Delta_1} = Y^{\mathrm{an}} \cap \mathcal{A}_{\Delta_1}$, it is a complex analytic space.

We will transport P in $\mathcal{A}^{\mathrm{an}}$ above along a loop γ in S^{an} based at s and keep the Betti coordinates fixed. After completing the loop we will have returned to the fiber $\mathcal{A}_s$. But P will have transformed according to the monodromy representation (5.3). The degeneracy condition imposed on P implies that this new point lies again in Y. This is guaranteed by the fact that the Betti fibers are complex analytic, see (ii) of Proposition 4.1 and our hypothesis $\dim S = 1$.

Let us check the details. We set $P_0 = P$ and $\xi = b_1(P_0)$ and define

$$Z_1 = b_1^{-1}(\xi).$$

So Z_1 is a complex analytic subset of the complex analytic space $\mathcal{A}_{\Delta_1}$ by (ii) of Proposition 4.1. Therefore, $Z_1 \cap Y_{\Delta_1}$ is complex analytic in Y_{Δ_1}. As P_0 is not isolated in $Z_1 \cap Y_{\Delta_1}$, we find $\dim_{P_0} Z_1 \cap Y_{\Delta_1} \geq 1$, see [21, Chapter 5] for the dimension theory of complex analytic spaces.

If $P = P_0$ happens to be a point of finite order N in $\mathcal{A}_{\pi(P)}(\mathbb{C})$, then all points of Z_1 have order N in their respective fibers as β is fiberwise a group isomorphism. From the degeneracy of P we conclude $\dim_P Y \cap \mathcal{A}[N] \geq 1$ and this yields the second claim of (i).

The natural projection $Z_1 \mapsto \Delta_1$ is holomorphic and a homeomorphism. So $\dim_Q Z_1 \leq \dim_{\pi(Q)} \Delta_1 = 1$ for all $Q \in Z_1$. So we conclude $\dim_{P_0} Z_1 \cap Y_{\Delta_1} = \dim_{P_0} Z_1 = 1$ and $\dim Z_1 = 1$. The singular points of Z_1 are isolated in Z_1, see [21, Chapter 6, §2.2]. Since Z_1 is homeomorphic to Δ_1 and the latter is homeomorphic to the open unit disc we conclude that the smooth locus of Z_1 is path connected. Therefore, we can apply the Identity Lemma [21, Chapter 9, §1.1] to conclude that $Z_1 \cap Y_{\Delta_1} = Z_1$, hence

$$Z_1 \subseteq Y_{\Delta_1}.$$

In particular, the point $P_1 = b_1^{-1}(\xi, \gamma(a_1)) \in Z_1$ also lies in Y_{Δ_1}.

Observe that we used the fact that S^{an} is a curve in a crucial way. Indeed, for higher dimensional S we cannot exclude $\dim Z_1 \cap Y_{\Delta_1} < \dim Z_1$ in the paragraph above. This makes applying the Identity Lemma impossible.

We have reached $\gamma(a_1)$ and will continue on the circuit along γ. However, by construction b_1^* and b_2^* agree on $\mathcal{A}_{s_1}^{\mathrm{an}}$ where $s_1 = \gamma(a_1)$. They also agree on $\mathcal{A}_s^{\mathrm{an}}$ for all s sufficiently close to s_1. Let $t_1, t_2, \ldots$ be a sequence of elements in $[0, a_1]$ with limit a_1. Then $b_1^{*-1}(\xi, \gamma(t_k))$ converges to P_1 as $k \to \infty$. For k sufficiently large we have $\gamma(t_k) \in \Delta_2$ and therefore $b_1^{*-1}(\xi, \gamma(t_k)) = b_2^{*-1}(\xi, \gamma(t_k))$. So $P_1 \in \pi_1^{-1}(\Delta_1 \cap \Delta_2)$ is not isolated in the fiber of $b_2 : \mathcal{A}_{\Delta_2} \to \mathbb{T}^{2g}$ restricted to Y_{Δ_2} above ξ.

Now we repeat the process and transport P_1 along $\gamma([a_1, a_2])$ to obtain $P_2 \in Y_{\Delta_2}$ with $\pi(P_2) = \gamma(a_2)$ that is not isolated in $b_3|_{Y_{\Delta_3}}$. Eventually, we will have returned to the fiber $\mathcal{A}_s$. The final point lies in Y_s^{an} and it is obtained from $P_0 \in Y^{\mathrm{an}}$ by a continuous group automorphism of $\mathcal{A}_s^{\mathrm{an}}$ that depends on the homotopy class of γ relative to $\{0, 1\}$. More precisely, by construction the final point is $\widetilde{\rho}([\gamma])(P_0)$ where

$$\widetilde{\rho} \colon \pi_1(S^{\mathrm{an}}, s) \to \{\text{homeomorphisms } \mathcal{A}_s^{\mathrm{an}} \to \mathcal{A}_s^{\mathrm{an}} \text{ that are group homomorphisms}\}$$

is a group homomorphism that is compatible with the monodromy representation (5.3), indeed

$$\widetilde{\rho}(h)_* = \rho(h) \quad \text{for all} \quad h \in \pi_1(S^{\mathrm{an}}, s)$$

and part (i) follows.

The proof of (ii) relies on (5.5) and some basic functoriality. Let $s \in S^{\mathrm{an}}$. A homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ of abelian schemes over S induces a group homomorphism $(\alpha|_{\mathcal{A}_s^{\mathrm{an}}})_* : H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z}) \to H_1(\mathcal{B}_s^{\mathrm{an}}, \mathbb{Z})$. Moreover, this group homomorphism is equivariant with respect to the action of $\pi_1(S^{\mathrm{an}}, s)$ on both homology groups. By abuse of notation let $\widetilde{\rho}$ denote the continuous action of $\pi_1(S^{\mathrm{an}}, s)$ on $\mathcal{A}_s$ and $\mathcal{B}_s$ and ρ the induced action on homology. We find

$$\left(\widetilde{\rho}(h)\alpha|_{\mathcal{A}_s^{\mathrm{an}}}\right)_* = \rho(h)(\alpha|_{\mathcal{A}_s^{\mathrm{an}}})_* = (\alpha|_{\mathcal{A}_s^{\mathrm{an}}})_*\rho(h) = (\alpha|_{\mathcal{A}_s^{\mathrm{an}}}\widetilde{\rho}(h))_*$$

for all $h \in \pi_1(S^{\mathrm{an}}, s)$, the first and third equality follow from (5.5), the second one follows since the monodromy action commutes with homomorphisms of abelian varieties. Both self-maps $\widetilde{\rho}(h)\alpha|_{\mathcal{A}_s^{\mathrm{an}}}$ and $\alpha|_{\mathcal{A}_s^{\mathrm{an}}}\widetilde{\rho}(h)$ are continuous group endomorphisms of $\mathcal{A}_s^{\mathrm{an}}$, which is homeomorphic to $\mathbb{T}^{2g}$. As their induced maps on homology coincide, they must coincide as well. $\square$

5.3. **Monodromy on Abelian Schemes.** Let S be an irreducible and smooth curve over $\mathbb{C}$ and let $\mathcal{A}$ be an abelian scheme over S of relative dimension $g \geq 1$. We write $\overline{\mathbb{C}(S)}$ for an algebraic closure of the function field $\mathbb{C}(S)$ of S.

For a base point $s \in S(\mathbb{C})$ the monodromy representation is (5.3). Let G_s denote the Zariski closure of $\Gamma_s = \rho(\pi_1(S^{\mathrm{an}}, s))$ in $\mathrm{Aut}_{\mathbb{Q}} H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Q})$ and let G_s^0 be its connected component containing the unit element. Deligne proved in [16, Corollaire 4.2.9] that G_s^0 is a semisimple algebraic group.

The next lemma uses a Theorem of Tits connected to his famous "alternative".

Lemma 5.5. *In the notation above suppose that G_s^0 is not trivial, then any finite index subgroup of Γ_s has a free subgroup on 2 generators.*

Proof. Let Γ' be a finite index subgroup of Γ_s. As G_s^0 is of finite index in G_s we see that $\Gamma' \cap G_s^0(\mathbb{Q})$ lies Zariski dense in G_s^0. Our lemma follows from [48, Theorem 3] applied to G_s^0 and $\Gamma' \cap G_s^0(\mathbb{Q})$. □

Certainly, G_s and G_s^0 etc. depend on s. However their isomorphism classes do not and the index $[G_s : G_s^0]$ is independent of $s \in S^{\mathrm{an}}$, see the comments before Zarhin's [64, Theorem 3.3].

Lemma 5.6. *Let A be the generic fiber of $\mathcal{A} \to S$, it is an abelian variety over $\mathbb{C}(S)$. If $s \in S(\mathbb{C})$ and $H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})$ has a non-zero element that is invariant under the monodromy action (5.3), then the $\mathbb{C}(S)/\mathbb{C}$-trace of A is non-zero.*

Proof. We write $H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})^\rho$ for the elements in $H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})$ that are invariant under (5.3). A conclusion of Deligne's Theorem of the Fixed Part, see [16, Corollaire 4.1.2], implies that the weight -1 Hodge structure on $H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})$ restricts to a Hodge structure on $H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})^\rho$.

It is well-known that Hodge substructures of $H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})$ come from abelian subvarieties of $\mathcal{A}_s$. Hence $H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})^\rho$ gives rise to an abelian subvariety $B \subseteq \mathcal{A}_s$ of dimension $\frac{1}{2}\mathrm{Rank}\, H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})^\rho$. As $H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})^\rho \neq 0$ by hypothesis we have $\dim B \geq 1$.

Then $\mathcal{B} = B \times_{\mathrm{Spec}\,(\mathbb{C})} S$ is a constant abelian scheme over S. The monodromy representation $\pi_1(S^{\mathrm{an}}, s) \to \mathrm{Aut}(H_1(\mathcal{B}_s^{\mathrm{an}}, \mathbb{Z}))$ is certainly trivial. The inclusion $\mathcal{B}_s \to \mathcal{A}_s$ induces a homomorphism $H_1(\mathcal{B}_s^{\mathrm{an}}, \mathbb{Z}) \to H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})$ and the restriction of ρ from (5.3) to the image of this homomorphism is trivial. A theorem of Grothendieck [24] implies, that any element in

$$\mathrm{Hom}(\mathcal{B}_s, \mathcal{A}_s) \cap \mathrm{Hom}(H_1(\mathcal{B}_s^{\mathrm{an}}, \mathbb{Z}), H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z}))$$

is induced by the restriction of a morphism $\varphi\colon \mathcal{B} \to \mathcal{A}$ over S to $\mathcal{B}_s$ such that $\varphi \circ 0_{\mathcal{B}} = 0_{\mathcal{A}}$ where $0_{\mathcal{A}}\colon S \to \mathcal{A}$ and $0_{\mathcal{B}}\colon S \to \mathcal{B}$ are the zero sections. See also [16, 4.1.3.2].

The restriction of φ to the generic fiber of $\mathcal{B}$ is a homomorphism $B \otimes_{\mathbb{C}} \mathbb{C}(S) \to \mathcal{A} \times_S \mathrm{Spec}\,\mathbb{C}(S) = A$ of abelian varieties over $\mathbb{C}(S)$. If the $\mathbb{C}(S)/\mathbb{C}$-trace of A is trivial, then the said homomorphism is trivial. In this case, the morphism φ and the zero section both extend $B \otimes_{\mathbb{C}} \mathbb{C}(S) \to A$ to a morphism $\mathcal{B} \to \mathcal{A}$. As the generic fiber lies Zariski dense in $\mathcal{A}$ we find that φ is the zero section. But then B must be trivial and this is a contradiction. □

For us, an abelian subscheme of $\mathcal{A}$ is the image of an endomorphism of $\mathcal{A}$. We call $s \in S(\mathbb{C})$ *extendable* for $\mathcal{A}$ if any abelian subvariety $B_s \in \mathcal{A}_s$ extends to an abelian subscheme $\mathcal{B}$ of $\mathcal{A}$, *i.e.* there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B} \cap \mathcal{A}_s = B_s$.

For readers who are familiar with Hodge theory, extendable points of S are closely related to Hodge generic points. We shall not go into details and but state the following corollary of a result of Deligne for our purpose.

Lemma 5.7. *In the notation above suppose $G_s^0 = G_s$ for some $s \in S(\mathbb{C})$. There is an at most countable infinite subset of $S(\mathbb{C})$ whose complement consists only of extendable points of $\mathcal{A}$.*

Proof. We refer to [64, Corollary 3.5 and the preceding comments] for this result. In fact in the reference, it is pointed out that the extendable points are precisely the Hodge generic points under this mild assumption ($G_s^0 = G_s$ for some $s \in S(\mathbb{C})$).

More precisely [64, Corollary 3.5 and the preceding comments] says that any $s \in S(\mathbb{C})$ outside an at most countably infinite set Σ satisfies the following property: For any $\alpha_s \in \mathrm{End}(\mathcal{A}_s)$ there exists $n \in \mathbb{N}$ such that $n\alpha_s$ is the restriction of an endomorphism of $\mathcal{A}$. Now for any $s \in S(\mathbb{C}) \setminus \Sigma$, any abelian subvariety B_s of $\mathcal{A}_s$ is the image of some $\alpha_s \in \mathrm{End}(\mathcal{A}_s)$. There exists $n \in \mathbb{N}$ such that $n\alpha_s$ is the restriction of an element $\alpha \in \mathrm{End}(\mathcal{A})$. And then we can take $\mathcal{B}$ to be the image of α. $\square$

Let Y be an irreducible closed subvariety of $\mathcal{A}$ that dominates S. Then Y is flat over S by [31, Proposition III.9.7]. We write Y_s for the fiber of $Y \to S$ above s with the reduced induced structure. By [31, Corollary III.9.6] we see that Y_s is equidimensional of dimension $\dim Y - 1$.

We say that Y is *virtually monodromy invariant* above $s \in S(\mathbb{C})$ if there exists an irreducible component Z of Y_s and a subgroup $G \subseteq \pi_1(S^{\mathrm{an}}, s)$ of finite index such that

$$\widetilde{\rho}(\gamma)(Z(\mathbb{C})) = Z(\mathbb{C}) \quad \text{for all} \quad \gamma \in G$$

for the representation $\widetilde{\rho}$ defined in Proposition 5.4.

Lemma 5.8. *In the notation above we suppose Y is an irreducible closed subvariety of $\mathcal{A}$ that dominates S. We assume that there is an uncountable set $M \subseteq S(\mathbb{C})$ satisfying all of the following properties:*

(i) *for all irreducible S' that are finite and étale over S the generic fiber of $\mathcal{A} \times_S S' \to S'$ has trivial $\mathbb{C}(S')/\mathbb{C}$-trace,*
(ii) *all elements in M are extendable for $\mathcal{A}$ (see Lemma 5.7 and above for definition),*
(iii) *and the variety Y is virtually monodromy invariant above all elements in M.*

Then there exist an abelian scheme $\mathcal{C}$ over S and a homomorphism $\mathcal{A} \to \mathcal{C}$ of abelian schemes over S whose kernel contains Y and has dimension $\dim Y$.

Proof. Our proof is by induction on

$$\dim \mathcal{A}.$$

The small possible value is 2 as we require $g \geq 1$. We call this the minimal case and we treat it directly below.

Let $s \in S(\mathbb{C})$ be arbitrary for the moment. If G_s^0, defined near the beginning of this subsection, is trivial then the image of $\pi_1(S^{\mathrm{an}}, s)$ under (5.3) is finite. By the Riemann Existence Theorem there is an irreducible curve S' that is finite and étale over S such that the monodromy representation of the fundamental group of S' at some base point $s' \in S'(\mathbb{C})$ on $H_1((\mathcal{A} \times_S S')_{s'}^{\mathrm{an}}, \mathbb{Z})$ is trivial. Recall that $g \geq 1$. By Lemma 5.6 the generic fiber of $\mathcal{A} \times_S S' \to S'$ has non-zero $\mathbb{C}(S')/\mathbb{C}$-trace, contradicting our hypothesis. Therefore, $\dim G_s^0 \geq 1$.

Let $s \in M$ with M as in the hypothesis. A finite index subgroup of $\pi_1(S^{\mathrm{an}}, s)$ acts on $Z_s(\mathbb{C})$ via (5.4) where Z_s is an irreducible component of Y_s. We write Γ'_s for the image of this finite index subgroup under the monodromy representation (5.3). Then Γ'_s has a free subgroup on 2 generators by Lemma 5.5. We invoke Lemma 5.6 by using hypothesis (i) and passing to a covering of S and find that no non-zero element of $H_1(\mathcal{A}_s^{\mathrm{an}}, \mathbb{Z})$ is invariant under the action of Γ'_s.

We aim to apply Proposition 5.3. But first let us verify that Γ'_s is of monodromy type with respect to corresponding Betti map. Indeed, an abelian subvariety B of $\mathcal{A}_s$ extends to an abelian subscheme $\mathcal{B}$ of $\mathcal{A}$ by hypothesis (ii). Then $\widetilde{\rho}_{\mathcal{A}}(\gamma)(B^{\mathrm{an}}) = \widetilde{\rho}_{\mathcal{A}}(\gamma)(\iota(B^{\mathrm{an}})) = \iota(\widetilde{\rho}_{\mathcal{B}}(\gamma)(B^{\mathrm{an}})) = B^{\mathrm{an}}$ by Proposition 5.4(ii) for all $\gamma \in \pi_1(S^{\mathrm{an}}, s)$ where $\iota : \mathcal{B} \to \mathcal{A}$ is the inclusion.

By Proposition 5.3. we are in one of two cases for any given $s \in M$. Let $M_{1,2}$ be the set of $s \in M$ such that we are in case $1, 2$, respectively. As $M = M_1 \cup M_2$ one among M_1, M_2 is uncountable.

Case 1: The set M_1 is uncountable.

For all $s \in M_1$ the subvariety Z_s is contained in the translate of a proper abelian subvariety B_s of $\mathcal{A}_s$ by a point P_s of finite order $N_s \in \mathbb{N}$. As M_1 is uncountable and $\mathbb{N}$ is countable, we may replace M_1 by an uncountable subset and assume that there exists $N \in \mathbb{N}$ such that $[N]P_s = 0$ for all $s \in M_1$.

Let us treat the minimal case $\dim \mathcal{A} = 2$ now. Then $B_s = \{0\}$ and thus $Z_s = \{P_s\}$ for all $s \in M_1$. But then Y contains an infinite, and hence Zariski dense, set of points lying in $\ker([N] \colon \mathcal{A} \to \mathcal{A})$. This completes the proof in the minimal case as we can take $\mathcal{C} = \mathcal{A}$ and $[N] \colon \mathcal{A} \to \mathcal{A}$.

We now treat the non-minimal case $\dim \mathcal{A} \geq 3$. By condition (ii) there exists an abelian subscheme $\mathcal{B}(s)$ of $\mathcal{A}$ such that $\mathcal{B}(s) \cap \mathcal{A}_s = B_s$ for any $s \in M_1$. But M_1 is uncountable and $\mathcal{A}$ has only countably many abelian subschemes, so we may replace M_1 by an uncountable subset and assume that there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A}$ with $\mathcal{B}(s) = \mathcal{B}$, *i.e.* $\mathcal{B} \cap \mathcal{A}_s = B_s$, for all $s \in M_1$.

We have $[N]Z_s \subseteq \mathcal{B} \cap \mathcal{A}_s$ for all $s \in M_1$. But $\bigcup_{s \in M_1} [N]Z_s$ is Zariski dense in $[N]Y$ by dimension reasons, so $[N]Y \subseteq \mathcal{B}$ by taking the Zariski closures on both sides.

Clearly $\mathcal{B}$ satisfies the analog trace condition (i) of the current lemma by basic properties of the trace. Any $s \in M_1$ is extendable for $\mathcal{B}$ because it is extendable for $\mathcal{A}$ and $\mathcal{B}$ is an abelian subscheme of $\mathcal{A}$. Finally $[N]Y$, as a subvariety of $\mathcal{B}$, is virtually monodromy invariant at each $s \in M_1$. To see this it suffices to prove that $[N]Y$ is virtually monodromy invariant as a subvariety of $\mathcal{A}$ by Proposition 5.4(ii). But then it suffices to show that $[N]Z_s$ is an irreducible component of $[N]Y_s$. This is true because $[N]Z_s$ is Zariski closed (as $[N]$ is proper) and $\dim [N]Y_s = \dim [N]Y - 1 = \dim Y - 1 = \dim Z_s = \dim [N]Z_s$.

We observe that $\dim \mathcal{B} = \dim B_s + 1 \leq (\dim \mathcal{A}_s - 1) + 1 = \dim \mathcal{A} - 1$. By induction there is an abelian scheme $\mathcal{C}$ over S and a homomorphism $\psi \colon \mathcal{B} \to \mathcal{C}$ of abelian schemes over S whose kernel contains $[N]Y$ and $\dim \ker \psi = \dim [N]Y = \dim Y$. Then $(\ker \psi)^\circ$, the identity component [8, §6.4] of $\ker \psi$, has dimension $\dim Y$ and is an abelian subscheme of $\mathcal{B}$ and hence of $\mathcal{A}$. There exists an integer $m \in \mathbb{N}$ such that $[m]\ker\psi \subseteq (\ker\psi)^\circ$. In particular $[mN]Y \subseteq (\ker\psi)^\circ$. Note that $\dim(\ker\psi)^\circ = \dim\ker\psi = \dim Y$.

Now it suffices to take $\mathcal{A} \to \mathcal{C}$ to be the composition $\mathcal{A} \xrightarrow{[mN]} \mathcal{A} \to \mathcal{A}/(\ker\psi)^\circ$.

$\boxed{\text{Case 2:}}$ The set M_2 is uncountable.

For all $s \in M_2$ there exists an abelian subvariety $B_s \subseteq \mathcal{A}_s$ with $\dim B_s \geq 1$ and $Z_s + B_s = Z_s$. Note that $\dim Y \geq 2$ since $\dim Z_s \geq 1$, so we are not in the minimal case.

By condition (ii) there exists an abelian subscheme $\mathcal{B}(s)$ of $\mathcal{A}$ such that $\mathcal{B}(s) \cap \mathcal{A}_s = B_s$ for any $s \in M_2$. Since M_2 is uncountable and $\mathcal{A}$ has only countably many abelian subschemes, we may replace M_2 by an uncountable subset and assume that there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A}$ with $\mathcal{B}(s) = \mathcal{B}$, *i.e.* $\mathcal{B} \cap \mathcal{A}_s = B_s$, for all $s \in M_2$.

We shall work with the abelian scheme $\mathcal{A}/\mathcal{B}$ over S. Let $\varphi\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the natural quotient. Then any fiber of φ has dimension $\dim_S \mathcal{B} = \dim \mathcal{B} - 1$. The condition $Z_s + B_s = Z_s$ implies that the fibers of $\varphi|_{Z_s}$ have dimension $\dim B_s = \dim \mathcal{B} - 1$ for all $s \in M_2$. Since $\bigcup_{s \in M_2} Z_s$ is Zariski dense in Y by dimension reasons, we see that a general fiber of $\varphi|_Y$ has dimension $\dim \mathcal{B} - 1$. Thus by Fiber Dimension Theorem we have

$$\dim Y = \dim \mathcal{B} - 1 + \dim \varphi(Y).$$

Clearly $\mathcal{A}/\mathcal{B}$ satisfies the analog trace condition (i) of the current lemma by basic properties of the trace. Any $s \in M_2$ is extendable for $\mathcal{A}/\mathcal{B}$ because any abelian subvariety of $(\mathcal{A}/\mathcal{B})_s$ is the quotient of an abelian subvariety of $\mathcal{A}_s$ and s is extendable for $\mathcal{A}$. Finally $\varphi(Y)$ is virtually monodromy invariant above all points in M_2 by Proposition 5.4(ii).

Now since $\dim(\mathcal{A}/\mathcal{B}) = \dim \mathcal{A} - \dim B_s \leq \dim \mathcal{A} - 1$, there exist by induction an abelian scheme $\mathcal{C}$ over S and a homomorphism $\psi\colon \mathcal{A}/\mathcal{B} \to \mathcal{C}$ whose kernel has dimension $\varphi(Y)$ and contains $\varphi(Y)$. Then $Y \subseteq \ker(\psi \circ \varphi)$ since $\varphi(Y) \subseteq \ker(\psi)$. But

$$\dim\ker(\psi \circ \varphi) = \dim_S \mathcal{B} + \dim\ker(\psi) = \dim \mathcal{B} - 1 + \dim \varphi(Y) = \dim Y.$$

So $\psi \circ \varphi\colon \mathcal{A} \to \mathcal{C}$ is what we desire. $\square$

5.4. **End of the Proof of Theorem 5.1.** Now we are ready to prove Theorem 5.1.

Let Y be an irreducible closed subvariety that is degenerate. We want to prove that Y is generically special.

Note that being generically special is a property on the geometric generic fiber. Moreover, it is enough to show that one irreducible component on the geometric generic fiber of Y has the property stated in Definition 1.2. We will remove finitely points from S and replace S by a finite and étale covering S' which we assume to be irreducible throughout this proof. Observe that the base change Y' of Y may no longer be irreducible. But it is étale over Y and thus reduced. In particular, Y' is flat over Y and thus over S. It follows that Y' is equidimensional of dimension $\dim Y$ by [31, Corollary III.9.6]. Note that if U is an open subset of Y^{an} consisting of degenerate points for Y, then its preimage will be open in Y'^{an} and consist of degenerate points.

So to ease notation we will write $S' = S$ below and take Y to be an irreducible component of Y'.

Let A be the generic fiber of $\mathcal{A}$. After possibly removing finitely many points from S and replacing by a finite étale covering we may assume that $A^{\overline{\mathbb{C}(S)}/\mathbb{C}} = A^{\mathbb{C}(S)/\mathbb{C}}$. We also assume that all abelian subvarieties of $A \otimes_{\mathbb{C}(S)} \overline{\mathbb{C}(S)}$ are defined over $\mathbb{C}(S)$. By passing to a further finite étale covering we may assume that $\mathcal{A}$ satisfies the hypothesis of Lemma 5.7. Let $\Sigma \subseteq S(\mathbb{C})$ be a countable subset such that any element in $S(\mathbb{C}) \setminus \Sigma$ is extendable for $\mathcal{A}$.

Let U be a non-empty, open subset of $Y^{\mathrm{an}} \cap \mathcal{A}_\Delta$ consisting of degenerate points for Y; here is Δ as above Theorem 5.1.

If $s \in S^{\mathrm{an}}$, then $\pi_1(S^{\mathrm{an}}, s)$ acts on $\mathcal{A}_s^{\mathrm{an}}$ via (5.4). We have proven in Proposition 5.4 that $\widetilde{\rho}(\gamma)(P) \in Y^{\mathrm{an}}$ for all $P \in U$ and all $\gamma \in \pi_1(S^{\mathrm{an}}, s)$. This property continues to hold with U replaced by the union $\bigcup_\gamma \widetilde{\rho}(\gamma)(U)$ over $\pi_1(S^{\mathrm{an}}, s)$. Note that U is open and invariant under the action of the fundamental group.

Let Z be an irreducible component of Y_s with $Z^{\mathrm{an}} \cap U \neq \emptyset$. The representation $\widetilde{\rho}$ maps $Z^{\mathrm{an}} \cap U$ into Y_s^{an}. As everything is real analytic we see that for each $\gamma \in \pi_1(S^{\mathrm{an}}, s)$ there is an irreducible component Z' of Y_s such that $\widetilde{\rho}(\gamma)(Z^{\mathrm{an}} \cap U) \subseteq Z'^{\mathrm{an}} \cap U$. Because all irreducible components of Y_s have dimension equal to $\dim Y - 1$ and by the Invariance of Domain Theorem we conclude that Z' is uniquely determined by $\widetilde{\rho}(\gamma)(Z^{\mathrm{an}} \cap U) \subseteq Z'^{\mathrm{an}} \cap U$ among all irreducible components of Y_s. We conclude that $\pi_1(S^{\mathrm{an}}, s)$ acts on the finite set of irreducible components of Y_s that meet U. Therefore, $\widetilde{\rho}(\gamma)(Z^{\mathrm{an}} \cap U) \subseteq Z^{\mathrm{an}} \cap U$ for all γ in a finite index subgroup of $\pi_1(S^{\mathrm{an}}, s)$.

The smooth locus of Z^{an} is path-connected, lies dense in Z^{an}, and contains a point of $Z^{\mathrm{an}} \cap U$. By fixing piece-wise real analytic paths we find that $\widetilde{\rho}(\gamma)(Z^{\mathrm{an}}) \subseteq Z^{\mathrm{an}}$ for all γ in the finite index subgroup mentioned before.

The arguments above show that Y is virtually monodromy invariant above s. Clearly, $U \setminus \Sigma$ is an uncountable set as U is open in S^{an} and non-empty.

Let us suppose $A^{\overline{\mathbb{C}(S)}/\mathbb{C}} = 0$ for the moment. We can apply Lemma 5.8 to $Y, \mathcal{A}$, and M equal to the set of s obtained from $U \setminus \Sigma$ and conclude that Y is an irreducible component of a subgroup scheme of $\mathcal{A}$ that is generically special. This completes the proof of Theorem 5.1 in the current case.

Let us turn to the general case. Recall that $\pi\colon \mathcal{A} \to S$ is an abelian scheme with generic fiber A whose $\mathbb{C}(S)/\mathbb{C}$-trace is $A^{\mathbb{C}(S)/\mathbb{C}}$. We take A_0 to be $A^{\mathbb{C}(S)/\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}(S) \subseteq A$. So the $\mathbb{C}(S)/\mathbb{C}$-trace of A/A_0 vanishes, *cf.* [14, Theorem 6.4 and the following comment]. Moreover, $(A/A_0)^{\overline{\mathbb{C}(S)}/\mathbb{C}} = 0$ as $A^{\overline{\mathbb{C}(S)}/\mathbb{C}} = A^{\mathbb{C}(S)/\mathbb{C}}$.

By [8, Proposition 3 §7.5], the Néron model $\mathcal{B}$ of A/A_0 is an abelian scheme over S and sits in the short exact sequence of abelian schemes over S

$$0 \to A^{\mathbb{C}(S)/\mathbb{C}} \times S \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \to 0.$$

In A we fix an abelian subvariety C that meets A_0 in a finite set and with $A_0 + C = A$. Let $\mathcal{C}$ be the Néron model of C. It is an abelian scheme over S and we may assume $\mathcal{C} \subseteq \mathcal{A}$. The restriction $\varphi|_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{B}$ is dominant and proper, hence surjective. It is fiberwise an isogeny of abelian varieties. We conclude $(A^{\mathbb{C}(S)/\mathbb{C}} \times S) + \mathcal{C} = \mathcal{B}$.

As Y is degenerate there exists an open and non-empty U subset of Y^{an} of degenerate points. By shrinking U we may assume that $\varphi|_Y : Y \to \varphi(Y)$ is smooth at all points of U. So $\varphi(U)$ is open in $\varphi(Y)^{\mathrm{an}}$. This set consists of degenerate points for $\varphi(Y)$. By the previous case ($A^{\overline{\mathbb{C}(S)}/\mathbb{C}} = 0$), the set of $P \in U$ such that $\varphi(P)$ has finite order in the corresponding fiber of $\mathcal{B}$ lies Zariski dense in Y.

We consider such a P and suppose $\varphi(P)$ has order N and write $P = Q + R$ with $Q \in (A^{\mathbb{C}(S)/\mathbb{C}} \times S)(\mathbb{C})$ and $R \in \mathcal{C}(\mathbb{C})$, where Q, R lie in the same fiber above S as P.

So $0 = [N](\varphi(P)) = \varphi([N](R))$. As $R \in \mathcal{C}(\mathbb{C})$ it must have finite order N'. Moreover, $R \in Y - \sigma_Q$ where σ_Q is the image of a constant section $S \to A^{\mathbb{C}(S)/\mathbb{C}} \times S$ with value Q.

The Betti map is constant on sufficiently small open subsets of σ_Q as $A^{\mathbb{C}(S)/\mathbb{C}} \times S$ is a constant abelian scheme. Therefore, R is a degenerate point of $Y - \sigma_Q$.

Recall that the order of a point is constant on a fiber of the Betti map. By the second claim in Proposition 5.4(i) there exists an irreducible component $C \subseteq \mathcal{C}[N']$ containing R with $C \subseteq Y - \sigma_Q$.

We conclude that P is a point of $\sigma_Q + C$, a generically special subvariety of $\mathcal{A}$. As this holds for a Zariski dense set of P in Y we conclude from Proposition 1.3 that Y is generically special.

6. Construction of the Auxiliary Variety

In this section we work in the category of schemes over an algebraically closed subfield F of $\mathbb{C}$. We abbreviate $\mathbb{P}^m_F$ by $\mathbb{P}^m$ throughout this section. Suppose S is a smooth irreducible algebraic curve. Let $\mathcal{A}$ be an abelian scheme of relative dimension $g \geq 1$ over S with structural morphism $\pi\colon \mathcal{A} \to S$. For a closed subvariety $X \subseteq \mathcal{A}$ and $s \in S(F)$ we write $X_s = \pi^{-1}(s)$.

We assume that $\mathcal{A}$ comes equipped with an admissible immersion $\mathcal{A} \to \mathbb{P}^M \times \mathbb{P}^m$ as in §2.2, *i.e.*, it satisfies conditions (A1), (A2), and (A3) in §2.2. In particular, each fiber $\mathcal{A}_s$ of π with $s \in S(F)$ is an abelian variety in $\mathbb{P}^M$. On this projective space we let $\deg(\cdot)$ denote the degree of an algebraic set.

In this section X will denote an irreducible, closed subvariety of $\mathcal{A}$ that dominates S and with $X \neq \mathcal{A}$. Hence $\pi|_X\colon X \to S$ is surjective as $\pi|_X$ is proper. We write $\dim X = \dim \mathcal{A} - n = g + 1 - n$ where $n \geq 1$ is the codimension of X in $\mathcal{A}$.

Let $\Delta \subseteq S^{\mathrm{an}}$ be a non-empty open subset with Betti map $b\colon \pi^{-1}(\Delta) = \mathcal{A}_\Delta \to \mathbb{T}^{2g}$. See Proposition 4.1, we recall that $\mathbb{T}$ denotes the circle group. It is convenient to write $X_\Delta = X^{\mathrm{an}} \cap \mathcal{A}_\Delta$.

The following convention will be used in this section. If P is a point on a real (resp. complex) analytic manifold Y, then $T_P(Y)$ denotes the tangent space of Y at P. This is an $\mathbb{R}$- resp. $\mathbb{C}$-vector space, depending on whether Y is a real or complex analytic manifold. If Z is another real (resp. complex) analytic manifold and $f\colon Y \to Z$ is a real (resp. complex) analytic mapping, then $T_P(f)$ denotes the differential $T_P(Y) \to T_{f(P)}(Z)$. It is $\mathbb{R}$- (resp. $\mathbb{C}$-)linear. Let $\mathrm{im}(T_P(f))$ denote the image of $T_P(f)$ in $T_{f(P)}(Z)$.

Recall that $X^{\mathrm{sm,an}}$ is the complex analytic space attached to the smooth locus X^{sm} of X. If $P \in \mathcal{A}_\Delta \cap \mathcal{A}(F)$ then $b|_{X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta}\colon X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ is a real analytic map. The condition in the proposition below concerns the image of its differential.

Proposition 6.1. *We keep the notation from above and assume that X is not generically special. Suppose $P \in \mathcal{A}_\Delta \cap \mathcal{A}(F)$ with $\pi(P) = s$ such that P is a smooth point of X_s and of X with*

$$\dim \mathrm{im}(T_P(b|_{X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta})) = 2 \dim X. \tag{6.1}$$

Then there exists a closed irreducible subvariety $Z \subseteq \mathcal{A}$ over F with the following properties.

(i) *We have $\dim Z = n$ and Z dominates S.*
(ii) *We have that P is a smooth point of Z_s and of Z.*

(iii) *The fiber* Z_s *does not contain any positive dimensional coset in* $\mathcal{A}_s$.
(iv) *There exists* $D \geq 1$ *such that* $\deg Z_t \leq D$ *for all* $t \in S(\mathbb{C})$.
(v) *We have* $\mathrm{im}(T_P(b|_{X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta})) \cap \mathrm{im}(T_P(b|_{(Z_s)^{\mathrm{sm,an}}})) = 0$ *in* $T_{b(P)}(\mathbb{T}^{2g})$.

Moreover, the set

$$\{t \in S(\mathbb{C}) : Z_t \textit{ contains a positive dimensional coset in } \mathcal{A}_t\}.$$

is finite.

Condition (v) implies that Z_s and X_s intersect transversally in $\mathcal{A}_s$. Condition (i) implies that Z_t is equidimensional of dimension $n-1$ for all $t \in S(F)$ by [31, Corollary III.9.6 and Proposition III.9.7].

We will prove this proposition in the next few subsections, see §6.1-6.2 for the construction of Z and §6.4 for the "Moreover" part. But first, let us relate its hypothesis (6.1) to our notion of generically special. A crucial point is to use Theorem 5.1.

Lemma 6.2. *Suppose that* X *is not generically special. Then there exists* $P \in X^{\mathrm{sm}}(F)$ *with* $\pi(P) \in \Delta$ *and* $P \in (X_{\pi(P)})^{\mathrm{sm}}(F)$ *that satisfies (6.1).*

Proof. Let us consider the restriction

$$b|_{X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} : X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta \to \mathbb{T}^{2g}.$$

Observe that domain and target are smooth manifolds of dimension $2\dim X$ and $2g$, respectively.

Let $r \in \{0, \ldots, 2g\}$ denote the largest possible rank of $T_P(b|_{X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta})$ as P ranges over the domain. Then there exists an open and non-empty subset $\mathfrak{U}$ of $X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ on which the rank is r. It follows from [56, Appendix II, Corollary 7F], that any fiber of $b|_{\mathfrak{U}} \colon \mathfrak{U} \to b(\mathfrak{U})$ is a smooth manifold of dimension $2\dim X - r$.

By hypothesis and Theorem 5.1 the variety X is not degenerate. In particular, there exists $P \in \mathfrak{U}$ that is not degenerate for X. So the fiber of $b|_{\mathfrak{U}}$ through P contains P as an isolated point. So we have $r = 2\dim X$.

By continuity we may assume that (6.1) holds for all $P \in \mathfrak{U}$, after possibly shrinking $\mathfrak{U}$.

On the other hand the set $U = \{P \in X^{\mathrm{sm}}(F) : \pi|_X \colon X \to S \text{ is smooth at } P\}$ is Zariski open and dense in X. So $U(F) \cap \mathfrak{U} \neq \emptyset$ because F is algebraically closed and dense in $\mathbb{C}$. Now any point $P \in U(F) \cap \mathfrak{U}$ satisfies the desired properties as S is smooth. □

For the further construction of Z we assume that P is as in this lemma.

6.1. **The First Four Properties.** We show how to construct Z satisfying the first four properties in the proposition. Indeed, our construction will show that a generic choice, in a suitable sense, of Z will suffice for (i)-(iv). Later on we will see how to obtain in addition (v) and deduce the final statement.

Let P be as in the hypothesis of Proposition 6.1. Recall that $\mathcal{A}$ comes with an admissible immersion $\mathcal{A} \to \mathbb{P}^M \times \mathbb{P}^m$ as in §2.2. Observe that $\mathcal{A}_s \subseteq \mathbb{P}^M$ is Zariski closed, irreducible, and contains P as a smooth point as it is an abelian variety. By property (A3) a generic homogeneous linear form $f \in F[X_0, \ldots, X_M]$ vanishing at P satisfies the following property. The intersection of the zero locus $\mathscr{Z}(f)$ of f with $\mathcal{A}_s$ contains no positive dimensional cosets in $\mathcal{A}_s$. Here generic means that we may allow the coefficients of f to come from a Zariski open dense subset of all possible coefficient vectors.

According to Bertini's Theorem there are linearly independent homogeneous linear forms $f_1, \ldots, f_{g+1-n} \in F[X_0, \ldots, X_M]$ such that their set of common zeros $\mathscr{Z}(f_1, \ldots, f_{g+1-n})$ in $\mathbb{P}^M$ intersects $\mathcal{A}_s$ in a Zariski closed set Z' that is smooth at P and of dimension $\dim \mathcal{A}_s - (g+1-n) = g - (g+1-n) = n-1$. If $n \geq 2$ we may arrange that Z' is irreducible by applying a suitable variant of Bertini's Theorem. By the previous paragraph, we can arrange that Z' contains no positive dimensional cosets in $\mathcal{A}_s$. We will see that this establishes (ii), (iii), and (iv) with our choice of Z below.

Note that a generic choice of $(f_1, \ldots, f_{g+1-n})$ in $F[X_0, \ldots, X_M]^{\oplus(g+1-n)}$, where each entry has degree one, that vanishes at P will have the property described in the previous paragraph. Here generic means that we may allow the coefficient vector attached to $(f_1, \ldots, f_{g+1-n})$ to come from a Zariski open dense subset of all possible coefficient vectors that lead to linear forms with coefficients in F vanishing at P. We may arrange f_1 to be an f as in the last paragraph, so Z' contains no coset of positive dimension.

Each irreducible component of

$$(\mathscr{Z}(f_1, \ldots, f_{g+1-n}) \times \mathbb{P}^m) \cap \mathcal{A} \tag{6.2}$$

has dimension at least n. Suppose Z is an irreducible component of (6.2) that contains P. By the Fiber Dimension Theorem we find $\dim Z_s \geq \dim Z - \dim \pi(Z) \geq \dim Z - 1$. As $\dim Z' = n-1$ and $Z_s \subseteq Z'$ we conclude $\dim Z \leq n$. Thus $\dim Z = n, \dim \pi(Z) = 1$, and $\dim Z_s = n-1$. This implies both claims in (i).

If $n = 1$, then $\dim Z_s = 0$ and hence P is smooth in Z_s. If $n \geq 2$, then $Z_s = Z'$ and hence P is smooth in Z_s by construction. Now as P is smooth in Z_s and s is smooth in S, P is also smooth in Z. This establishes (ii).

If $n = 1$, then $\dim Z_s = 0$ and (iii) clearly holds. If $n \geq 2$, then by construction Z_s satisfies (iii). In both cases, Z_t is a union of irreducible components of $\mathscr{Z}(f_1, \ldots, f_{g+1-n}) \cap \mathcal{A}_t$ for all but at most finitely many $t \in S(\mathbb{C})$. For these t we conclude $\deg Z_t \leq \deg \mathcal{A}_t$ from Bézout's Theorem. But $\mathcal{A} \to S$ is a flat family embedded in $\mathbb{P}^M \times S \to S$, so $\deg \mathcal{A}_t \leq D$ for some $D \geq 1$ depending only on $\mathcal{A}$ and the immersion. We can take care of the remaining finitely many fibers by increasing D if necessary. Thus we have established (iv).

6.2. **The Fifth Property.**

6.2.1. *Linear Algebra.* For a $\mathbb{C}$-vector space T we write $T_\mathbb{R}$ for T with its natural structure as an $\mathbb{R}$-vector space. For example, if T is finite dimensional, then $\dim T_\mathbb{R} = 2\dim T$. A vector subspace V_0 of $T_\mathbb{R}$ is naturally an $\mathbb{R}$-vector space. We denote by $\mathbb{C}V_0$ the smallest vector subspace of T containing V_0. For example, if $V_0 = \mathbb{R}v_1 + \cdots + \mathbb{R}v_k$, then $\mathbb{C}V_0 = \mathbb{C}v_1 + \cdots + \mathbb{C}v_k$. Let J denote the multiplication by $\sqrt{-1}$ map $J\colon T \to T$. Then $(\mathbb{C}V_0)_\mathbb{R} = V_0 + JV_0$. A vector subspace of $T_\mathbb{R}$ is a vector subspace of T if and only if it is J-invariant.

In this section $g \geq 1$ is an integer. We show that an even dimensional real subspace of $\mathbb{C}^g$ intersects some complex subspace of complementary real dimension transversally.

Lemma 6.3. *Let T be a $\mathbb{C}$-vector space of dimension g and suppose W is a vector subspace of T with $\dim W = m$. Let V_0 be a vector subspace of $T_\mathbb{R}$ of dimension $2m+2k$ that contains W. Then there exists a vector subspace V of T of dimension $g-(m+k)$ such that $V \cap V_0 = 0$.*

Before proving this lemma, let us do the following preparation.

Lemma 6.4. *Let $\mathbb{C}^{2k}$ be the standard complex vector space of dimension $2k$ and let $\mathbb{R}^{2k} \subseteq \mathbb{C}^{2k}$ be the real part of $\mathbb{C}^{2k}$, i.e. $\mathbb{C}^{2k} = \mathbb{R}^{2k} \oplus \sqrt{-1}\mathbb{R}^{2k}$. Then there exists a vector subspace V of $\mathbb{C}^{2k}$ of dimension k such that $V \cap \mathbb{R}^{2k} = 0$.*

Proof. For any $j = 1, \ldots, 2k$, we let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^{2k}$ be the vector with the j-th entry being 1 and the other entries being 0.

For any $i = 1, \ldots, k$, we let $v_i = e_{2i-1} + \sqrt{-1}e_{2i} \in \mathbb{C}^{2k}$. We show that the complex vector space $V = \mathbb{C}v_1 + \cdots + \mathbb{C}v_k$ satisfies the desired property.

Indeed, $\dim V = k$ as $v_1, \ldots, v_k$ are $\mathbb{C}$-linearly independent. So it remains to show $V \cap \mathbb{R}^{2k} = 0$. Any vector in V is of the form $c_1v_1 + \cdots + c_kv_k$ for some $c_1, \ldots, c_k \in \mathbb{C}$. If $c_1v_1 + \cdots + c_kv_k \in \mathbb{R}^{2g}$, then we have

$$c_i \in \mathbb{R} \text{ and } \sqrt{-1}c_i \in \mathbb{R} \text{ for all } i = 1, \ldots, k.$$

Thus $c_1 = \cdots = c_k = 0$. □

Lemma 6.5. *Let U be a $\mathbb{C}$-vector space of dimension $2k$ and let V_0 be a vector subspace of $U_{\mathbb{R}}$ of dimension $2k$ such that $\mathbb{C}V_0 = U$. Then there exists a vector subspace V of U of dimension k such that $V \cap V_0 = 0$.*

Proof. We take a basis of V_0 which is an $\mathbb{R}$-vector space, and call it $e_1, \ldots, e_{2k}$. Since $\mathbb{C}V_0 = U$, we have $U = \mathbb{C}e_1 + \cdots + \mathbb{C}e_{2k}$. But $\dim U = 2k$, so $e_1, \ldots, e_{2k}$ form a basis of U.

Now under the identification $U = \mathbb{C}^{2k}$ via the basis $e_1, \ldots, e_{2k}$, the vector subspace V_0 of U becomes the real part of $\mathbb{C}^{2k}$. We can apply the previous lemma to conclude. □

Now we are ready to prove Lemma 6.3.

Proof of Lemma 6.3. We begin by showing that we can reduce to the case $m = 0$. If the lemma is known when $m = 0$, then we apply it to the $\mathbb{C}$-vector space T/W and the image of the $\mathbb{R}$-vector space V_0 in this quotient to get a vector subspace V' of T/W of dimension $g - (m + k)$. Let $W^\perp$ be a vector subspace of T with $W + W^\perp = T$ and $W \cap W^\perp = 0$. Then the natural linear map $W^\perp \to T/W$ is an isomorphism. The preimage of V' under this map is the vector subspace which we desire.

Now we treat the case $m = 0$, note that $W = 0$ in this case. As above we write J for multiplication by $\sqrt{-1}$ on T. Then $(\mathbb{C}V_0)_{\mathbb{R}} = V_0 + JV_0$.

$\boxed{\text{Case (i)}}$ The $\mathbb{R}$-vector space V_0 contains no non-zero vector subspace of T.

In this case $V_0 \cap JV_0$, being a J-invariant vector subspace of V_0, must be trivial. So $\dim(\mathbb{C}V_0)_{\mathbb{R}} = \dim V_0 + \dim JV_0 = 2k + 2k = 4k$ and hence $\dim \mathbb{C}V_0 = 2k \leq g$. Thus we can apply the previous lemma to $U = \mathbb{C}V_0$ and V_0 to get a vector subspace V' of $\mathbb{C}V_0$ of dimension k such that $V' \cap V_0 = 0$. Then it suffices to take $V = V' + V''$ for any vector subspace $V'' \subseteq T$ with $\mathbb{C}V_0 + V'' = T$ and $\mathbb{C}V_0 \cap V'' = 0$.

$\boxed{\text{Case (ii)}}$ General case.

We write V_0^J for the largest J-invariant vector subspace of V_0. As it is J-invariant by definition, we consider it as a $\mathbb{C}$-vector space. Then $T' = T/V_0^J$ is a $\mathbb{C}$-vector space of dimension $g - \dim V_0^J$, and $V_0' = V_0/V_0^J$ is a vector subspace of $T'_{\mathbb{R}}$ of dimension $2(k - \dim V_0^J)$.

We claim that V_0' contains no non-zero vector subspace of the $\mathbb{C}$-vector space T'. If U' is a vector subspace of T' with $U' \subseteq V_0'$, then its preimage under the quotient $T \to T' = T/V_0^J$ is a vector subspace of T that is contained in V_0 and that contains V_0^J. The maximality of V_0^J yields $U' = 0$.

Now we can apply case (i) to T' and $V_0' \subseteq T_{\mathbb{R}}'$ to get a vector subspace V' of T' of dimension $(g - \dim V_0^J) - (k - \dim V_0^J) = g - k$ such that $V' \cap V_0' = 0$. Let V'' be the preimage of V' under the quotient $T \to T' = T/V_0^J$. Then V'' is a vector subspace of T with dimension $g - k + \dim V_0^J$ such that $V'' \cap V_0 = V_0^J$. Recall that V_0^J is a vector subspace of the $\mathbb{C}$-vector space T, and hence a vector subspace of V''. Now it suffices to let V be any complement of V_0^J in V''. $\square$

Let g and T be as in Lemma 6.3 and suppose $k \geq 0$ is an integer. Let $\mathrm{Gr}(T_{\mathbb{R}}, 2k)$ denote the set of all $2k$-dimensional vector subspaces of $T_{\mathbb{R}}$. On identifying $T_{\mathbb{R}}$ with $\mathbb{R}^{2g}$ we may use Plücker coordinates to identify $\mathrm{Gr}(T_{\mathbb{R}}, 2k)$ with a closed subset of $\mathbb{P}^N(\mathbb{R})$ where $N = \binom{2g}{2k} - 1$.

Note that $\mathbb{P}^N(\mathbb{R})$ is equipped with the archimedean topology that makes it a compact Hausdorff space. We will use this topology and its induced subspace topology on $\mathrm{Gr}(T_{\mathbb{R}}, 2k)$.

Multiplication by $\sqrt{-1}$ induces an $\mathbb{R}$-linear automorphism $T_{\mathbb{R}} \to T_{\mathbb{R}}$ and hence a selfmap $\mathrm{Gr}(T_{\mathbb{R}}, 2k) \to \mathrm{Gr}(T_{\mathbb{R}}, 2k)$. By the Cauchy-Binet Formula this selfmap can be described on $\mathrm{Gr}(T_{\mathbb{R}}, 2k) \subseteq \mathbb{P}^N(\mathbb{R})$ by linear forms. Its fixpoints are precisely the $2k$-dimensional vector subspaces of $T_{\mathbb{R}}$ that are k-dimensional vector subspaces of T. We write $\mathrm{Gr}(T, k)$ for the set of these fix points. It is a closed subset of $\mathrm{Gr}(T_{\mathbb{R}}, 2k)$. In this notation we can use Lemma 6.3 to prove the following.

Lemma 6.6. *Let T be a $\mathbb{C}$-vector space of dimension g and suppose W is a vector subspace of T with $\dim W = m \leq g - 1$. Let V be a vector subspace of $T_{\mathbb{R}}$ of dimension $2(m+1)$ that contains W. There exists a non-empty open (in the archimedean topology) subset $\mathfrak{U} \subseteq \mathrm{Gr}(T, g - m - 1)$ such that $V' \cap V = 0$ for all $V' \in \mathfrak{U}$.*

Proof. By the Cauchy-Binet Formula the set

$$\{V' \in \mathrm{Gr}(T_{\mathbb{R}}, 2(g - m - 1)) : V' \cap V = 0\}$$

is the complement in $\mathrm{Gr}(T_{\mathbb{R}}, 2(g - m - 1))$ of the zero set of a homogeneous linear polynomial defined on the projective coordinates of $\mathbb{P}^N$. So the set in question is open in $\mathrm{Gr}(T_{\mathbb{R}}, 2(g - m - 1))$. Its intersection $\mathfrak{U}$ with $\mathrm{Gr}(T, g - m - 1)$ lies open in $\mathrm{Gr}(T, g - m - 1)$. But $\mathfrak{U} \neq \emptyset$ by Lemma 6.3 applied to the case $k = 1$. $\square$

6.2.2. *Verification of (v) in Proposition 6.1.* We retain the conventions made in §6 and §6.1. So P is as in the hypothesis. We set up the various vector spaces needed in Lemma 6.6.

For T we take the tangent space $T_P(\mathcal{A}_s^{\mathrm{an}})$ which is a $\mathbb{C}$-vector space of dimension g. Note that P and π are defined over F, so this complex space T also descends to F.

For W we take the image of $T_P((X_s)^{\mathrm{sm,an}})$ under the linear map

$$T_P((X_s)^{\mathrm{sm,an}}) \to T_P(\mathcal{A}_s^{\mathrm{an}})$$

induced by the inclusion $(X_s)^{\mathrm{sm}} \to \mathcal{A}_s$ (recall that P is a smooth point of X_s). Its dimension equals $\dim X_s = \dim X - 1 = g - n \leq g - 1$ which we define as m.

Finally, we take $V = \mathrm{im}(T_P(b|_{X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta}))$ which is a vector subspace of the $\mathbb{R}$-vector space $T_{b(P)}(\mathbb{T}^{2g})$. As $b|_{\mathcal{A}_s^{\mathrm{an}}} : \mathcal{A}_s^{\mathrm{an}} \to \mathbb{T}^{2g}$ is an isomorphism of real analytic spaces, we can identify $T_P(\mathcal{A}_s^{\mathrm{an}}) = T$ with $T_{b(P)}(\mathbb{T}^{2g})$ as $\mathbb{R}$-vector spaces. Therefore, $V \subseteq T_{\mathbb{R}}$ as in the setup of Lemma 6.6. Note that V does not carry a complex structure as we treat $\mathbb{T}^{2g}$ as a real analytic space.

Our hypothesis (6.1) implies $\dim V = 2\dim X = 2(m+1)$. So the hypothesis of Lemma 6.6 is satisfied.

So there exists $\mathfrak{U}$ open in $\mathrm{Gr}(T, g-m-1)$, the latter is a compact Hausdorff space. Its points correspond to $(g-m-1)$-dimensional vector subspaces of the $\mathbb{C}$-vector space T.

In §6.1 we saw that a generic choice of $f_1, \ldots, f_{g+1-n}$ vanishing at P yields properties (i)-(iv) in the proposition. To obtain (v) we must make sure that $V \cap \mathrm{im}(T_P(b|_{(Z_s)^{\mathrm{sm,an}}})) = 0$. According to Lemma 6.6 this holds if $\mathrm{im}(T_P(b|_{(Z_s)^{\mathrm{sm,an}}}))$, a vector subspace of the $\mathbb{C}$-vector space $T_P(\mathcal{A}_s^{\mathrm{an}})$, under the identification of $T_{b(P)}(\mathbb{T}^{2g})$ with $T_P(\mathcal{A}_s^{\mathrm{an}})$ made above, lies in $\mathfrak{U}$. Ranging over all possible choices of $f_1, \ldots, f_{g+1-n}$ as in §6.1 yields points in $U(F)$ for some Zariski open dense subset $U \subseteq \mathrm{Gr}(T, g-m-1)$. As $\mathfrak{U}$ is open in the archimedean topology and since $U(F)$ lies dense in $\mathrm{Gr}(T, g-m-1)$ we have $\mathfrak{U} \cap U(F) \neq \emptyset$. Any element in $\mathfrak{U} \cap U(F)$ is sufficient and this completes the proof. □

6.3. **A Detour to Bézout's Theorem.** In this subsection we prove the following degree bound on long intersections. It will be used to prove the "Moreover" part of Proposition 6.1 in the next subsection. In this subsection we temporarily allow F to be any algebraically closed field of characteristic 0.

Proposition 6.7. *Suppose $V_1, \ldots, V_m$ are irreducible closed subvarieties of $\mathbb{P}^n_F = \mathbb{P}^n$ such that $\deg(V_i) \leq \delta$ for all $i \in \{1, \ldots, m\}$. Let $C_1, \ldots, C_r$ be all the irreducible components of $V_1 \cap \cdots \cap V_m$ of top dimension which we denote by k, then*

$$\sum_{i=1}^{r} \deg(C_i) \leq \delta^{n-k}. \tag{6.3}$$

The crucial aspect of (6.3) is that the right-hand side is independent of m.

Lemma 6.8. *Let V be an irreducible closed subvariety of $\mathbb{P}^n$ of degree δ. Then there exist finitely many irreducible hypersurfaces of $\mathbb{P}^n$ of degree at most δ such that V is their intersection.*

Proof. This is Faltings's [19, Proposition 2.1]. □

Proof of Proposition 6.7. By Lemma 6.8, we may assume that every V_i is an irreducible hypersurface for all $i \in \{1, \ldots, m\}$. Then $V_i = \mathscr{Z}(f_i)$ is the zero locus of an irreducible homogeneous polynomial $f_i \in F[X_0, \ldots, X_n]$ of degree at most δ.

We shall prove inductively on $s \in \{1, \ldots, n-k\}$ that there exist hypersurfaces $H_1, \ldots, H_{n-k}$ (possibly reducible) of degree at most δ such that for all $s \in \{1, \ldots, n-k\}$,

(i) each irreducible component of $\bigcap_{j=1}^{s} H_j$ has dimension at most $n - s$
(ii) and $C_i \subseteq \bigcap_{j=1}^{s} H_j$ for each $i \in \{1, \ldots, r\}$.

Assume this for $s = n - k$. Then each C_i, being of dimension k, is an irreducible component of $\bigcap_{j=1}^{n-k} H_j$, and thus

$$\sum_{i=1}^{r} \deg C_i \leq \prod_{j=1}^{n-k} \deg H_j \leq \delta^{n-k}$$

by Bézout's Theorem, *cf.* [20, Example 8.4.6] which holds here even though the hypersurfaces H_j may be reducible.

Let us take $H_1 = V_1$. Then $\deg H_1 \leq \delta$.

Now suppose we have constructed $H_1, \ldots, H_{s-1}$ for some $2 \leq s \leq n-k$.

Let $W_1, \ldots, W_t$ be the irreducible components of $H_1 \cap \cdots \cap H_{s-1}$. Let $l \in \{1, \ldots, t\}$. Since $s \leq n-k$, we have $\dim W_l > k$. By assumption each irreducible component of $\bigcap_{i=1}^{m} \mathscr{Z}(f_i) = \bigcap_{i=1}^{m} V_i$ has dimension at most k, so there exists some $i_0 \in \{1, \ldots, m\}$ such that $\widetilde{f_l} = f_{i_0}$ does not vanish on W_l. Then $\widetilde{f_l} \in F[X_0, \ldots, X_n]$ has degree at most δ and vanishes on $C_1 \cup \cdots \cup C_r$. We may assume that $\deg \widetilde{f_l} = \delta$ after possibly multiplying $\widetilde{f_l}$ with a homogeneous polynomial of a suitable degree in general position.

Let $F[X_0, \ldots, X_n]_\delta$ be the union of 0 and the homogeneous polynomials in $F[X_0, \ldots, X_n]$ of degree δ. We can identify $F[X_0, \ldots, X_n]_\delta$ with $\mathbb{A}^{\binom{n+\delta}{n}}(F) = F^{\binom{n+\delta}{n}}$. Then

$$\{f \in \mathbb{A}^{\binom{n+\delta}{n}}(F) : f \text{ vanishes on } C_1 \cup \cdots \cup C_r\}$$

is the set of F-points of a linear subvariety $L \subseteq \mathbb{A}^{\binom{n+\delta}{n}}$, and $\{f \in L(F) : f|_{W_l} \neq 0\}$ defines a Zariski open U_l in L that is non-empty as $\widetilde{f_l} \in U_l(F)$. Now L is irreducible and so U_l is Zariski open and dense in L. In particular, the intersection

$$\Theta = \bigcap_{l=1}^{t} U_l(F)$$

is non-empty.

Now fix any $f_s \in \Theta$ and let $H_s = \mathscr{Z}(f_s)$. Then H_s has degree at most δ and no irreducible component of $H_1 \cap \cdots \cap H_{s-1} \cap H_s$ is an irreducible component of $H_1 \cap \cdots \cap H_{s-1}$. So the irreducible components of $H_1 \cap \cdots \cap H_s$ have dimension at most $n-s$ using (i) in the case $s-1$. Property (ii) clearly holds by the construction of the U_l. □

6.4. **Control of Bad Fibers.** In this subsection we prove the "Moreover" part of Proposition 6.1.

Recall our setting: $\pi \colon \mathcal{A} \to S$ is an abelian scheme of relative dimension $g \geq 1$ over a smooth irreducible curve, defined over $F \subseteq \mathbb{C}$. For simplicity we assume $F = \mathbb{C}$. We have constructed an auxiliary subvariety Z of $\mathcal{A}$ in Proposition 6.1. It remains to show that

$$\{t \in S(\mathbb{C}) : Z_t \text{ contains a positive dimensional coset in } \mathcal{A}_t\}.$$

is finite. It fact, we show that it follows from condition (iii) of Proposition 6.1. More precisely we shall prove the following result.

Proposition 6.9. *Let Z be an irreducible closed subvariety of $\mathcal{A}$ dominating S. Suppose $s \in S(\mathbb{C})$ such that Z_s contains no positive dimensional cosets in $\mathcal{A}_s$. Then*

$$\{t \in S(\mathbb{C}) : Z_t \textit{ contains a positive dimensional coset in } \mathcal{A}_t\}$$

is finite.

Proof. Let ℓ be a prime that we will choose in terms of $\mathcal{A}, Z$, and s later on.

We begin by introducing full level ℓ structure. We will take care to ensure that various quantaties are uniform in ℓ.

Let S' be an irreducible, quasi-projective curve over $\mathbb{C}$ that is also finite and étale over S such the base change $\mathcal{A}' = \mathcal{A} \times_S S'$ admits all ℓ^{2g} torsion sections

$$S' \to \mathcal{A}'[\ell].$$

We write $Z' = Z \times_S S'$ which comes with a closed immersion $Z' \to \mathcal{A}'$. Observe that Z may now longer be irreducible. But $Z' \to S'$ is flat as $Z \to S$ is, cf. [31, Proposition III.9.7]. So all irreducible components of Z' dominate S'. The morphism $Z' \to Z$ is finite and flat since $S' \to S$ is. Therefore Z' is equidimensional of dimension $\dim Z$ by [31, Corollary III.9.6]. Finally, Z' is reduced since $Z' \to Z$ is étale and Z is reduced. For any $t' \in S'(\mathbb{C})$ above $t \in S(\mathbb{C})$, we may identify the fiber Z_t with $Z'_{t'}$ and the fiber $\mathcal{A}_t$ with $\mathcal{A}_{t'}$. Let $Z'_1, \ldots, Z'_r$ be the irreducible components of Z'. Both $(Z'_i)_{t'}$ and Z_t are equidimensional of dimension $\dim Z' - 1 = \dim Z - 1$ as $Z' \to S'$ and $Z \to S$ are flat, cf. [31, Corollary III.9.6]. Since $(Z'_i)_{t'} \subseteq Z_t$ an irreducible component of $(Z'_i)_{t'}$ is also an irreducible component of Z_t. Moreover, Z_t contains a positive dimensional coset in $\mathcal{A}_t$ if and only if one among $(Z'_1)_{t'}, \ldots, (Z'_r)_{t'}$ contains a positive dimensional coset in $\mathcal{A}'_{t'}$.

To prove the proposition we may thus suppose that $S = S'$, $\mathcal{A} = \mathcal{A}'$, and Z is some Z'_i. In particular, ℓ^{2g} distinct torsion sections $S \to \mathcal{A}[\ell]$ exist.

For any non-zero torsion section $\sigma\colon S \to \mathcal{A}[\ell]$ we define

$$Z(\sigma) = Z \cap (Z - \sigma) \cap (Z - [2] \circ \sigma) \cap \cdots \cap (Z - [\ell - 1] \circ \sigma)$$

by identifying a section $S \to \mathcal{A}$ with its image in $\mathcal{A}$. Then $Z(\sigma)$ is Zariski closed in $\mathcal{A}$.

Now suppose $t \in S(\mathbb{C})$ such that Z_t contains $P + B$ where $P \in \mathcal{A}_t(\mathbb{C})$ and $B \subseteq \mathcal{A}_t$ is an abelian subvariety of positive dimension. Therefore, $B[\ell]$ is a non-trivial group and there exists a section $\sigma : S \to \mathcal{A}[\ell]$ such that $\sigma(t) \in B[\ell]\backslash\{0\}$. Hence $\sigma(t) + B = B$ and we find

$$P + B = P + B - [k](\sigma(t)) \subseteq Z_t - [k](\sigma(t)) \quad \text{for all} \quad k \in \mathbb{Z}.$$

This implies $P + B \subseteq Z(\sigma)_t$. In particular, $t \in \pi(Z(\sigma))$.

Now π is a proper morphism and so $\pi(Z(\sigma))$ is Zariski closed in S for all of the finitely many σ as above. In order to prove the proposition it suffices to show that s from Proposition 6.1 does not lie in any $\pi(Z(\sigma))$ if $\sigma \neq 0$, for then all $\pi(Z(\sigma))$ are finite. We will prove that $Z(\sigma)_s = \emptyset$ for all non-zero sections $\sigma\colon S \to \mathcal{A}[\ell]$.

Recall that the admissible immersion from the beginning of this section induces a polarization on $\mathcal{A}_s$ and, as usual, we use $\deg(\cdot)$ to denote the degree. This polarization and $\mathcal{A}_s$ do not depend on the base changed defined using ℓ. Let us assume $Z(\sigma)_s \neq \emptyset$. This will lead to a contradiction for ℓ large in terms of $Z_s, \mathcal{A}_s$, and the polarization.

Note that

$$Z(\sigma)_s = Z_s \cap (Z_s - \sigma(s)) \cap \cdots \cap (Z_s - [\ell - 1] \circ \sigma(s)) \tag{6.4}$$

is Zariski closed in $\mathcal{A}_s$ and stable under translation by the subgroup of $\mathcal{A}_s(\mathbb{C})$ of order ℓ that is generated by $\sigma(s)$. Observe that if W' is an irreducible component of $Z(\sigma)_s$ of maximal dimension, then $\sigma(s) + W'$ is also an irreducible component of $Z(\sigma)_s$. We define W to be the union of the top dimensional irreducible components of $Z(\sigma)_s$. The group generated by $\sigma(s)$ acts on the set of irreducible components of W.

Recall that Z_s and thus each $Z_s-[k](\sigma(s))$ with $k\in\mathbb{Z}$ is equidimensional of dimension $\dim Z-1$. All irreducible components that appear have degree bounded by a constant independent of the auxiliary prime ℓ. By (6.4) and Proposition 6.7 the degree $\deg(W)$ is bounded from above by a constant $c\geq 1$ that is independent of ℓ.

The number N of irreducible components of W is at most $\deg(W)\leq c$. If we assume $\ell>N$, then the symmetric group on N symbols contains no elements of order ℓ. So if we assume $\ell>c$, as we may, then $\sigma(s)+W'=W'$ for all irreducible components W' of W.

Now let us fix such an irreducible component W'. Then the subgroup generated by $\sigma(s)$ lies in the stabilizer $\mathrm{Stab}(W')$ of W'. By [15, Lemme 2.1(ii)], the degree of the stabilizer $\mathrm{Stab}(W')$ is bounded from above solely in terms $\deg(W')$ and $\dim W'\leq g$. Note also that $\deg(W')\leq\deg(W)\leq c$. Thus if ℓ is large in terms of c and g, then we can arrange $\ell>\deg\mathrm{Stab}(W')$. But $\mathrm{Stab}(W')$ contains $\sigma(s)$ which has order ℓ. Therefore B, the connected component of $\mathrm{Stab}(W')$ containing the neutral element, has positive dimension. Fix any $P\in W'(\mathbb{C})$. Then

$$P+B\subseteq W'\subseteq W\subseteq Z_s$$

contradicts the hypothesis that Z_s does not contain a positive dimensional coset. $\square$

7. Lattice Points

For our abelian scheme $\mathcal{A}\to S$ and subvariety $X\subseteq\mathcal{A}$, we want to count the number of points in $[N]X\cap Z$ for each $N\gg 1$ where $Z\subseteq\mathcal{A}$ is of complimentary dimension of X (as constructed in Proposition 6.1). It is equivalent to count the intersection points of $[N]X-Z$ and the zero section of $\mathcal{A}\to S$. Via the Betti map and a local lift with respect to $\mathbb{R}^{2g}\to\mathbb{T}^{2g}$, we obtain a subset $\tilde{U}_N\subseteq\mathbb{R}^{2g}$ from $[N]X-Z$ and we are led to counting lattice point in $\tilde{U}_N$. The goal of this section is to settle the lattice point counting problem.

Suppose $m,m'\in\mathbb{N}$ and let ψ be a function defined on a non-empty open subset U of $\mathbb{R}^{m'}$ with values in $\mathbb{R}^m$. We suppoes that the coordinate functions of ψ lie in $C^1(U)$, the $\mathbb{R}$-vector space of real valued functions on U that are continuously differentiable. We write $D_z(\psi)\in\mathrm{Mat}_{mm'}(\mathbb{R})$ for the jacobian matrix of ψ evaluated at $z\in U$. We also set

$$|\psi|_{C^1}=\max\left\{\sup_{x\in U}|\psi(x)|,\sup_{x\in U}\left|\frac{\partial\psi}{\partial x_1}(x)\right|,\ldots,\sup_{x\in U}\left|\frac{\partial\psi}{\partial x_{m'}}(x)\right|\right\}\in\mathbb{R}\cup\{\infty\}$$

here $|\cdot|$ is the maximum norm on $\mathbb{R}^m$. We write $\mathrm{vol}(\cdot)$ for the Lebesgue measure on $\mathbb{R}^m$. Recall that all open subsets of $\mathbb{R}^m$ are measurable.

For $i\in\{0,1,2\}$ let $m_i\in\mathbb{N}$ and suppose U_i is a non-empty open subset of $\mathbb{R}^{m_i}$. Let $\pi_1\colon\mathbb{R}^{m_0+m_1+m_2}\to\mathbb{R}^{m_0+m_1}$ be defined by $\pi_1(w,x,y)=(w,x)$ and $\pi_2\colon\mathbb{R}^{m_0+m_1+m_2}\to\mathbb{R}^{m_0+m_2}$ by $\pi_2(w,x,y)=(w,y)$.

We now suppose $m_0=2$ and $m=2+m_1+m_2$. Let $\phi_1\colon U_0\times U_1\to\mathbb{R}^m$ and $\phi_2\colon U_0\times U_2\to\mathbb{R}^m$ have continuously differentiable coordinate functions and satisfy $|\phi_1|_{C^1}<\infty$ and $|\phi_2|_{C^1}<\infty$. Define

$$\psi_N(w,x,y)=N\phi_1(w,x)-\phi_2(w,y) \tag{7.1}$$

where $w \in U_0, x \in U_1$, and $y \in U_2$. Thus ψ_N has target $\mathbb{R}^m$ and coordinate functions in $C^1(U)$ where $U = U_0 \times U_1 \times U_2$.

We write ϕ_{1j} and ϕ_{2j} for the coordinate functions of ϕ_1 and ϕ_2, respectively. By abuse of notation we sometimes write $\phi_{1j}(z)$ for $\phi_{1j}(w,x)$ and $\phi_{2j}(z)$ for $\phi_{2j}(w,x)$ if $z = (w,x,y)$ with $w \in U_0, x \in U_1, y \in U_2$.

The jacobian matrix $D_{(w,x,y)}(\psi_N) \in \mathrm{Mat}_m(\mathbb{R})$ equals

$$\left(\begin{array}{cccccccc} N\frac{\partial\phi_1}{\partial w_1} - \frac{\partial\phi_2}{\partial w_1} & N\frac{\partial\phi_1}{\partial w_2} - \frac{\partial\phi_2}{\partial w_2} & N\frac{\partial\phi_1}{\partial x_1} & \cdots & N\frac{\partial\phi_1}{\partial x_{m_1}} & -\frac{\partial\phi_2}{\partial y_1} & \cdots & -\frac{\partial\phi_2}{\partial y_{m_2}} \end{array} \right).$$

evaluated at $(w,x,y) \in U$. For fixed (w,x,y), the determinant $\det D_{(w,x,y)}(\psi_N)$ is a polynomial in N of degree at most $2+m_1$. More precisely, we have

$$\det D_{(w,x,y)}(\psi_N) = \delta_0(w,x,y)N^{2+m_1} + \delta_1(w,x,y)N^{1+m_1} + \delta_2(w,x,y)N^{m_1}$$

where the crucial term is

$$\delta_0(w,x,y) = \det \left(\begin{array}{cccccccc} \frac{\partial\phi_1}{\partial w_1} & \frac{\partial\phi_1}{\partial w_2} & \frac{\partial\phi_1}{\partial x_1} & \cdots & \frac{\partial\phi_1}{\partial x_{m_1}} & -\frac{\partial\phi_2}{\partial y_1} & \cdots & -\frac{\partial\phi_2}{\partial y_{m_2}} \end{array} \right)\Bigg|_{(w,x,y)}. \tag{7.2}$$

If x is in any power of $\mathbb{R}$ and $r > 0$ we let $B_r(x)$ denote the open ball of radius r around x with respect to $|\cdot|$.

Lemma 7.1. *In the notation above let $z_0 \in U$ with $\delta_0(z_0) \neq 0$. There exist two bounded open neighborhoods $U'' \subseteq U'$ of z_0 in U and a constant $c \in (0,1]$ with the following properties:*

(i) *the map ϕ_1 is injective when restricted to $\pi_1(U') \subseteq \mathbb{R}^{2+m_1}$,*

and for all real numbers $N \geq c^{-1}$

(ii) *the map $\psi_N|_{U'} \colon U' \to \mathbb{R}^m$ is injective and open,*
(iii) *we have $\mathrm{vol}(\psi_N(U'')) \geq cN^{2+m_1}$, and*
(iv) *we have $B_c(\psi_N(U'')) \subseteq \psi_N(U')$.*

Proof. As the first order partial derivatives of all ϕ_{ij} are continuous we can find an open neighborhood U' of $z_0 = (w,x,y)$ in U such that the determinant of

$$\begin{pmatrix} \frac{\partial\phi_{11}}{\partial w_1}(\tilde{z}_1) & \frac{\partial\phi_{11}}{\partial w_2}(\tilde{z}_1) & \frac{\partial\phi_{11}}{\partial x_1}(\tilde{z}_1) & \cdots & \frac{\partial\phi_{11}}{\partial x_{m_1}}(\tilde{z}_1) & -\frac{\partial\phi_{21}}{\partial y_1}(\tilde{z}_1) & \cdots & -\frac{\partial\phi_{21}}{\partial y_{m_2}}(\tilde{z}_1) \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial\phi_{1m}}{\partial w_1}(\tilde{z}_m) & \frac{\partial\phi_{1m}}{\partial w_2}(\tilde{z}_m) & \frac{\partial\phi_{1m}}{\partial x_1}(\tilde{z}_m) & \cdots & \frac{\partial\phi_{1m}}{\partial x_{m_1}}(\tilde{z}_m) & -\frac{\partial\phi_{2m}}{\partial y_1}(\tilde{z}_m) & \cdots & -\frac{\partial\phi_{2m}}{\partial y_{m_2}}(\tilde{z}_m) \end{pmatrix} \tag{7.3}$$

has absolute value at least $\epsilon = |\delta(z_0)|/2 > 0$ for all $\tilde{z}_1, \ldots, \tilde{z}_m \in U'$.

Observe that $D_{\pi_1(z_0)}(\phi_1)$ is an $m \times (2+m_1)$-matrix consisting of the first $2+m_1$ columns as in the determinant (7.2). Our hypothesis $\delta_0(z_0) \neq 0$ implies that $D_{\pi_1(z_0)}(\phi_1)$ has maximal rank $2+m_1$. By the Inverse Function Theorem we may, after shrinking U', assume that ϕ_1 restricted to $\pi_1(U')$ is injective. This implies (i).

We may shrink U' further and assume that

$$U'' = B_\delta(z_0) \subseteq U' = B_{2\delta}(z_0) \subseteq U, \tag{7.4}$$

for some $\delta > 0$, a property we will need later on. Our constant c will depend on δ but not on N.

To show injectivity in (ii), let $z, z' \in U'$ and $N \in \mathbb{R}$ be such that $\psi_N(z) = \psi_N(z')$. Let $j \in \{1,\ldots,m\}$, then $N\phi_{1j}(z) - \phi_{2j}(z) = N\phi_{1j}(z') - \phi_{2j}(z')$. By the Mean Value

Theorem there exists $\tilde{z}_j \in U'$ on the line segment connecting z and z' such that the column vector $z - z'$ lies in the kernel of

$$\left(N\frac{\partial \phi_{1j}}{\partial w_1} - \frac{\partial \phi_{2j}}{\partial w_1}, N\frac{\partial \phi_{1j}}{\partial w_2} - \frac{\partial \phi_{2j}}{\partial w_2}, N\frac{\partial \phi_{1j}}{\partial x_1}, \ldots, N\frac{\partial \phi_{1j}}{\partial x_{m_1}}, -\frac{\partial \phi_{2j}}{\partial y_1}, \ldots, -\frac{\partial \phi_{2j}}{\partial y_{m_2}} \right) \Big|_{\tilde{z}_j}$$

Thus $z - z'$ lies in the kernel of $M(\tilde{z}_1, \ldots, \tilde{z}_m) \in \mathrm{Mat}_m(\mathbb{R})$ whose rows are these expressions as $j \in \{1, \ldots, m\}$.

The determinant of this matrix can be expressed as

$$\tilde{\delta}_0(\tilde{z}_1, \ldots, \tilde{z}_m) N^{2+m_1} + \tilde{\delta}_1(\tilde{z}_1, \ldots, \tilde{z}_m) N^{1+m_1} + \tilde{\delta}_2(\tilde{z}_1, \ldots, \tilde{z}_m) N^{m_1}$$

where $\tilde{\delta}_0(\tilde{z}_1, \ldots, \tilde{z}_m)$ is the determinant of (7.3). In particular, $|\tilde{\delta}_0(\tilde{z}_1, \ldots, \tilde{z}_m)| \geq \epsilon$.

We recall that $|\phi_{1,2}|_{C^1} < \infty$. So for $i = 1, 2$ we find $|\tilde{\delta}_i(\tilde{z}_1, \ldots, \tilde{z}_m)| \leq C$ where C depends only on ϕ_1 and ϕ_2. For all sufficiently large $N \geq 1$ we have

$$\begin{aligned} |\tilde{\delta}_0(\tilde{z}_1, \ldots, \tilde{z}_m) N^{2+m_1} + \cdots + \tilde{\delta}_2(\tilde{z}_1, \ldots, \tilde{z}_m) N^{m_1}| &\geq \epsilon N^{2+m_1} - 2C N^{1+m_1} \\ &\geq \frac{\epsilon}{2} N^{2+m_1}. \end{aligned} \tag{7.5}$$

And so in particular, $\det M(\tilde{z}_1, \ldots, \tilde{z}_m) \neq 0$. As $z - z'$ lies in the kernel of the relevant matrix, we conclude $z = z'$. Therefore, $\psi_N|_{U'}$ is injective for all large N. We conclude injectivity (ii)

If N is sufficiently large, then (7.5) implies $|\det M(z, \ldots, z)| \geq \epsilon N^{2+m_1}/2$ for all $z \in U'$. In particular, $D_z(\psi_N) = M(z, \ldots, z)$ is invertible for all $z \in U'$. Hence ψ_N is locally invertible on U' and $\psi_N|_{U'}$ is an open map. This completes the proof of (ii).

As $\psi_N|_{U''}$ is injective and for N large, Integration by Substitution implies

$$\mathrm{vol}(\psi_N(U'')) = \int_{\psi_N(U'')} du = \int_{U''} |\det D_z(\psi_N)| dz \geq \frac{\epsilon}{2} N^{2+m_1} \mathrm{vol}(U'').$$

This yields our claim in (iii) for small enough c.

To prove (iv) it suffices to verify that if $z \in U''$, then the distance $\Delta(z)$ of $\psi_N(z)$ to $\mathbb{R}^m \setminus \psi_N(U') \neq \emptyset$ is at least c, for $c > 0$ sufficiently small and independent of N.

As the set $\mathbb{R}^m \setminus \psi_N(U')$ is closed in $\mathbb{R}^m$ it contains v which depends on z and N, such that $\Delta(z) = |\psi_N(z) - v|$. As v realizes the minimal distance, the ball $B_{1/n}(v)$ must meet $\psi_N(U')$ for all $n \in \mathbb{N}$. Let us fix $z_n \in U'$ with $|\psi_N(z_n) - v| < 1/n$. Now U' is bounded, so after passing to a convergent subsequence we may assume that z_n converges towards $z' \in \overline{U'} = \overline{B_{2\delta}(z_0)}$.

We claim that $z' \notin U'$. Indeed, otherwise $\psi_N(z_n)$ would converge towards $\psi_N(z') \in \psi_N(U')$. But then $\psi_N(z') = v \in \mathbb{R}^m \setminus \psi_N(U')$ is a contradiction. We conclude

$$|z' - z_0| = 2\delta. \tag{7.6}$$

By the Mean Value Theorem we find

$$\psi_N(z) - \psi_N(z_n) = M(\tilde{z}_1, \ldots, \tilde{z}_m)(z - z_n) \tag{7.7}$$

where $M(\cdot)$ is the matrix above and $\tilde{z}_1, \ldots, \tilde{z}_n$ lie on the line segment between z and z_n and thus in U'. As above, the absolute determinant of this matrix is at least $\epsilon N^{2+m_1}/2$ for N large enough. The entries of the adjoint matrix have absolute value bounded by a fixed multiple of N^{2+m_1}. We find $|M(\tilde{z}_1,\ldots,\tilde{z}_m)^{-1}| \le c_1$ for the maximum norm where $c_1 > 0$ is independent of N and $\tilde{z}_1,\ldots,\tilde{z}_m$. We find that (7.7) implies

$$|z - z_n| = |M(\tilde{z}_1,\ldots,\tilde{z}_m)^{-1}(\psi_N(z) - \psi_N(z_n))| \le c_2|\psi_N(z) - \psi_N(z_n)|$$

where $c_2 > 0$ is independent of N. Hence

$$|z - z_n| \le c_2(|\psi_N(z) - v| + |v - \psi_N(z_n)|) = c_2(\Delta(z) + |\psi_N(z_n) - v|)$$

by our choice of v. Recall that $|\psi_N(z_n) - v| < 1/n$ and $z \in U''$ which was defined in (7.4), so

$$|z_n - z_0| - \delta \le |z_n - z_0| - |z_0 - z| \le |z - z_n| \le c_2(\Delta(z) + 1/n).$$

By taking the limit as $n \to \infty$ we can replace z_n by z' on the left. We recall (7.6) and conclude $\Delta(z) \ge \delta/c_2$. Part (iv) follows as we may assume $\delta/c_2 \ge c$. □

Our aim is to find many integral points in $\psi_N(U)$. If $\psi_N(U)$ has volume v, one could hope that $\psi_N(U)$ contains at least v points in $\mathbb{Z}^m$. Of course, simple examples show that this does not need to be true in general. Blichfeldt's Theorem guarantees that we can find at least this number of lattice points after possibly translating by a point in $\mathbb{R}^m$. In our situation we will be able to translate by a rational point of controlled denominator. For the reader's convenience we repeat the hypothesis in the next proposition.

Proposition 7.2. *Let $U_0 \subseteq \mathbb{R}^2, U_1 \subseteq \mathbb{R}^{m_1}$, and $U_2 \subseteq \mathbb{R}^{m_2}$ be non-empty open subsets and suppose $\phi_1\colon U_0 \times U_1 \to \mathbb{R}^m$ and $\phi_2\colon U_0 \times U_2 \to \mathbb{R}^m$ have coordinate functions in $C^1(U_0 \times U_1)$ and $C^1(U_0 \times U_2)$, respectively, where $m = 2 + m_1 + m_2$. We suppose that $|\phi_{1,2}|_{C^1} < \infty$. Let $z_0 \in U = U_0 \times U_1 \times U_2$ with $\delta_0(z_0) \ne 0$. For $N \in \mathbb{R}$ we define ψ_N as in (7.1). There exists a bounded open neighborhood U' of z_0 in U and a constant $c \in (0,1]$ with the following property. For all integers $N_0 \ge c^{-1}$ and all real numbers $N \ge c^{-1}$ we have*

$$\#\left(\psi_N(U') \cap N_0^{-1}\mathbb{Z}^m\right) \ge cN^{2+m_1}.$$

Moreover, $\phi_1|_{\pi_1(U')}$ is injective, and $\psi_N|_{U'}$ is injective for all $N \ge c^{-1}$.

Proof. Let $U'' \subseteq U'$ and $c_1 > 0$ be as in Lemma 7.1 and suppose $N \ge c_1^{-1}$. Below we will use $\mathrm{vol}(\psi_N(U'')) \ge c_1 N^{2+m_1}$ and $B_{c_1}(\psi_N(U'')) \subseteq \psi_N(U')$.

By Blichfeldt's Theorem [12, Chapter III.2, Theorem I], there exists $x \in \mathbb{R}^m$ which may depend on N, such that $\#(-x + \psi_N(U'')) \cap \mathbb{Z}^m \ge \mathrm{vol}(\psi_N(U'')) \ge c_1 N^{2+m_1}$. So there exist an integer $M \ge c_1 N^{2+m_1}$, $a_1,\ldots,a_M \in \mathbb{Z}^m$, and $z_1,\ldots,z_M \in U''$ such that

$$-x + \psi_N(z_i) = a_i \in \mathbb{Z}^m \quad \text{for all} \quad i \in \{1,\ldots,M\}$$

and the a_i are pairwise distinct.

There exists $c_2 > 0$ such that if N_0 is any integer with $N_0 \ge c_2^{-1}$ then $B_{c_1}(x') \cap N_0^{-1}\mathbb{Z}^m \ne \emptyset$ for all $x' \in \mathbb{R}^m$.

Let us fix $q \in B_{c_1}(x) \cap N_0^{-1}\mathbb{Z}^m$ where x comes from Blichfeldt's Theorem. Then

$$q + a_i = (q - x) + x + a_i = (q - x) + \psi_N(z_i) \in B_{c_1}(\psi_N(z_i)) \subseteq \psi_N(U').$$

Observe $q + a_i \in N_0^{-1}\mathbb{Z}^m$ for all $i \in \{1,\ldots,M\}$.

We have proved $\#(\psi_N(U') \cap N_0^{-1}\mathbb{Z}^m) \ge M \ge c_1 N^{2+m_1}$ for all $N \ge c_1^{-1}$ and all $N_0 \ge {c_2}^{-1}$. The proposition follows by taking $c = \min\{c_1, c_2\}$ and by the injectivity statements in (i) and (ii) of Lemma 7.1. □

8. Intersection Numbers

Let F be an algebraically closed subfield of $\mathbb{C}$. Let S be a smooth irreducible curve over F and let $\pi\colon \mathcal{A} \to S$ be an abelian scheme over F of relative dimension $g \geq 1$. In this section we abbreviate $\mathbb{P}^m_F$ by $\mathbb{P}^m$ for integers $m \geq 1$.

We will use the basic setup introduced in §2.2. In particular $\mathcal{A} \subseteq \mathbb{P}^M \times \mathbb{P}^m$ is an admissible immersion.

Proposition 8.1. *Suppose X is an irreducible, closed subvariety of $\mathcal{A}$ defined over F that dominates S and is not generically special. Then there exist*

- *a constant $c > 0$,*
- *a finite and étale covering $S' \to S$ where S' is an irreducible curve over F,*
- *and finitely many closed (not necessarily irreducible) subvarieties $Y_1, \ldots, Y_R$ of $\mathcal{A}' = \mathcal{A} \times_S S'$*

such that the following holds for $X' = X \times_S S'$. For each integer $N \geq c^{-1}$, there exists $Y \in \{Y_1, \ldots, Y_R\}$ such that $X' \cap [N]^{-1}(Y)$ contains at least $r \geq cN^{2\dim X}$ irreducible components of dimension 0.

Note that X' from the proposition is a closed subvariety of the abelian $\mathcal{A}'$ scheme. It may not be irreducible, but it is equidimensional of dimension $\dim X$ since $S' \to S$ is finite and étale. Note also that each irreducible component of $X' \cap [N]^{-1}(Y)$ consists of one F-rational point as X' and $[N]^{-1}(Y)$ are defined over F.

We will prove Proposition 8.1 in the next few subsections.

8.1. Constructing a Covering $S' \to S$.

Further down we will need to pass to a finite and étale covering S' of S. In this subsection we make some preparations and mention some facts.

We recall our convention $F \subseteq \mathbb{C}$ and fix $P \in X(F)$ as in Lemma 6.2.

By assumption on X and P, we have an irreducible closed subvariety $Z \subseteq \mathcal{A}$ defined over F satisfying the conclusion of Proposition 6.1. In particular, $\dim Z = \mathrm{codim}_{\mathcal{A}} X = n$.

We fix a prime number ℓ satisfying

$$\ell > D^{2^{g+1}(g+1)} \tag{8.1}$$

where D comes from (iv) of Proposition 6.1. Later on, we will impose a second lower bound on ℓ.

There is a finite étale covering $S' \to S$ such that $\mathcal{A}'/S'$ admits all the ℓ^{2g} torsion sections $S' \to \mathcal{A}'[\ell]$ where $\mathcal{A}'$ is the abelian scheme $\mathcal{A} \times_S S'$ over S'. We may assume that S' is irreducible. The prescribed closed immersion $\mathcal{A} \to \mathbb{P}^M_S$ induces a closed immersion $\mathcal{A}' \to \mathbb{P}^M_{S'}$.

Observe that the induced morphism $\rho\colon \mathcal{A}' \to \mathcal{A}$ is finite and étale. So the pre-image of any irreducible subvariety Y of $\mathcal{A}'$ is equidimensional of dimension $\dim Y$.

Let $Z' = Z \times_S S'$, this is a closed subvariety of $\mathcal{A}'$ that may not be irreducible. It is equidimensional of dimension $\dim Z$. A further and crucial observation for our argument is that the fibers Z_t and $Z'_{t'}$ are equal if $t' \in S'(F)$ maps to $t \in S(F)$. So by Proposition 6.1(iv)

$$\text{the degree of any } Z'_{t'} \subseteq \mathbb{P}^M \text{ is bounded by } D \text{ for all } t' \in S'(F) \tag{8.2}$$

and by the "Moreover" part of Proposition 6.1

$$\text{(8.3)} \qquad \begin{array}{c}\text{at most finitely many fibers of } Z' \to S' \text{ over } S'(\mathbb{C}) \text{ contain a coset}\\ \text{of positive dimension in the respective fiber of } \mathcal{A}' \to S'.\end{array}$$

8.2. **Local Parametrization and Lattice Points.** We keep the notation introduced above and prove the following intermediate counting result.

Lemma 8.2. *Let X be as in Proposition 8.1. Then there exist*

- *a constant $c > 0$,*
- *a prime number ℓ satisfying* (8.1),
- *and a finite étale covering $S' \to S$, with S' irreducible, admitting all the ℓ^{2g} torsion sections $S' \to \mathcal{A}'[\ell]$, with $\pi'\colon \mathcal{A}' \to S'$ the canonical morphism, $X' = X \times_S S'$, and $Z' = Z \times_S S'$*

such that for all integers $N \geq c^{-1}$ the following holds. There exist $r \geq cN^{2\dim X}$ pairs $(P_1', Q_1'), \ldots, (P_r', Q_r') \in X'(\mathbb{C}) \times Z'(\mathbb{C})$ such that the $P_1', \ldots, P_r'$ are pairwise distinct with the following properties for all $i \in \{1, \ldots, r\}$:

(i) *We have $\pi'(P_i') = \pi'(Q_i')$ and $[\ell N](P_i') = [\ell](Q_i')$.*
(ii) *The Zariski closed subset $Z'_{\pi'(P_i')}$ of $\mathcal{A}'_{\pi'(P_i')}$ does not contain any coset of positive dimension.*
(iii) *If Y' is an irreducible closed subvariety of $\mathcal{A}'$ such that $Q_i' \in Y'(\mathbb{C})$ and P_i' is not isolated in $X' \cap [\ell N]^{-1}([\ell](Y'))$, then there exists an ℓ-torsion section $\sigma\colon S' \to \mathcal{A}'[\ell]$ with $\sigma \neq 0$ and $Q_i' \in Y'(\mathbb{C}) \cap (Y' - \operatorname{im}\sigma)(\mathbb{C})$.*

Proof. We make use of the lattice point counting technique from §7.

By Lemma 6.2 we have that P is a smooth point of X^{an} and $X^{\mathrm{an}}_{\pi(P)}$. We have $\dim X = g + 1 - n$, so we can trivialize the family $X^{\mathrm{an}} \to S^{\mathrm{an}}$ in a neighborhood of P in X^{an} using a smooth map $\tilde{\phi}_1$ defined on $U_0 \times U_1$ where $U_0 \subseteq \mathbb{R}^2$ and $U_1 \subseteq \mathbb{R}^{2(g-n)}$ are both open and non-empty. We may assume that $\tilde{\phi}_1(0) = P$. After possibly shrinking U_0 and U_1 we compose $\tilde{\phi}_1$ with $\tilde{b}$, the Betti map $b\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ composed by a local inverse of $\mathbb{R}^{2g} \to \mathbb{T}^{2g}$. This yields a smooth map $\phi_1\colon U_0 \times U_1 \to \mathbb{R}^{2g}$ that produces the Betti coordinates relative to the local parametrization of X^{an}.

Now P is also a smooth point of Z^{an}. Recall that $\dim Z = n$. The same construction yields a non-empty and open subset $U_2 \subseteq \mathbb{R}^{2(n-1)}$ and a smooth map $\tilde{\phi}_2\colon U_0 \times U_2 \to Z^{\mathrm{an}}$ with $\tilde{\phi}_2(0) = P$. We restrict if necessary and write $\phi_2 = \tilde{b} \circ \tilde{\phi}_2\colon U_0 \times U_2 \to \mathbb{R}^{2g}$. The subsets U_0, U_1 and U_2 can be chosen to be bounded.

In this setting ϕ_1 parametrizes $\tilde{b}(U)$ where $U \subseteq X^{\mathrm{an}}$ is a neighborhood of P in X and $\tilde{b}$ is a local lift of the Betti map to $\mathbb{R}^{2g}$. Similarly ϕ_2 parametrizes $\tilde{b}(V)$ where V is a neighborhood of P in Z^{an}.

We may assume, after shrinking U_0, U_1, and U_2 if necessary, that $|\phi_1|_{C^1} < \infty$ and $|\phi_2|_{C^1} < \infty$. By Proposition 6.1(iii) the fiber $Z_{\pi(P)}$ contains no positive dimensional cosets in $\mathcal{A}_{\pi(P)}$. By (8.3) and up to shrinking U_0 we may assume that

$$\text{(8.4)} \qquad Z_t \text{ contains no positive dimensional cosets in } \mathcal{A}_t \text{ for all } t \in \pi(\tilde{\phi}_1(U_0 \times U_1)).$$

For any $N \in \mathbb{N}$ and $(w, x, y) \in U_0 \times U_1 \times U_2$ we define a map

$$\psi_N(w, x, y) = N\phi_1(w, x) - \phi_2(w, y) \in \mathbb{R}^{2g}.$$

Let $\pi_i\colon U_0 \times U_1 \times U_2 \to U_0 \times U_i$ be the natural projection for $i = 1, 2$ and $\delta_0(w, x, y)$ as above Lemma 7.1.

Condition (v) of Proposition 6.1 implies that $\delta_0(0) \neq 0$. So we can apply Proposition 7.2. There exists a bounded open neighborhood U' of 0 in $U_0 \times U_1 \times U_2$ and a constant $c \in (0, 1]$ with the following property. For all integers $N_0 \geq c^{-1}$ and $N \geq c^{-1}$ we have that $\phi_1|_{\pi_1(U')}$ and $\psi_N|_{U'}$ are injective and

$$\# \left(\psi_N(U') \cap N_0^{-1}\mathbb{Z}^{2g}\right) \geq cN^{2+2(g-n)} = cN^{2\dim X}. \tag{8.5}$$

Proposition 7.2 allows us to increase N_0. From now on we fix N_0 to be a prime number ℓ that satisfies (8.1) and $\ell \geq c^{-1}$. As in §8.1 we fix a finite étale covering $S' \to S$ with S' irreducible such that $\mathcal{A}' := \mathcal{A} \times_S S' \to S'$ admits all the ℓ^{2g} torsion sections $S' \to \mathcal{A}'[\ell]$.

Recall that $|\phi_2|_{C^1} < \infty$, so is $\phi_2(U_0 \times U_2)$ is bounded and

$$\# \left(\phi_2(U_0 \times U_2) - \phi_2(U_0 \times U_2)\right) \cap \ell^{-1}\mathbb{Z}^{2g} \leq C \text{ for some } C \text{ independent of } N. \tag{8.6}$$

Suppose $(w, x, y) \in U'$ satisfies $\psi_N(w, x, y) \in \ell^{-1}\mathbb{Z}^{2g}$. Then $\ell N\phi_1(w, x) - \ell\phi_2(w, y) \in \mathbb{Z}^{2g}$. For the Betti coordinates we find on $\mathcal{A}$ that

$$[\ell N](\tilde{\phi}_1(w, x)) = [\ell](\tilde{\phi}_2(w, y)) \in [\ell](Z)(\mathbb{C}).$$

Thus we get a mapping

$$\psi_N(U') \cap \ell^{-1}\mathbb{Z}^{2g} \ni \psi_N(w, x, y) \quad \mapsto \quad (\tilde{\phi}_1(w, x), \tilde{\phi}_2(w, y)) \in X(\mathbb{C}) \times Z(\mathbb{C}). \tag{8.7}$$

The image points are of the form (P_i, Q_i) and lie in the same fiber above S and with $[\ell N](P_i) = [\ell](Q_i)$. By (8.5) these points arise from at least $cN^{2\dim X}$ elements of $\psi_N(U') \cap \ell^{-1}\mathbb{Z}^{2g}$ for all large N. We claim that up to adjusting c the number of different P_i is also at least $cN^{2\dim X}$.

So let $(w, x, y) \in U'$ with $\psi_N(w, x, y) \in \ell^{-1}\mathbb{Z}^{2g}$ whose image under (8.7) is (P_i, Q_i). Let (P_j, Q_j) be a further pair with $P_i = P_j$ that comes from $\psi_N(w', x', y') \in \ell^{-1}\mathbb{Z}^{2g}$ where $(w', x', y') \in U'$. Hence $(w', x') = (w, x)$ as $\phi_1|_{\pi_1(U')}$ is injective. Thus $\phi_2(w', y') - \phi_2(w, y) = \psi_N(w, x, y) - \psi_N(w', x', y') \in \ell^{-1}\mathbb{Z}^{2g}$. By (8.6) there are at most C possibilities for $\psi_N(w', x', y')$, when (x, y, z) is fixed. So there are at most C possibilities for (w', x', y') as ψ_N is injective on U'; recall that C may depend on ℓ but not on N.

After omitting pairs with duplicate P_i and replacing c by c/C we have found (P_i, Q_i) with pairwise different P_i for $1 \leq i \leq r$ and $r \geq cN^{2\dim X}$.

Let $\rho\colon \mathcal{A}' \to \mathcal{A}$ denote the canonical morphism. We fix lifts $P_i', Q_i' \in \mathcal{A}'(\mathbb{C})$ of P_i, Q_i respectively in the same fiber of $\mathcal{A}' \to S'$. So

$$[\ell N](P_i') = [\ell](Q_i') \quad \text{for all} \quad i \in \{1, \ldots, r\}. \tag{8.8}$$

This yields claim (i) of the lemma.

As our points P_i lie above points in $\pi(\tilde{\phi}_1(U_0 \times U_1))$ we deduce (ii) from (8.4).

It remains to prove part (iii). Let Y' be as in (iii), namely $Q_i' \in Y'(\mathbb{C})$ and P_i' is not isolated in $X' \cap [\ell N]^{-1}([\ell](Y'))$ for some $i \in \{1, \ldots, r\}$. To simplify notation we write $P' = P_i'$ and $Q' = Q_i'$.

Then there is a sequence $(P^\alpha)_{\alpha \in \mathbb{N}}$ of pairwise distinct points of $X'(\mathbb{C})$ that converges in X'^{an} to P' with $[\ell N](P^\alpha) \in [\ell](Y')(\mathbb{C})$ for all $\alpha \in \mathbb{N}$. We fix $Q^\alpha \in Y'(\mathbb{C})$ with $[\ell N](P^\alpha) = [\ell](Q^\alpha)$ for all $\alpha \in \mathbb{N}$. Thus $\pi'(P^\alpha) = \pi'(Q^\alpha)$ and by continuity the sequence $[\ell](Q^\alpha)$ converges. Since $[\ell]$ induces a proper map $(\mathcal{A}')^{\mathrm{an}} \to (\mathcal{A}')^{\mathrm{an}}$ we may assume, after passing to a subsequence, that the Q^α converge in $(Y')^{\mathrm{an}}$ to some $Q'' \in Y'(\mathbb{C})$.

Taking the limit we see by continuity and (8.8) that $[\ell](Q'') = [\ell N](P') = [\ell](Q')$ and in particular $\pi'(Q'') = \pi'(Q')$. Hence

$$Q'' = Q' + T \in Y'(\mathbb{C})$$

for some T that is either trivial or of finite prime order ℓ in $\mathcal{A}'_{\pi'(Q')}(\mathbb{C})$.

All ℓ^{2g} torsion sections $S' \to \mathcal{A}'[\ell]$ exist, so there is one σ with $\sigma(\pi'(Q')) = T$. Hence $Q' \in Y'(\mathbb{C}) \cap (Y' - \operatorname{im}\sigma)(\mathbb{C})$.

To complete the proof it remains to verify $\sigma \neq 0$, *i.e.* $T \neq 0$. For this we assume the converse and derive a contradiction. For α large enough, the sequence member $\rho(P^\alpha)$ will be close enough to $\rho(P') = P_i$ as to lie in $\tilde{\phi}_1(U')$. As $Q'' = Q'$ the analog statement holds for the sequence of $\rho(Q^\alpha)$, *i.e.* $\rho(Q^\alpha) \in \tilde{\phi}_2(U')$ for all sufficiently large α. For α sufficiently large we may write $\rho(P^\alpha) = \tilde{\phi}_1(w^\alpha, x^\alpha)$ and $\rho(Q^\alpha) = \tilde{\phi}_2(w^\alpha, y^\alpha)$ with $(w^\alpha, x^\alpha, y^\alpha) \in U'$. The condition $[\ell N](P^\alpha) = [\ell](Q^\alpha)$ implies $[\ell N](\rho(P^\alpha)) = [\ell](\rho(Q^\alpha))$ and hence

$$\psi_N(w^\alpha, x^\alpha, w^\alpha) = N\phi_1(w^\alpha, x^\alpha) - \phi_2(w^\alpha, y^\alpha) \in \ell^{-1}\mathbb{Z}^{2g}.$$

By continuity, the sequence $\psi_N(w^\alpha, x^\alpha, w^\alpha)$ is eventually constant. But ψ_N is injective on U' by Proposition 7.2 and hence $(w^\alpha, x^\alpha, w^\alpha)$ is eventually constant. So $\rho(P^\alpha)$ is eventually constant and, as ρ is finite, P^α attains only finitely many values. But this contradicts the fact that the P^α are pairwise distinct and concludes the proof of (iii). □

8.3. **Induction and Isolated Intersection Points.** Let $X, \ell, \mathcal{A}' \to S', X'$, and Z' be as in Lemma 8.2. The conclusion of Lemma 8.2 is already close to what we are aiming at in Proposition 8.1. However, we must first deal with the possibility that most P'_i from the lemma are not isolated in $X' \cap [\ell N]^{-1}([\ell](Z'))$; otherwise we could just take $Y_1 = [\ell]^{-1}([\ell](Z'))$. We will handle this in the current subsection by introducing additional auxiliary subvarieties derived from Z'.

Recall that D was introduced in §8.1 and ultimately comes from Proposition 6.1(iv). For brevity we write $\mathcal{A}'[\ell](S')$ for the group of torsion sections $S' \to \mathcal{A}'[\ell]$ of order dividing ℓ. Recall also that X' is equidimensional of dimension $\dim X = g + 1 - n$.

We now describe a procedure to construct a finite set Σ of auxiliary subvarieties. To be more precise we will construct for each $k \in \{0, \ldots, n\}$ a finite set Σ_k with the following properties:

(i) If $Y' \in \Sigma_k$, then Y' is an irreducible closed subvariety of Z' with $\dim Y' \leq n - k$.
(ii) If $Y' \in \Sigma_k$ and $t \in S'(\mathbb{C})$ such that $Y'_t \neq \emptyset$, then $\deg Y'_t \leq D^{2^k}$.
(iii) If $k \leq n - 1$, then for all $Y' \in \Sigma_k$ and all $\sigma \in \mathcal{A}'[\ell](S')$ such that $Y' \not\subseteq Y' - \operatorname{im}\sigma$, all irreducible components of $Y' \cap (Y' - \operatorname{im}\sigma)$ are elements of Σ_{k+1}.

We define Σ_0 to be the set of irreducible components of Z'. Clearly, (i) is satisfied as Z' is equidimensional of dimension $\dim Z = n$. Moreover, (ii) is satified due to (8.2).

We construct the remaining $\Sigma_1, \ldots, \Sigma_n$ and verify the properties inductively. Suppose $k \in \{0, \ldots, n-1\}$ and that Σ_k has already been constructed.

Consider the set of all $Y' \in \Sigma_k$ and $\sigma \in \mathcal{A}'[\ell](S')$ with $Y' \not\subseteq Y' - \operatorname{im}\sigma$. There are only finitely many such pairs (Y', σ) and we take as Σ_{k+1} all irreducible components of all $Y' \cap (Y' - \operatorname{im}\sigma)$ that arise this way. This choice makes (iii) automatically hold true for all $k \in \{0, \ldots, n-1\}$. If $Y'' \in \Sigma_{k+1}$ is such an irreducible component, then

$Y'' \subsetneq Y' \subseteq Z'$ and $\dim Y'' \le \dim Y' - 1 \le n-(k+1)$ by (i) applied to k. This implies (i) for $k+1$.

We now verify (ii). If Y'' does not dominate S', then the image of Y'' in S' is a point t, hence $Y'' = Y''_t$. In this case Y''_t is an irreducible component of $Y'_t \cap (Y'_t - \sigma(t))$. By Bézout's Theorem and since $\deg Y'_t = \deg(Y'_t - \sigma(t))$ we find $\deg Y''_t \le (\deg Y'_t)^2$. By (ii) applied to Y'_t this implies $\deg Y''_t \le (D^{2^k})^2 = D^{2^{k+1}}$. So (ii) holds for all Y'' that do not dominate S'. If Y'' dominates S', then for all but at most finitely many $t \in S'(\mathbb{C})$ all irreducible components of Y''_t are also irreducible components of $Y'_t \cap (Y'_t - \sigma(t))$. For such a t we have, again by Bézout's Theorem, $\deg Y''_t \le (\deg Y'_t)^2 \le D^{2^{k+1}}$ which implies (ii). For any remaining $t \in S'(\mathbb{C})$, observe that Y'' is flat over S' by [31, Proposition III.9.7] since $\dim S' = 1$ and Y'' is irreducible. As cycles of $\mathbb{P}^M$ the fibers of Y'' are pairwise algebraically equivalent. So $\deg Y''_t \le D^{2^{k+1}}$ for all $t \in S(\mathbb{C})$, see Fulton [20, Chapters 10.1 and 10.2] on the conservation of numbers. More precisely let $H_1, \ldots, H_{\dim Y''-1}$ be generic hyperplane sections of $\mathbb{P}^M \times S \to S$ such that $Y'' \cap \bigcap_{i=1}^{\dim Y''-1} H_i$ is flat of relatively dimension 0 over S, and then apply [20, Corollary 10.2.2] to, using the notation of *loc.cit.*, $Y = \mathbb{P}^M$, $T = S$, $\alpha_1 = [Y'']$ and $\alpha_i = [H_{i-1}]$ for $i = 2, \ldots, \dim Y''$. So we conclude (ii) for all fibers of Y'' regardless whether it dominates S' or not.

We define

$$\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_n.$$

It is crucial for us that the following bound involving $Y' \in \Sigma$ is independent of ℓ:

$$\deg Y'_t \le D^{2^n} \le D^{2^{g+1}} \text{ for all } t \in S'(\mathbb{C}) \text{ with } Y'_t \ne \emptyset. \tag{8.9}$$

We are now ready to prove the main result of this section.

Proof of Proposition 8.1. Let $X, c, \ell, \mathcal{A}' \to S', X'$, and Z' be as in Lemma 8.2. We will prove that $\{[\ell]^{-1}([\ell](Y')) : Y' \in \Sigma\}$ is the desired set of closed subvarieties of $\mathcal{A}'$.

For $N \ge c^{-1}$, Lemma 8.2 produces $r \ge cN^{2\dim X}$ pairs $(P'_1, Q'_1), \ldots, (P'_r, Q'_r) \in X'(\mathbb{C}) \times Z'(\mathbb{C})$ with the stated properties.

Note that each Q'_i is a point of some element of Σ. Indeed, it is a point of Z' whose irreducible components are in Σ_0. To each $i \in \{1, \ldots, r\}$ we assign an auxiliary variety in Σ_k containing Q'_i and with maximal k.

By the Pigeonhole Principle there exist k and an auxiliary variety $Y' \in \Sigma_k$ that is hit at least $r/\#\Sigma$ times. As $\#\Sigma$ is independent of N we may assume, after adjusting c, that $r \ge cN^{2\dim X}$ and $Q'_i \in Y'(\mathbb{C})$ for all $i \in \{1, \ldots, r\}$.

Let $Y = [\ell]^{-1}([\ell](Y'))$. We prove that Y is what we want, *i.e.* $X' \cap [N]^{-1}(Y)$ contains at least $r \ge cN^{2\dim X}$ isolated points.

Observe that $P'_1, \ldots, P'_r$ are points of $X' \cap [N]^{-1}(Y)$. If they are all isolated in this intersection then the proposition follows as they are pairwise distinct.

We assume that some P'_i is not isolated in $X' \cap [N]^{-1}(Y)$ and will arrive at a contradiction. By Lemma 8.2(iii) there exists a non-trivial $\sigma \in \mathcal{A}'[\ell](S')$ such that $Y' \cap (Y' - \operatorname{im}\sigma)$ contains Q'_i. In particular, Y' cannot be a point.

Let us assume $Y' \not\subseteq Y' - \operatorname{im}\sigma$ for now. Thus properties (i) and (iii) of Σ_k listed above imply $1 \le \dim Y' \le n-k$ and that Q'_i lies in an element of Σ_{k+1}. This contradicts the maximality of k. Hence $Y' \subseteq Y' - \operatorname{im}\sigma$ and in particular $Y'_t \subseteq Y'_t - \sigma(t)$ where t is the image of Q'_i under $\mathcal{A}' \to S'$.

If Y' dominates S', then Y'_t is equidimensional of dimension $\dim Y' - 1$. If Y' does not dominate S', then $Y' = Y'_t$ is irreducible and in particular equidimensional. In both cases we find $Y'_t = Y'_t - \sigma(t)$ and the group generated by $\sigma(t)$ acts on the set of irreducible components of Y'_t.

The number of irreducible components of Y'_t is at most $\deg Y'_t \le D^{2^{g+1}}$ by (8.9). Furthermore, $\sigma(t)$ has precise order ℓ since $\sigma \neq 0$ and ℓ is prime. By (8.1) we have $\ell > D^{2^{g+1}}$. As ℓ is a prime there is no non-trivial group homomorphism from $\mathbb{Z}/\ell\mathbb{Z}$ to the symmetric group on $\deg Y'_t$ symbols. We conclude that $\sigma(t) + W = W$ for all irreducible components W of Y'_t.

The stabilizer $\mathrm{Stab}(W)$ in $\mathcal{A}'_t$ of any irreducible component of Y'_t of W, satisfies $\deg \mathrm{Stab}(W) \le \deg(W)^{\dim W + 1} \le \deg(W)^{g+1}$ by [15, Lemme 2.1(ii)]. We obtain $\deg W \le \deg Y'_t \le D^{2^{g+1}}$ again from (8.9). Putting these bounds together gives $\deg \mathrm{Stab}(W) \le D^{2^{g+1}(g+1)}$. But $\mathrm{Stab}(W)$ contains $\sigma(t)$, a point of order $\ell > \deg \mathrm{Stab}(W)$ by (8.1). In particular, $\mathrm{Stab}(W)$ has positive dimension. But this implies that Y'_t contains a positive dimensional coset. By property (i) in the construction of Σ we have $Y'_t \subseteq Z'_t$ and therefore Z'_t contains a coset of positive dimension. This contradicts Lemma 8.2(ii). □

9. Height Inequality in the Total Space

In this section, and if not stated otherwise, we work with the category of schemes over $\overline{\mathbb{Q}}$ and abbreviate $\mathbb{P}^m_{\overline{\mathbb{Q}}}$ by $\mathbb{P}^m$ for integers $m \ge 1$. Let S be a smooth irreducible curve defined over $\overline{\mathbb{Q}}$. Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme over $\overline{\mathbb{Q}}$ of relative dimension $g \ge 1$.

We will use the basic setup introduced in §2.2. In particular $\mathcal{A} \subseteq \mathbb{P}^M \times \mathbb{P}^m$ is an admissible immersion.

Our principal result is the following proposition. It makes use of the naive height given by (2.2).

Proposition 9.1. *Suppose X is an irreducible closed subvariety of $\mathcal{A}$ that dominates S and is not generically special, then there exists a constant $c > 0$ depending on X and the data introduced above with the following property. For any integer $N \ge c^{-1}$ there exist a non-empty Zariski open subset $U_N \subseteq X$ and a constant $c'(N)$, both of which depend on N, such that*

$$h([2^N]Q) \ge c4^N h(Q) - c'(N) \quad \textit{for all } Q \in U_N(\overline{\mathbb{Q}}).$$

9.1. Polynomials Defining Multiplication-by-2 on $\mathcal{A}$.

Let $\underline{X} = [X_0 : \cdots : X_M]$ denote the projective coordinates on $\mathbb{P}^M$ and let $\underline{Y} = [Y_0 : \cdots : Y_m]$ denote the projective coordinates on $\mathbb{P}^m$. Recall condition (ii) of the admissible setting $\mathcal{A} \subseteq \mathbb{P}^M \times \mathbb{P}^m$: the morphism $[2]$ is represented globally on $\mathcal{A} \subseteq \mathbb{P}^M \times \mathbb{P}^m$ by $M+1$ bi-homogeneous polynomials, homogeneous of degree 4 in $\underline{X}$ and homogeneous of a certain degree, say c_0, in $\underline{Y}$. In other words, there exist $G_0, \ldots, G_M \in \overline{\mathbb{Q}}[\underline{X}; \underline{Y}]$, each G_i being homogeneous of degree 4 on the variables $\underline{X}$ and homogeneous of degree c_0 on the variables $\underline{Y}$, such that $[2](a) = ([G_0(a_1; a_2) : \cdots : G_M(a_1; a_2)]; a_2)$ for any $a \in \mathcal{A}(\mathbb{C})$. Here we write $a = (a_1; a_2)$ under $\mathcal{A} \subseteq \mathbb{P}^M \times \mathbb{P}^m$. Note that c_0 depends only on the immersion $\mathcal{A} \subseteq \mathbb{P}^M \times \mathbb{P}^m$.

9.2. An Auxiliary Rational Map.

Lemma 9.2. *Let X and Y be locally closed algebraic subsets of $\mathbb{P}^M$ and suppose that X is irreducible. There exists $\delta \in \mathbb{N}$ that depends only on Y with the following property.*

Suppose $r \geq 1$ and $Q_1, \ldots, Q_r \in X(\overline{\mathbb{Q}}) \cap Y(\overline{\mathbb{Q}})$ for all $i \in \{1, \ldots, r\}$. There exist homogeneous polynomials $\varphi_0, \varphi_1, \ldots, \varphi_{\dim X} \in \overline{\mathbb{Q}}[X_0, \ldots, X_M]$ of degree δ, whose set of common zeros is denoted by $Z \subseteq \mathbb{P}^M$, such that the rational map $\varphi = [\varphi_0 : \cdots : \varphi_{\dim X}] \colon \mathbb{P}^M \dashrightarrow \mathbb{P}^{\dim X}$ satisfies:

(i) *We have $\varphi(Y \backslash Z) = [1 : 0 : \cdots : 0]$ and $Q_i \notin Z(\overline{\mathbb{Q}})$ for all $i \in \{1, \ldots, r\}$.*
(ii) *If C is an irreducible subvariety of X and $i \in \{1, \ldots, r\}$ with $Q_i \in C(\overline{\mathbb{Q}})$ such that $\varphi|_{C \backslash Z}$ is constant, then $C \subseteq \overline{Y}$, where $\overline{Y}$ is the Zariski closure of Y in $\mathbb{P}^M$.*

Proof. The Zariski closure $\overline{Y}$ of Y in $\mathbb{P}^M$ is the zero set of finitely many homogeneous polynomials $g_1, \ldots, g_m \in \overline{\mathbb{Q}}[X_0, \ldots, X_M]$. We may assume that $g_1, \ldots, g_m$ all have the same degree δ. Note that δ depends only on Y.

We may fix further $g_0 \in \overline{\mathbb{Q}}[X_0, \ldots, X_M]$, also of degree δ, such that $g_0(Q_i) \neq 0$ for all $i \in \{1, \ldots, r\}$. For example, we can take g_0 to be the δ-th power of a linear polynomial whose zero set avoids the Q_i's. The set of common zeros Z_G of all g_i does not contain any Q_i and thus not all of Y.

We obtain a rational map $G = [g_0 : \cdots : g_m] \colon \mathbb{P}^M \dashrightarrow \mathbb{P}^m$. Observe that $G(Y \backslash Z_G) = [1 : 0 : \cdots : 0]$. Observe also that $G(X \backslash Z_G)$ is constructible in $\mathbb{P}^m$ and its Zariski closure is of dimension at most $\dim X$. This image also contains $[1 : 0 : \cdots : 0]$, so there exist homogenous linear forms $l_1, \ldots, l_{\dim X} \in \overline{\mathbb{Q}}[X_0, \ldots, X_m]$ such that

$$[1 : 0 : \cdots : 0] \text{ is isolated in } \mathscr{Z}(l_1, \ldots, l_{\dim X}) \cap G(X \backslash Z_G). \tag{9.1}$$

We set $l_0 = X_0$ and consider $[l_0 : \cdots : l_{\dim X}]$ as a rational map $\mathbb{P}^m \dashrightarrow \mathbb{P}^{\dim X}$. It is well-defined at $[1 : 0 : \cdots : 0] \in \mathbb{P}^m(\overline{\mathbb{Q}})$ and maps this point to $[1 : 0 : \cdots : 0] \in \mathbb{P}^{\dim X}(\overline{\mathbb{Q}})$.

We set $\varphi_0 = l_0(g_0, \ldots, g_m), \ldots, \varphi_{\dim X} = l_{\dim X}(g_0, \ldots, g_m)$. Then φ_i is homogeneous of degree δ for all $i \in \{0, 1, \ldots, \dim X\}$. Let Z be the set of common zeros of the φ_i. Then $Q_i \notin Z(\overline{\mathbb{Q}})$ for all i by construction. The rational map $\varphi = [\varphi_0 : \cdots : \varphi_{\dim X}] \colon \mathbb{P}^M \dashrightarrow \mathbb{P}^{\dim X}$ is defined on Q_i and maps $Y \backslash Z$ to $[1 : 0 : \cdots : 0]$. We conclude (i).

Let C be as in claim (ii). Then $C \backslash Z$ is non-empty and is mapped to $[1 : 0 : \cdots : 0] \in \mathbb{P}^{\dim X}(\overline{\mathbb{Q}})$ under φ. So $l_1, \ldots, l_{\dim X}$ vanish on $G(C \backslash Z)$. As the image $G(C \backslash Z) \subseteq G(C \backslash Z_G)$ contains $G(Q_i) = [1 : 0 : \cdots : 0] \in \mathbb{P}^m(\overline{\mathbb{Q}})$ we infer from (9.1) that the $g_1, \ldots, g_m$ vanish on $C \backslash Z$. Hence $C \backslash Z \subseteq \overline{Y}$ and so $C \subseteq \overline{Y}$ since $C \backslash Z$ is Zariski dense in the irreducible C. □

The map φ depends on the collection of Q_i in the previous proposition, but the degree δ does not. Also note that the Q_i's are not necessarily pairwise distinct.

9.3. **Height Change under Scalar Multiplication.** The following lemmas are proven by the second-named author in [28]. Lemma 9.4 is our main tool to deduce the desired height inequality (Proposition 9.1) from the "division intersection points" counting (Proposition 8.1).

Lemma 9.3. *Let X be an irreducible variety over $\mathbb{C}$ and let $\varphi \colon X \dashrightarrow \mathbb{P}^{\dim X}_{\mathbb{C}}$ be a rational map. Then for any $Q \in \mathbb{P}^{\dim X}(\mathbb{C})$, the number of zero-dimensional irreducible components of $\varphi^{-1}(Q)$ is at most $\deg \varphi$. By convention we say that $\deg \varphi = 0$ if φ is not dominant.*

Proof. This is [28, Lemma 4.2]. The crucial point is that $\mathbb{P}^{\dim X}_{\mathbb{C}}$ is a normal variety. □

Lemma 9.4. *Let $X \subseteq \mathbb{P}^M$ be an irreducible closed subvariety over $\overline{\mathbb{Q}}$ of positive dimension. Let $\varphi\colon X \dashrightarrow \mathbb{P}^{\dim X}$ be the rational map given by $\varphi = [\varphi_0 : \cdots : \varphi_{\dim X}]$ where φ_i are homogeneous polynomials with coefficients in $\overline{\mathbb{Q}}$ that are not all identically zero on X and have equal degree at most $D \geq 1$. Then there exist a constant $c = c(X, \varphi)$ and a Zariski open dense subset U of X such that $\varphi_0, \ldots, \varphi_{\dim X}$ have no common zeros on U and*

$$h(\varphi(P)) \geq \frac{1}{4^{\dim X}\deg(X)}\frac{\deg\varphi}{D^{\dim X-1}}h(P) - c$$

for any $P \in U(\overline{\mathbb{Q}})$.

Proof. This is [28, Lemma 4.3]. □

Now we are ready to prove Proposition 9.1.

To X, recall that we associated in Proposition 8.1 a finite étale covering $S' \to S$. Set $X' = X \times_S S'$. Then X' is a closed subvariety of $\mathcal{A}' = \mathcal{A} \times_S S'$ and equidimensional of dimension $\dim X$. Let $\rho\colon \mathcal{A}' \to \mathcal{A}$ denote the natural projection, it is finite and étale. Let $\overline{S}$ be the Zariski closure of S in $\mathbb{P}^m$. We fix a smooth projective curve $\overline{S'}$ that contains S' as a Zariski open subset, then $S' \to S$ extends to a morphism $\overline{S'} \to \overline{S}$. Some positive power of the pullback of $\mathcal{O}(1)$ under $\overline{S'} \to \overline{S} \to \mathbb{P}^m$ yields a closed immersion $\overline{S'} \to \mathbb{P}^{m'}$ for some $m' \in \mathbb{N}$. The pullback of the closed immersion $\mathcal{A} \to \mathbb{P}^M \times S$ yields a closed immersion $\mathcal{A}' \to \mathbb{P}^M \times S'$ and thus an immersion $\mathcal{A}' \to \mathbb{P}^M \times \mathbb{P}^{m'}$.

We recall that [2] on $\mathcal{A} \subseteq \mathbb{P}^M \times \mathbb{P}^m$ is presented globally by bihomogeneous polynomials $G_0, \ldots, G_M$ on $\mathcal{A}$ described in §9.1. The morphism $\overline{S'} \to \overline{S} \to \mathbb{P}^m$ is defined Zariski locally on $\overline{S'}$ by an $(m+1)$-tuple of homogeneous polynomials in $m'+1$ variables. In other words there is a finite open cover $\{S'_\alpha\}_{\alpha=1}^{n_1}$ of S' such that $\overline{S'} \to \overline{S}$ is represented on each S'_α by a tuple F_α of homogeneous polynomials of equal degree and no common zero on S'_α. Above each S'_α the morphism [2] is defined by $[2](a_1, a_2) = ([G_0(a_1, F_\alpha(a_2)) : \cdots : G_M(a_1, F_\alpha(a_2))], a_2)$; here $a = (a_1, a_2) \in \mathcal{A}'(\overline{\mathbb{Q}}) \subseteq \mathbb{P}^M(\overline{\mathbb{Q}}) \times \mathbb{P}^{m'}(\overline{\mathbb{Q}})$ lies above S'_α. Iterating [2] we find that for all integers $N \geq 1$ and above each S'_α the morphism $[2^N]$ is defined by bihomogeneous polynomials with degree in a_1 equal to 4^N and degree in a_2 at most $c_1(4^N - 1)/3$; here $c_1 = c_0 \deg F_\alpha$ and c_0 is as in §9.1. As for several constants below, c_1 may depend on $\mathcal{A}'$ and X', but not on N.

Let us embed $\mathcal{A}'$ in $\mathbb{P}^{(M+1)(m'+1)-1}$ by composing the immersion $\mathcal{A}' \to \mathbb{P}^M \times \mathbb{P}^{m'}$ with the Segre morphism $\mathbb{P}^M \times \mathbb{P}^{m'} \to \mathbb{P}^{(M+1)(m'+1)-1}$. After locally inverting the Segre morphism and increasing n_1 we obtain to an open cover $\{V_\alpha\}_{\alpha=1}^{n_1}$ of $\mathcal{A}'$, a refinement of $\{\mathcal{A}'|_{S'_\alpha}\}_\alpha$, such that $[2^N]|_{V_\alpha}\colon V_\alpha \to \mathcal{A}'$ is represented by a tuple of homogeneous polynomials of degree at most $c_2 4^N$ on each V_α. Here n_1 and c_2 are independent of N.

For any irreducible component X'_0 of X' the restriction $\rho|_{X'_0}\colon X'_0 \to X$ is dominant and $\dim X'_0 = \dim X$. So Silverman's height inequality [47] applies; here we could have also used the Height Machine and Lemma 9.4. To prove the proposition, it suffices to find a constant $c > 0$ that is independent of N with the following property. For any integer $N \geq c^{-1}$ there exist an irreducible component X'_0 of X', a non-empty Zariski open subset $U'_N \subseteq X'_0$, and a constant $c'(N, X'_0)$ such that $h([2^N]Q) \geq c4^N h(Q) - c'(N, X'_0)$ for all $Q \in U'_N(\overline{\mathbb{Q}})$. Then we can take U_N to be a non-empty Zariski open subset of X with $U_N \subseteq \rho(U'_N)$ and $c'(N) = c'(N, X'_0)$.

Let $c_3 > 0$ be the c in Proposition 8.1 and let $Y_1, \ldots, Y_R$ be the subvarieties of $\mathcal{A}'$ therein. The constant c_3 and the varieties $Y_1, \ldots, Y_R$ will depend on on X and $\mathcal{A}$, but not on N. We work with 2^N instead of N in Proposition 8.1. So for any sufficienty large (but fixed) N we let $P_1, \ldots, P_r \in X'(\overline{\mathbb{Q}})$ with $r \geq c_3 4^{N \dim X}$ be pairwise distinct points as in Proposition 8.1 and $Y \in \{Y_1, \ldots, Y_R\}$ such that $[2^N](P_i) \in Y(\overline{\mathbb{Q}})$ and P_i is isolated in $X' \cap [2^N]^{-1}(Y)$ for all i.

Suppose X' has n_2 irreducible components, then n_2 is independent on N. We apply the Pigeonhole Principle to find $\alpha \in \{1, \ldots, n_1\}$ and some irreducible component of X' such that at least $c_3 4^{N \dim X}/(n_1 n_2)$ points P_i lie on V_α and this component. Replace X' by the said component and replace c_3 by $c_3/(n_1 n_2)$. Now we may assume that there is a tuple of homogeneous polynomials of equal degree at most $c_2 4^N$ that define $[2^N]$ on a Zarski open subset of $X' \subseteq \mathcal{A}' \subseteq \mathbb{P}^{(M+1)(m'+1)-1}$ that contains all the P_i's.

We apply Lemma 9.2 to $[2^N](X') \subseteq \mathcal{A}' \subseteq \mathbb{P}^{(M+1)(m'+1)-1}$, Y, and the points $[2^N](P_1), \ldots, [2^N](P_r)$. Thus we obtain a rational map $\varphi\colon \mathbb{P}^{(M+1)(m'+1)-1} \dashrightarrow \mathbb{P}^{\dim X'}$ defined at all $[2^N](P_i)$ that arises from homogenous polynomials of equal degree δ_Y. Observe that δ_Y depends only on $Y \in \{Y_1, \ldots, Y_R\}$. Let $\delta = \max_{Y \in \{Y_1, \ldots, Y_R\}} \delta_Y$. Now that $\{Y_1, \ldots, Y_R\}$ is fixed as N varies, this does not endanger our application. There exists a constant $c_4(\varphi) \geq 0$ such that

$$h(\varphi(Q)) \leq \delta h(Q) + c_4(\varphi) \tag{9.2}$$

for any Q outside the set of common zeros of the polynomials involved in φ, see [33, Theorem B.2.5.(a)]. To emphasize that φ may depend on N we write $c_4(N)$ for $c_4(\varphi)$.

For N as before we define $\varphi^{(N)} = \varphi \circ [2^N]\colon X' \dashrightarrow \mathbb{P}^{\dim X'}$. Then by Lemma 9.2(i), each P_i is mapped via $\varphi^{(N)}$ to $[1 : 0 : \cdots : 0]$.

We would like to invoke Lemma 9.3 to bound $\deg \varphi^{(N)}$ from below by r. To do this we must verify that each P_i is isolated in the fiber of $\varphi^{(N)}$ above $[1 : 0 : \cdots : 0]$. Let us suppose $C \subseteq X'$ is irreducible, contains P_i, and is inside a fiber of $\varphi^{(N)}$. Apart from finitely many points, $[2^N](C)$ is in a fiber of φ. Now we apply Lemma 9.2(ii) to conclude that $[2^N](C)$ is contained in the Zariski closure of Y inside $\mathbb{P}^{(M+1)(m'+1)-1}$. But $Y \subseteq \mathcal{A}'$, so $C \subseteq [2^N]^{-1}(Y)$. Now Proposition 8.1 implies $C = \{P_i\}$.

This settles our claim that P_i is isolated in the fiber of $\varphi^{(N)}$ and we conclude $\deg \varphi^{(N)} \geq r \geq c_3 4^{N \dim X}$.

Recall that there is a tuple of homogeneous polynomials of equal degree at most $c_2 4^N$ that define $[2^N]$ on a Zarski open subset of $X' \subseteq \mathcal{A}' \subseteq \mathbb{P}^{(M+1)(m'+1)-1}$ that contains all P_i. So we can describe $\varphi^{(N)}$ on this subset of X' using polynomials of degree at most $c_2 \delta 4^N$. We apply Lemma 9.4 to $\varphi^{(N)}$ and the Zariski closure of X' in $\mathbb{P}^{(M+1)(m'+1)-1}$ to conclude that there exist a constant $c_5 > 0$, independent of N, and a constant $c_6(N) \geq 0$ which may depend on N, such that

$$h(\varphi^{(N)}(P)) \geq c_5 4^N h(P) - c_6(N) \tag{9.3}$$

for all $P \in U'_N(\overline{\mathbb{Q}})$ where U'_N is non-empty Zariski open in X' and may depend on N.

Now by letting $Q = [2^N]P$ and dividing by δ in (9.2), we get by (9.3) that

$$h([2^N]P) \geq \frac{c_5}{\delta} 4^N h(P) - c_7(N)$$

for all $P \in U'_N(\overline{\mathbb{Q}})$ after possibly shrinking U'_N. The proof is complete as c_5/δ is independent of N. □

10. Néron–Tate Height and Height on the Base

The goal of this section is to prove Theorem 1.4 and the slightly stronger Theorem 1.4′.

We will use the basic setup introduced in §2.2. Thus S is a smooth, irreducible curve over $\overline{\mathbb{Q}}$, $\pi\colon \mathcal{A} \to S$ is an abelian scheme of relative dimension ≥ 1, and $\mathcal{A} \subseteq \mathbb{P}^M_{\overline{\mathbb{Q}}} \times \mathbb{P}^m_{\overline{\mathbb{Q}}}$ is an admissible immersion. All varieties in this sections are defined over $\overline{\mathbb{Q}}$.

10.1. **Auxiliary Proposition.** We prove the following proposition; recall that both heights below are defined as in §2.2.

Proposition 10.1. *Assume X is an irreducible closed subvariety of $\mathcal{A}$ that is not generically special. Then there exist a non-empty Zariski open subset $U \subseteq X$ defined over $\overline{\mathbb{Q}}$ and a constant $c > 0$ depending only on $\mathcal{A}/S$, X, and the admissible immersion such that*

$$h(P) \leq c\left(1 + \hat{h}_{\mathcal{A}}(P)\right) \tag{10.1}$$

for all $P \in U(\overline{\mathbb{Q}})$.

Proof. By the Theorem of Silverman-Tate [46, Theorem A], there exist a constant $c_1 \geq 0$ such that

$$|\hat{h}_{\mathcal{A}}(P) - h(P)| \leq c_1\big(1 + h(\pi(P))\big) \tag{10.2}$$

for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$; observe that this proof also holds without the smoothness assumption when working with line bundles instead of Weil divisors.

To prove the proposition we may thus assume that π is non-constant on X. Therefore, X dominates S.

Since X is not generically special, we can apply Proposition 9.1 to X. There exists a constant $c_2 > 0$, depending on $X, \mathcal{A}$, and its admissible immersion, such that the following holds. For any integer $N \geq c_2^{-1}$, there exist a Zariski open dense subset $U_N \subseteq X$ and a constant $c_3(N) \geq 0$ such that

$$h([2^N]P) \geq c_2 4^N h(P) - c_3(N) \tag{10.3}$$

for all $P \in U_N(\overline{\mathbb{Q}})$; we stress that U_N and $c_3(N) \geq 0$ may depend on N in addition to $X, \mathcal{A}$, and the immersion.

Now for any integer $N \geq c_2^{-1}$ and any $P \in U_N(\overline{\mathbb{Q}})$, we have

$$\begin{aligned}
\hat{h}_{\mathcal{A}}([2^N](P)) &\geq h([2^N](P)) - c_1\left(1 + h(\pi([2^N](P)))\right) \quad \text{by (10.2)}\\
&= h([2^N](P)) - c_1\left(1 + h(\pi(P))\right)\\
&\geq c_2 4^N h(P) - c_3(N) - c_1\left(1 + h(P)\right) \text{by (10.3) and } h(\pi(P)) \leq h(P) \text{ (2.2)}.
\end{aligned}$$

But $\hat{h}_{\mathcal{A}}([2^N]P) = 4^N \hat{h}_{\mathcal{A}}(P)$ and dividing by 4^N yields

$$\hat{h}_{\mathcal{A}}(P) \geq \left(c_2 - \frac{c_1}{4^N}\right) h(P) - \frac{c_3(N) + c_1}{4^N}$$

for all $N \geq c_2^{-1}$ and all $P \in U_N(\overline{\mathbb{Q}})$.

Recall that c_2 and c_3 are independent of N. We fix N to be the least integer such that $4^N \geq 2c_1/c_2$ and $N \geq c_2^{-1}$. Then

$$\hat{h}_{\mathcal{A}}(P) \geq \frac{c_2}{2} h(P) - \frac{c_3(N) + c_1}{4^N}$$

for all $P \in U_N(\overline{\mathbb{Q}})$. Since N is fixed now, the Zariski open dense subset U_N of X is also fixed. For an appropriate $c > 0$, depending on N, c_1, c_2, and $c_3(N)$ we conclude (10.1). $\square$

10.2. **Proof of Theorem 1.4.** The first inequality in (1.1) follows from the definition of $h_{\mathcal{A},\iota}(\cdot)$ and as the absolute logarithmic Weil height is non-negative. To prove the second inequality we may assume, by properties of the height machine, that the two height functions appearing in the conclusion arise from an admissible immersion. We do an induction on $\dim X$. When $\dim X = 0$, this result is trivial. So let us assume $\dim X \geq 1$.

If X is generically special then $X^* = \emptyset$ and there is nothing to show. Otherwise we may apply Proposition 10.1 and so the inequality (10.1) holds for any $x \in (X \setminus Z)(\overline{\mathbb{Q}})$ for some proper closed subvariety Z of X defined over $\overline{\mathbb{Q}}$. Let $Z = Z_1 \cup \cdots \cup Z_r$ be the decomposition into irreducible components. Since $\dim Z_i \leq \dim X - 1$ we may do induction on the dimension. By the induction hypothesis, the inequality (10.1) holds for all points in $Z_1^*(\overline{\mathbb{Q}}) \cup \cdots \cup Z_r^*(\overline{\mathbb{Q}})$. Therefore the inequality (10.1) holds for all points in $(X \setminus Z)(\overline{\mathbb{Q}}) \cup Z_1^*(\overline{\mathbb{Q}}) \cup \cdots \cup Z_r^*(\overline{\mathbb{Q}})$.

To prove that the inequality (10.1) holds for all points in $X^*(\overline{\mathbb{Q}})$, it suffices to verify

$$X^* \subseteq (X \setminus Z) \cup Z_1^* \cup \cdots \cup Z_r^*.$$

But this is equivalent to the inclusion

$$X \setminus X^* \supseteq Z \cap (X \setminus Z_1^*) \cap \cdots \cap (X \setminus Z_r^*) = (Z \setminus Z_1^*) \cap \cdots \cap (Z \setminus Z_r^*). \tag{10.4}$$

Finally, a generically special subvariety of $\mathcal{A}$ contained in some Z_i will be contained in X, and therefore (10.4) holds true.

Now the inequalities in Theorem 1.4 and Theorem 1.4′ hold since, by (2.2), $h(\pi(P)) \leq h(P)$ for any $P \in \mathcal{A}(\overline{\mathbb{Q}})$. $\square$

11. Application to the Geometric Bogomolov Conjecture

In this section we prove Theorem 1.1 over the base field $\overline{\mathbb{Q}}$ and abbreviate $\mathbb{P}^m_{\overline{\mathbb{Q}}}$ by $\mathbb{P}^m$.

More general base fields can be handled using the Moriwaki height version of Theorem 1.4. More details are presented in Appendix A.

There exists a smooth, irreducible, quasi-projective curve S over $\overline{\mathbb{Q}}$ whose function field is K. We fix an algebraic closure $\overline{K} \supseteq K$ of K. As in §2.2 we can find, up to removing finitely many points of S, an abelian scheme $\mathcal{A} \to S$ whose generic fiber is A from Theorem 1.1. We equip $\mathcal{A}$ with an admissible immersion $\mathcal{A} \to \mathbb{P}^M \times \mathbb{P}^m$, cf. §2.2. In particular, we have an immersion $\iota\colon S \to \mathbb{P}^m$. For $s \in S(\overline{\mathbb{Q}})$ we set

$$h_S(s) = \frac{1}{\deg S} h(\iota(s)) \tag{11.1}$$

where h on the right-hand side is the height on $\mathbb{P}^m(\overline{\mathbb{Q}})$ and $\deg S$ is the degree of the Zariski closure of $\iota(S)$ in $\mathbb{P}^m$. We use the same normalization as in Silverman's work [46,

§4], which will play an important role momentarily. In addition, we have the fiberwise Néron–Tate height $\hat{h}_{\mathcal{A}}\colon \mathcal{A}(\overline{\mathbb{Q}}) \to [0,\infty)$, *cf.* (2.3). On A we also have a Néron–Tate height $h_{K,A}\colon A(\overline{K}) \to [0,\infty)$, *cf.* the end of §2.1.

Before we get to the nuts and bolts we state Silverman's Height Limit Theorem.

Recall that we can represent a point $x \in A(\overline{K}) = (A \otimes_K \overline{K})(\overline{K})$ using a section $S' \to \mathcal{A} \times_S S'$ where S' is a smooth, irreducible curve and $\rho\colon S' \to S$ is generically finite morphism. We write σ_x for the composition $S' \to \mathcal{A} \times_S S' \to \mathcal{A}$. We may evaluate $\hat{h}_{\mathcal{A}}$ at $\sigma_x(t) \in \mathcal{A}_{\rho(t)}$ for all $t \in S'(\overline{\mathbb{Q}})$.

Theorem 11.1 (Silverman)**.** *In the notation above we have*

$$\lim_{\substack{t\in S'(\overline{\mathbb{Q}})\\ h_S(\rho(t))\to\infty}} \frac{\hat{h}_{\mathcal{A}}(\sigma_x(t))}{h_S(\rho(t))} = \hat{h}_{K,A}(x). \tag{11.2}$$

Proof. This follows from [46, Theorem B] via a base change argument as follows. The smoothness condition in this reference can be dropped when using line bundles instead of Weil divisors. Observe that Silverman's Theorem deals with the rational case $x \in A(K)$, which comes from a section $S \to \mathcal{A}$.

We have a morphism $\sigma_x\colon S' \to \mathcal{A}$, which composed with $\mathcal{A} \to S$ equals $\rho\colon S' \to S$. We write $K' = \overline{\mathbb{Q}}(S')$ and $A_{K'} = A \otimes_K K'$. Then $h_{S'}\colon S'(\overline{\mathbb{Q}}) \to [0,\infty)$ is defined analog to h_S via an immersion of S' into some projective space and then normalizing as in (11.1). Of course, $h_{S'}$ depends on the choice of this immersion. But we have $h_S(\rho(t))/h_{S'}(t) \to \deg\rho = [K' : K]$ for $t \in S'(\overline{\mathbb{Q}})$ as $h_{S'}(\rho(t)) \to \infty$ by quasi-equivalent of heights on curves, *cf.* [7, Corollary 9.3.10] and our choice of normalization. Silverman's Theorem applied to $x \in A(K')$ implies $\hat{h}_{\mathcal{A}}(\sigma_x(t))/h_{S'}(t) \to \hat{h}_{K',A_{K'}}(x)$ as $h_S(t) \to \infty$ for $t \in S'(\overline{\mathbb{Q}})$. Thus $\hat{h}_{\mathcal{A}}(\sigma_x(t))/h_S(\rho(t)) \to [K' : K]^{-1}\hat{h}_{K',A_{K'}}(x)$ for $t \in S'(\overline{\mathbb{Q}})$ and $h_S(\rho(t)) \to \infty$.

Now $\hat{h}_{K,A}$ and $\hat{h}_{K',A'}$ are related by $\hat{h}_{K',A_{K'}} = [K' : K]\hat{h}_{K,A}$, this follows from the related statement for naive heights, *cf.* [14, Remark 9.2], and passing to the limit. The factor $[K' : K]$ cancels out with the same factor coming from quasi-equivalence of heights and this yields (11.2). □

Now we complete the proof of Theorem 1.1.

It is enough to prove the theorem for the symmetric line bundle L attached to the closed immersion $A \to \mathbb{P}^M_K$.

Let $\mathcal{X}$ be the Zariski closure of X inside $\mathcal{A} \supseteq A \supseteq X$. Then $\mathcal{X}$ is irreducible and flat over S and X is the generic fiber of $\mathcal{X} \to S$.

Therefore by the assumption on X the variety $\mathcal{X}$ is not generically special. We will apply Proposition 10.1, so let $\mathcal{U}$ be the Zariski open and dense subset of $\mathcal{X}$ from this proposition.

We define $U = \mathcal{U} \cap X$, where the intersection is inside $\mathcal{A}$. This is a Zariski open and dense subset of X.

It suffices to prove that there exists $\epsilon > 0$ such that $x \in U(\overline{K})$ implies $\hat{h}_{K,A}(x) \geq \epsilon$.

Indeed, let $x \in U(K')$ where K' is a finite field extension of K contained in $\overline{K}$. As above, there are an irreducible, quasi-projective curve S' over $\overline{\mathbb{Q}}$ with function field K', a generically finite morphism $\rho\colon S' \to S$, and a section $S' \to \mathcal{A} \times_S S'$ determined by x. We write $\sigma_x\colon S' \to \mathcal{A}$ for this section composed with $\mathcal{A} \times_S S' \to \mathcal{A}$.

The Zariski closure $\mathcal{Y}$ in $\mathcal{A}$ of the image of σ_x is an irreducible closed curve in $\mathcal{A}$. We have $\mathcal{Y} \subseteq \mathcal{X}$ as $x \in X(K')$. Moreover, $\mathcal{Y} \cap \mathcal{U} \neq \emptyset$ since $x \in U(\overline{K})$. So $\mathcal{Y} \cap \mathcal{U}$ is a curve that differs from $\mathcal{Y}$ in only finitely many points.

We fix a sequence $t_1, t_2, \ldots \in S'(\overline{\mathbb{Q}})$ such that $\lim_{n\to\infty} h_S(\rho(t_n)) = \infty$. Silverman's Theorem implies

$$\lim_{n\to\infty} \frac{\hat{h}_{\mathcal{A}}(\sigma_x(t_n))}{h_S(\rho(t_n))} = \hat{h}_{K,A}(x). \tag{11.3}$$

For n large enough we have $\sigma_x(t_n) \in \mathcal{U}(\overline{\mathbb{Q}})$. By Proposition 10.1 there exists a constant $c > 0$, independent of x, σ_x, and n, such that

$$h(\sigma_x(t_n)) \leq c\left(1 + \hat{h}_{\mathcal{A}}(\sigma_x(t_n))\right) \quad \text{for all large integers } n. \tag{11.4}$$

By (2.2) the naive height $h(\sigma_x(t_n))$ is at least the height of $\iota(\pi(\sigma_x(t_n))) = \iota(\rho(t_n)) \in \mathbb{P}^m(\overline{\mathbb{Q}})$. By our choice (11.1) we have $h(\iota(\rho(t_n))) = \deg(S) h_S(\rho(t_n))$. We insert into (11.4) and divide by $h_S(\rho(t_n))$ to obtain

$$\deg S \leq c \frac{1 + \hat{h}_{\mathcal{A}}(\sigma_x(t_n))}{h_S(\rho(t_n))}.$$

Finally, we pass to the limit $n \to \infty$ and recall (11.3) to conclude $\hat{h}_{K,A}(x) \geq \deg(S)/c$. The theorem follows as c and $\deg(S)$ are independent of x. □

Appendix A. Passing from $\overline{\mathbb{Q}}$ to any Field of Characteristic 0

In this appendix, we sketch a proof of Theorem 1.1 for any k algebraically closed of characteristic 0. We do this by proving a Moriwaki height version of Theorem 1.4, allowing $\overline{\mathbb{Q}}$ to be replaced by any algebraically closed field of finite transcendence degree over $\overline{\mathbb{Q}}$. Then we repeat the proof of Theorem 1.1 for $k = \overline{\mathbb{Q}}$ (§11) with this new height function to get the result when $\mathrm{trdeg}_{\mathbb{Q}} k < \infty$. Finally we use essential minimum to reduce to this case.

A.1. **Moriwaki height.** In this subsection we review Moriwaki's height theory [39].

Let k_0 be a finitely generated field over $\mathbb{Q}$ with $\mathrm{trdeg}(k_0/\mathbb{Q}) = d$. Moriwaki [39] developped the following height theory, generalizing the classical height theory for $\overline{\mathbb{Q}}$.

Fix a polarization $\overline{\mathbf{B}} = (B; \overline{H}_1, \ldots, \overline{H}_d; \tau)$ of the field k_0, *i.e.* a flat and quasi-projective integral scheme over $\mathbb{Z}$, a collection of nef smooth hermitian line bundles $\overline{H}_1, \ldots, \overline{H}_d$ on B and an isomorphism of fields $\tau\colon \mathbb{Q}(B) \to k_0$. In most of the literature, including Moriwaki's paper, the isomorphism τ is omitted as it is fixed.

Let X be an irreducible projective variety over k_0 and let L be a line bundle on X that is defined over k_0. Moriwaki [39] defines a height function

$$h^{\overline{\mathbf{B}}}_{k_0,X,L}\colon X(\overline{k_0}) \to \mathbb{R} \tag{A.1}$$

which is well-defined modulo the set of bounded functions on $X(\overline{k_0})$. We will not repeat the exact definition of the Moriwaki height here, but will mention some properties. If the field k_0 is clear from the context, then we abbreviate $h^{\overline{\mathbf{B}}}_{k_0,X,L}$ to $h^{\overline{\mathbf{B}}}_{X,L}$. If furthermore X is clear, then we abbreviate it to $h^{\overline{\mathbf{B}}}_{L}$.

Before going on, let us make the following remark. If k_0' is a field with an isomorphism $\iota\colon k_0 \to k_0'$, then we have a polarization $\overline{\mathbf{B}}' = (B; \overline{H}_1, \ldots, \overline{H}_d; \iota \circ \tau)$ of k_0'. For any algebraic closure $\overline{k_0}'$ of k_0', ι extends (non-uniquely) to an isomorphism $\overline{k_0} \to \overline{k_0}'$, which we still denote by ι by abuse of notation. Then $h^{\overline{\mathbf{B}}'}_{k_0'} \circ \iota = h^{\overline{\mathbf{B}}}_{k_0}$.

As is pointed out by Moriwaki, if k_0 is a number field, then we recover the classical height functions. Just as the classical height, the Moriwaki height (A.1) satisfies the several properties.

Proposition A.1 (Height Machine for the Moriwaki height)**.** *We keep the notation from above.*

(1) *(Additivity) If M is another line bundle on X, then $h^{\overline{\mathbf{B}}}_{L\otimes M} = h^{\overline{\mathbf{B}}}_L + h^{\overline{\mathbf{B}}}_M$.*
(2) *(Functoriality) Let $q\colon X \to Y$ be a quasi-finite morphism of projective varieties over k_0 and let M be a line bundle on Y. Then*

$$h^{\overline{\mathbf{B}}}_{q^*M} = h^{\overline{\mathbf{B}}}_M \circ q$$

(3) *(Boundedness) The function $h^{\overline{\mathbf{B}}}_L$ is bounded below away from the base locus of L. In particular $h^{\overline{\mathbf{B}}}_L$ is bounded on $X(\overline{k_0})$ if $L = \mathcal{O}_X$.*
(4) *(Northcott) If L is ample, and if $\overline{\mathbf{B}}$ is big, i.e. the $\overline{H}_i$'s are nef and big. Then for any real numbers B, D, the set*

$$\{P \in X(\overline{k_0}) : h^{\overline{\mathbf{B}}}_L(P) \le B,\ [k_0(P) : k_0] \le D\}$$

is finite.
(5) *(Algebraic Equivalence) If L and M are algebraically equivalent and L is ample, then*

$$\lim_{h^{\overline{\mathbf{B}}}_L(P)\to\infty} \frac{h^{\overline{\mathbf{B}}}_M(P)}{h^{\overline{\mathbf{B}}}_L(P)} = 1.$$

Proof. Part (1), (3) and (4) are proven by Moriwaki [39, Proposition 3.3.7(2-4)]. See [54, Proposition 2(iv)] for a proof of part (2), note that the smoothness assumption is unnecessary and that q must be generically finite in [39, Proposition 1.3(2)]. Part (5) can be proven by a verbalized copy of [35, Chapter 4, Proposition 3.3 and Corollary 3.4] with the usual height function replaced by the Moriwaki height. □

Proposition A.1 enables us to transfer results involving only properties of the height listed in the Height Machine to the Moriwaki height.

Next we turn to abelian varieties. Let A be an abelian variety over k_0, and let L be a symmetric ample line bundle on A that is defined over k_0. The limit

$$\hat{h}^{\overline{\mathbf{B}}}_L(P) = \lim_{n\to\infty} 2^{-2n} h^{\overline{\mathbf{B}}}_L([2^n]P) \quad \text{for all} \quad P \in A(\overline{k_0}) \tag{A.2}$$

exists and is independent of the choice of a representative of the height function. Then $\hat{h}^{\overline{\mathbf{B}}}_L$ is called the *canonical* or *Néron–Tate height* on $A(\overline{k_0})$ attached of L with respect to $\overline{\mathbf{B}}$. If $k_0 = \mathbb{Q}$, then $\hat{h}^{\overline{\mathbf{B}}}_L$ coincides with the usual Néron–Tate height over $\overline{\mathbb{Q}}$. Moriwaki [39, §3.4] proved that the Néron–Tate height $\hat{h}^{\overline{\mathbf{B}}}_L$ is quadratic, *i.e.*

$$\hat{h}^{\overline{\mathbf{B}}}_L([N]P) = N^2 \hat{h}^{\overline{\mathbf{B}}}_L(P) \quad \text{for all} \quad P \in A(\overline{k_0}).$$

The following proposition is proven by Moriwaki [39, Proposition 3.4.1].

Proposition A.2. (i) *We have* $\hat{h}_L^{\overline{\mathbf{B}}}(P) \geq 0$ *for all* $P \in A(\overline{k_0})$.

(ii) *We have* $\hat{h}_L^{\overline{\mathbf{B}}}(P) = 0$ *for all* $P \in A(\overline{k_0})_{\mathrm{tor}}$.

(iii) *Assume* $\overline{\mathbf{B}}$ *is big, i.e.* $\overline{H}_i$*'s are nef and big. Then* $\hat{h}_L^{\overline{\mathbf{B}}}(P) = 0$ *if and only if* P *is a torsion point.*

A.2. **Height Inequality.** Let k_0 be a finitely generated field extension of $\mathbb{Q}$. Let $\overline{\mathbf{B}}$ be a polarization of k_0. Let $\overline{k_0}$ be an algebraic closure of k_0.

Let S be a smooth irreducible quasi-projective curve over k_0, and let $\pi\colon \mathcal{A} \to S$ be an abelian scheme over k_0 of relative dimension $g \geq 1$. We fix a smooth, irreducible projective curve $\overline{S}$ over k_0 that contains S as a Zariski open subset. Let $\mathcal{M}$ be an ample line bundle on $\overline{S}$ defined over k_0. Let $\mathcal{L}$ be a symmetric relatively ample line bundle on $\mathcal{A}/S$ defined over k_0. Then we have the following analogue of Theorem 1.4.

Theorem A.3. *Let* X *be a closed irreducible subvariety of* $\mathcal{A}$ *over* k_0 *and let* X^* *be as above Proposition 1.3 with* $k = \overline{k_0}$. *Then there exists* $c > 0$ *depending only on* $\overline{\mathbf{B}}, \mathcal{A}/S, X, \mathcal{L}$, *and* $\mathcal{M}$ *such that*

$$h_{S,\mathcal{M}}^{\overline{\mathbf{B}}}(P) \leq c\left(1 + \hat{h}_{\mathcal{A},\mathcal{L}}^{\overline{\mathbf{B}}}(P)\right) \quad \text{for all} \quad P \in X^*(\overline{k_0})$$

where $h_{S,\mathcal{M}}^{\overline{\mathbf{B}}}$ *is the Moriwaki height defined by* (A.1), *and* $\hat{h}_{\mathcal{A},\mathcal{L}}^{\overline{\mathbf{B}}}(x)$ *is the Néron–Tate height* $\hat{h}_{\mathcal{L}_{\pi(x)}}^{\overline{\mathbf{B}}}(x)$ *defined by* (A.2) *on the abelian variety* $\mathcal{A}_{\pi(x)}$.

If k_0 is a subfield of $\mathbb{C}$, then we can take $\overline{k_0} \subseteq \mathbb{C}$. In this case we can proceed as in the proof of Theorem 1.4, with the usual height over $\overline{\mathbb{Q}}$ replaced by the Moriwaki height over k_0. In fact we only used $\overline{\mathbb{Q}}$ in the arguments involving heights, *i.e.* Proposition 9.1 (in fact only Lemma 9.4 and below) and Proposition 10.1 (for this we need the Moriwaki height version of Silverman-Tate, which is [54, Theorem 2]). Now Proposition A.1 provides us with the Height Machine for $h_L^{\overline{\mathbf{B}}}$ and hence all the arguments are still valid.

For general k_0 finitely generated over $\mathbb{Q}$, we have that k_0 is isomorphic to a subfield k_0' of $\mathbb{C}$ via some ι. Let $\overline{k_0}'$ be the algebraic closure of k_0' in $\mathbb{C}$, then ι extends to some $\iota\colon \overline{k_0} \to \overline{k_0}'$. As explained in the paragraph below (A.1), we can get a polarization $\overline{\mathbf{B}}'$ of k_0' such that $h_{k_0'}^{\overline{\mathbf{B}}'} \circ \iota = h_{k_0}^{\overline{\mathbf{B}}}$. So we are reduced to the case where k_0 is a subfield of $\mathbb{C}$ and hence we are done.

A.3. **Geometric Bogomolov Conjecture.** Suppose that we are in the situation of Theorem 1.1. There exists a smooth, irreducible, quasi-projective curve S over k whose function field is K. We can find, up to removing finitely many points of S, an abelian scheme $\mathcal{A} \to S$ whose generic fiber is A. We write $\mathcal{X}$ for the Zariski closure of X under $A \subseteq \mathcal{A}$. Then $\mathcal{X}$ is a closed irreducible subvariety of $\mathcal{A}$. We fix an algebraic closure $\overline{K} \supseteq K$ of K and a smooth, irreducible, projective curve $\overline{S}$ that contains S as a Zariski open subset.

The symmetric ample line bundle L extends, up to removing finitely many points of S, to a symmetric relatively ample line bundle $\mathcal{L}$ on $\mathcal{A}/S$. There exists a field k_0 finitely generated over $\mathbb{Q}$ such that S, $\mathcal{A}/S$, $\mathcal{L}$ and $\mathcal{X}$ are defined over k_0. We treat these objects as being over k_0.

Let us take an ample line bundle on $\overline{S}$ defined over k_0.

By [23, EGA IV$_2$, Proposition 2.8.5] $\mathcal{X}$ is flat over S and $i^{-1}(\mathcal{X}) = X$, so X is the generic fiber of $\mathcal{X} \to S$. Therefore $\mathcal{X}$ is not generically special by the assumption on X. Hence $\mathcal{X}^*$ is Zariski open dense in $\mathcal{X}$ by Proposition 1.3.

Take algebraic closures $\overline{k_0}$ in k and $\overline{k_0(S)}$ in $\overline{K}$. From now on we see X, A, and L as defined over $k_0(S)$. We claim that there exists a constant $\epsilon > 0$ such that

$$\{P \in X(\overline{k_0(S)}) : \hat{h}_{\overline{k_0}(S),A,L}(P) \leq \epsilon\} \tag{A.3}$$

is not Zariski dense in X.

We indicate how to modify the proof in §11 to prove this claim. The only changes are

- The Zariksi open subset $\mathcal{U}$ of $\mathcal{X}$ is replaced by $\mathcal{X}^*$.
- Instead of Proposition 10.1, we use the generalized version of Theorem 1.4. More precisely let $\overline{\mathbf{B}}$ be a big polarization of k_0, *i.e.* the $\overline{H}_i$'s are nef and big. Apply Theorem A.3 to the subvariety $\mathcal{X} \subseteq \mathcal{A}$ and $(k_0, \overline{\mathbf{B}})$ to obtain a constant $c > 0$.
- The polarization $\overline{\mathbf{B}}$ is big, so there exists a sequence of points $t_1, t_2, \ldots \in S(\overline{k_0})$ such that $\lim_{n\to\infty} h^{\overline{\mathbf{B}}}_{S,\mathcal{M}}(t_n) = \infty$. Also the Moriwaki height version of Silverman's Theorem (11.2) still holds, see [54, Theorem 3]. In fact Proposition A.1 provides us with the Height Machine for $h^{\overline{\mathbf{B}}}_L$ and hence Silverman's original proof still works with the usual height function replaced by the Moriwaki height.

We are not done yet because we want to replace $\overline{k_0(S)}$ by $\overline{K}$ in (A.3). Indeed, $K = k_0(S) \otimes_{k_0} k$ contains k which is an arbitrary field of characteristic 0 and therefore possibly not finitely generated over $\mathbb{Q}$. To proceed we prove the following statement on the essential minimum

$$\mu_{\text{ess}}(X) = \inf\left\{\epsilon > 0 : \{P \in X(\overline{k_0(S)}) : \hat{h}_{\overline{k_0}(S),A,L}(P) \leq \epsilon\} \text{ is Zariski dense in } X\right\}.$$

The analog $\mu_{\text{ess}}(X_K)$ where $X_K = X \otimes_{k_0(S)} K$ is defined similarly but involves K.

Claim: If $\mu_{\text{ess}}(X_K) = 0$, then $\mu_{\text{ess}}(X) = 0$.

S. Zhang proved two inequalities relating the essential minima and the height of a subvariety of an abelian variety in the number field case [65]. To prove our claim we require Gubler's [26, Corollary 4.4] version of Zhang's inequalities for function fields. See §3 of [26] for the definition of the height of a subvariety of A.

More precisely, $\mu_{\text{ess}}(X) = 0$ if and only if $\hat{h}_{\overline{k_0}(S),A,L}(X) = 0$. Moreover, $\mu_{\text{ess}}(X_K) = 0$ if and only if $\hat{h}_{K,A_K,L}(X_K) = 0$. Finally, we sketch how to prove the equality $\hat{h}_{\overline{k_0}(S),A,L}(X) = \hat{h}_{K,A_K,L_K}(X_K)$ which settles our claim. The base change involves only an extension of the field of constants $k/\overline{k_0}$ under which the naive height on projective height remains unchanged. The height of a subvariety of some projective space can be defined as the height of a Chow form and is thus invariant under base change as well. Finally, the canonical height of a subvariety of A is a limit as in Tate's argument and is thus invariant under base change.

Appendix B. Proposition 1.3 for Higher Dimensional Base

In this appendix, we explain how to generalize Proposition 3.1 to higher dimensional base. We work under the frame of §3 except that we do not make any assumption on $\dim S$. In other words, out setting is as follows.

Let k be an algebraically closed field of characteristic 0. Let S be a smooth irreducible quasi-projective variety over k and fix an algebraic closure $\overline{K}$ of $K = k(S)$. Let A be an abelian variety over $\overline{K}$.

We start with the following proposition.

Proposition B.1. *Assume $A^{\overline{K}/k} = 0$. The order of any point in*

$$\{P \in A(\overline{K})_{\mathrm{tor}} : [K(P) : K] \leq D\} \tag{B.1}$$

is bounded in terms of A and D only.

Proof. Fix an irreducible projective variety $\overline{S}$ over k whose function field is K. We may take S as a Zariski open dense subset of $\overline{S}$ such that A extends to an abelian scheme $\mathcal{A} \to S$, namely the generic fiber of $\mathcal{A} \to S$ is A. We may furthermore shrink S such that S is smooth and that $\overline{S} \setminus S$ is purely of codimension 1. We denote by $i\colon A \to \mathcal{A}$ the natural morphism.

Any $P \in A(\overline{K})_{\mathrm{tor}}$ defines a morphism $\sigma_P\colon \mathrm{Spec}\overline{K} \to A$. Suppose the order of P is N. Then the Zariski closure of $\mathrm{Im}(i \circ \sigma_P)$, which we denote by $\mathcal{T}$, is irreducible, dominates S and satisfies $[N]\mathcal{T} = 0$. So $\mathcal{T}$ is an irreducible component of the kernel of $[N]\colon \mathcal{A} \to \mathcal{A}$ by comparing dimensions. In particular $\mathcal{T} \hookrightarrow \ker[N]$ is an open and closed immersion. But $\ker[N] \to S$ is finite étale, so is $\mathcal{T} \to S$. Thus $\mathcal{T} \to S$ is an étale covering of degree $[K(P) : K]$.

In other words, any torsion point $P \in A(\overline{K})_{\mathrm{tor}}$ yields an étale covering of S of degree $[K(P) : K]$.

By Lemma B.2 below, the compositum F in $\overline{K}$ of all such extensions $K(P)$ of K of degree at most D is a finite field extension of K. For S a curve we cited [53, Corollary 7.11]. In particular, $P \in A(F)$ for all P in (B.1).

Now $A^{\overline{K}/k} = 0$. So the Lang–Néron Theorem, cf. [36, Theorem 1] or [14, Theorem 7.1], implies that $A(F)$ is a finitely generated group. Thus $[M](P) = 0$ for some $M \in \mathbb{N}$ that is independent of P. Our claim follows. $\square$

Lemma B.2. *Let k be an algebraically closed field of characteristic 0 and let S be a smooth irreducible quasi-projective variety over k. Then for any integer $D > 0$ there are at most finitely many étale coverings of S of degree $\leq D$.*

Proof. It suffices to prove that the étale fundamental group $\pi_1(S)$ is topologically finitely generated.

Let k_0 be an algebraically closed subfield of k that is of finite transcendence degree over $\overline{\mathbb{Q}}$ such that S is defined over k_0. Then we can fix an embedding $k_0 \hookrightarrow \mathbb{C}$. We write S_0 for the descent of S to k_0, namely $S = S_0 \otimes_{k_0} k$. We also write $S_{\mathbb{C}} = S_0 \otimes_{k_0} \mathbb{C}$ for the base change of S_0 to $\mathbb{C}$.

Let $s_0\colon \mathrm{Spec}k_0 \to S_0$ be a geometric point. Denote by $s\colon \mathrm{Spec}k \to S$, resp. $s_{\mathbb{C}}\colon \mathrm{Spec}\mathbb{C} \to S_{\mathbb{C}}$, the corresponding geometric points.

It is a classical result that the topological fundamental group $\pi_1(S_{\mathbb{C}}^{\mathrm{an}}, s_{\mathbb{C}})$ is finitely generated. Hence by Riemann's Existence Theorem [25, Exposé XII, Théorème 5.1] the étale fundamental group $\pi_1(S_{\mathbb{C}}, s_{\mathbb{C}})$ is topologically finitely generated. But then $\pi_1(S_{\mathbb{C}}, s_{\mathbb{C}}) \cong \pi_1(S_0, s_0)$ by [10, Corollary 6.5 and Remark 6.8]. So $\pi_1(S_0, s_0)$ is topologically finitely generated. Then again by [10, Corollary 6.5 and Remark 6.8], we have $\pi_1(S, s) \cong \pi_1(S_0, s_0)$. So $\pi_1(S, s)$ is topologically finitely generated. $\square$

Now we are ready to prove:

Proposition B.3. *let V_0 be an irreducible variety defined over k and $V = V_0 \otimes_k \overline{K}$. By abuse of notation we consider $V_0(k)$ as a subset of $V(\overline{K})$. Define $\Sigma = V_0(k) \times A_{\mathrm{tor}} \subseteq V(\overline{K}) \times A(\overline{K})$.*

Let Y be an irreducible closed subvariety of $V \times A$ such that $Y(\overline{K}) \cap \Sigma$ lies Zariski dense in Y. If $A^{\overline{K}/k} = 0$ then $Y = (W_0 \otimes_k \overline{K}) \times (t + B)$ where $W_0 \subseteq V_0$ is an irreducible closed subvariety, $t \in A(\overline{K})_{\mathrm{tor}}$ and B is an abelian subvariety of A.

The only difference of this proposition with Proposition 3.1 is that we do not make any assumption on $\dim S$.

As we have pointed out, in the proof of Proposition 3.1 the only place where we used the assumption $\dim S = 1$ is to prove the statement involving (3.2). But for S of arbitrary dimension this follows from Proposition B.1.

Appendix C. Hyperbolic Hypersurfaces of Abelian Varieties

Suppose F is an algebraically closed field of characteristic zero. For each integer $d \geq 0$ we let $F[X_0, \ldots, X_M]_d$ be the vector space of homogeneous polynomials of degree d in $F[X_0, \ldots, X_M]$ together with 0. In this section we identify $F[X_0, \ldots, X_M]_d$ with $\mathbb{A}^{\binom{M+d}{M}}(F)$, where we abbreviate $\mathbb{A}^M = \mathbb{A}^M_F$ and $\mathbb{P}^M = \mathbb{P}^M_F$.

Brotbek's deep result [9] implies that a generic sufficiently ample hypersurface in a smooth projective variety over $\mathbb{C}$ is hyperbolic. In the very particular case of an abelian variety we give an independent proof that involves an explicit bound on the degree. Recall that an irreducible subvariety of an abelian variety is hyperbolic if and only if it does not contain a coset of positive dimension by the Bloch–Ochiai Theorem. The main results of this paper do not depend on the Bloch–Ochiai Theorem.

Proposition C.1. *Let A be an abelian variety over F of dimension $g \geq 1$ with $A \subseteq \mathbb{P}^M$ and suppose $d \geq g-1$. There exists a Zariski open and dense subset $U \subseteq \mathbb{A}^{\binom{M+d}{M}}$, whose complement in $\mathbb{A}^{\binom{M+d}{M}}$ has codimension at least $d+2-g$, such that if $f \in U(F)$, then $A \cap \mathscr{Z}(f)$ does not contain any positive dimensional coset.*

A direct corollary of this proposition is the following statement. Let L be a very ample line bundle on A giving rise to a projectively normal, closed immersion $A \hookrightarrow \mathbb{P}^M$ and say $P \in A(F)$. Then the hypersurface defined by a generic choice of a section in $H^0(\mathbb{P}^n, \mathcal{O}(d))$ vanishing at P is hyperbolic for $d \geq g$.

Suppose V is an irreducible, closed subvariety of $\mathbb{P}^M$ with ideal $I \subseteq F[X_0, \ldots, X_M]$. We write $I_d = I \cap F[X_0, \ldots, X_M]_d$. Then $F[X_0, \ldots, X_M]_d / I_d$ is a finite dimensional F-vector space and the Hilbert function of V is defined as

$$\mathscr{H}_V(d) = \dim(F[X_0, \ldots, X_M]_d / I_d)$$

for all $d \geq 0$.

Lower bounds for $\mathscr{H}_V(d)$ were obtained by Nesterenko, Chardin, Sombra, and others. We only require a very basic inequality.

Let $r = \dim V \geq 0$. After permuting coordinates, which does not affect the problem, we suppose that $X_0 \notin I$ and that $X_1/X_0, \ldots, X_r/X_0$ are algebraically independent elements when taken as in the function field of V. It follows that the composition

$$F[X_0,\dots,X_r]_d \xrightarrow{\text{inclusion}} F[X_0,\dots,X_M]_d \to F[X_0,\dots,X_M]_d/I_d$$

is injective. Therefore,

$$\mathscr{H}_V(d) \ge \dim F[X_0,\dots,X_r]_d = \binom{r+d}{r}$$

for all $d \ge 0$.

We assume $r \ge 1$ is an integer. By basic properties of the binomial coefficients we have $\binom{r+d}{r} \ge \binom{1+d}{1} = d+1$. So

$$\mathscr{H}_V(d) \ge \binom{\dim V + d}{\dim V} \ge d+1 \tag{C.1}$$

if $\dim V \ge 1$.

We begin with a preliminary lemma that involves cosets in A with fixed stabilizer.

Lemma C.2. *Let M, A, and g be as in the proposition above with $g \ge 2$. Let B be an abelian subvariety of A of positive dimension such that $d \ge \max\{1, g - \dim B\}$. There exists a Zariski open and dense subset $U \subseteq \mathbb{A}^{\binom{M+d}{M}}$, whose complement in $\mathbb{A}^{\binom{M+d}{M}}$ has codimension at least $d + 1 + \dim B - g$, such that if $f \in U(F)$, then $A \cap \mathscr{Z}(f)$ does not contain any translate of B.*

Proof. Say $N = \binom{M+d}{M} \ge M + d \ge g \ge 2$. For the proof we abuse notation and consider elements in $\mathbb{P}^{N-1}(F)$ as classes of homogeneous polynomials in $F[X_0,\dots,X_M] \setminus \{0\}$ of degree d up-to scalar multiplication. So $f(P) = 0$ is a well-defined statement for $f \in \mathbb{P}^{N-1}(F)$ and $P \in \mathbb{P}^M(F)$ and the incidence set

$$\{(f,P) \in \mathbb{P}^{N-1}(F) \times A(F) : f(P) = 0\}$$

determines a Zariski closed subset $Z \subseteq \mathbb{P}^{N-1} \times A$.

We consider the two projections $\pi\colon \mathbb{P}^{N-1} \times A \to \mathbb{P}^{N-1}$ and $\rho\colon \mathbb{P}^{N-1} \times A \to A$.

Then $\rho|_Z\colon Z \to A$ is surjective and each fiber of $\rho|_Z$ is linear variety of dimension $N-2$.

Say B is an abelian subvariety of A with $\dim B \ge 1$. Let $\varphi\colon A \to A/B$ denote the quotient map. We write $\boldsymbol{\varphi} = (\mathrm{id}_{\mathbb{P}^{N-1}}, \varphi)\colon \mathbb{P}^{N-1} \times A \to \mathbb{P}^{N-1} \times (A/B)$; this morphism sends (f,P) to $(f,\varphi(P))$. The fibers of $\boldsymbol{\varphi}$ have dimension $\dim B$ and so

$$\begin{aligned}&\{(f,P) \in Z(F) : \dim_{(f,P)} \boldsymbol{\varphi}|_Z{}^{-1}(\boldsymbol{\varphi}(f,P)) \ge \dim B\}\\ &= \{(f,P) \in Z(F) : P + B \subseteq \mathscr{Z}(f)\}\end{aligned} \tag{C.2}$$

defines a Zariski closed subset Z_φ of Z. If Z_φ is empty then the lemma follows with $U = \mathbb{A}^N \backslash \{0\}$. Otherwise, let $W_1,\dots,W_r$ be the irreducible components of Z_φ.

If $P \in A(F)$, then the fiber of $\rho|_{Z_\varphi}$ above P is empty or has dimension

$$\dim I(P+B)_d - 1 = \dim F[X_0,\dots,X_M]_d - 1 - \mathscr{H}_{P+B}(d) \le N - 1 - (d+1).$$

where we used (C.1). So any non-empty fiber of $\rho|_{W_i}$ has dimension at most $N-1-(d+1)$. and by the Fiber Dimension Theorem we conclude

$$\dim W_i \le N - 1 - (d+1) + \dim \rho(W_i) \le N - d + g - 2 \tag{C.3}$$

for all $i \in \{1,\dots,r\}$ as $\dim \rho(W_i) \le \dim A \le g$.

If $(f,P) \in W_i(F)$, then $\{f\} \times (P+B) \subseteq Z_\varphi$. So if (f,P) is not contained in any W_j with $i \neq j$, then $\{f\} \times (P+B) \subseteq W_i$ by the irreducibility of $P+B$. We conclude that a general fiber of $\pi|_{W_i}$ has dimension at least $\dim B$. By the Fiber Dimension Theorem we find that $\dim W_i \geq \dim B + \dim \pi(W_i)$ for all $i \in \{1,\ldots,r\}$; note that $\pi(W_i)$ is Zariski closed in $\mathbb{P}^{N-1}$.

Together with (C.3) we conclude $\dim \pi(Z_\varphi) = \max_{1\leq i\leq r} \dim \pi(W_i) \leq N-d+g-2-\dim B$ and thus

$$\operatorname{codim}_{\mathbb{P}^{N-1}} \pi(Z_\varphi) \geq d+1-g+\dim B.$$

As $\pi(Z_\varphi)$ is Zariski closed in $\mathbb{P}^{N-1}$ we conclude that $U' = \mathbb{P}^{N-1} \setminus \pi(Z_\varphi)$ is Zariski open and dense in $\mathbb{P}^{N-1}$ if $d \geq g - \dim B$. If $f \in U'(F)$, then there is no $P \in A(F)$ with $P+B \subseteq \mathscr{Z}(f)$. Otherwise, we have in particular $f(P)=0$ and so $(f,P) \in Z(F)$ which entails the contradiction $(f,P) \in Z_\varphi(F)$ by (C.2). The lemma follows if we take U to be the preimage of U' under the cone map $\mathbb{A}^N \setminus \{0\} \to \mathbb{P}^{N-1}$. □

Proof of Proposition C.1. If $g=1$, then $A \cap \mathscr{Z}(f)$ is finite for a generic f that is homogenous and of degree d. The proposition is clearly true in this case.

Now say $g \geq 2$. If $f \in F[X_0,\ldots,X_M]_d$, then a coset contained in $\mathscr{Z}(f) \cap A$ is already contained in some irreducible component X of $\mathscr{Z}(f) \cap A$. By Bézout's Theorem, $\deg X$ is bounded solely in terms of d and A; here $\deg(\cdot)$ denotes the usual degree as a subvariety of $\mathbb{P}^M$.

By a theorem of Bogomolov, [6, Theorem 1], the maximal cosets contained in X are translates of abelian subvarieties whose degree are bounded in terms of $\deg X, A$, and the chosen polarization only. Observe that the proof of Bogomolov's Theorem works for algebraically closed fields in characteristic zero. As A contains only finitely many abelian subvarieties of given degree, Bogomolov produces a finite set of abelian subvarieties that depends only on $\deg X$ and $A \subseteq \mathbb{P}^M$, thus only on d and $A \subseteq \mathbb{P}^M$.

For any abelian subvariety $B \subsetneq A$ of positive dimension that arises in this set we write U_B for the Zariski open and dense set produced by Lemma C.2. To rule out that X contains a coset of positive dimension it suffices to take $f \in U(F)$ where $U = \bigcap_B U_B$ is the intersection over the finite set from Bogomolov's Theorem. □

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