UNIFORMITY IN RATIONAL TORSION AND SMALL POINTS ON ABELIAN VARIETIES

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Abstract. In this paper, we propose a method to study the *Uniform Boundedness Conjecture* and the *Lang–Silverman Conjecture*, which is a uniform lower bound on the heights of non-torsion rational points, for abelian varieties A defined over a global field K. Our method is inspired by Vojta's proof of the Mordell Conjecture (Faltings's Theorem).

Over function fields of characteristic 0, a recent breakthrough of Looper–Yap [LY26] proves both conjectures with inexplicit bounds. In our paper, we give a new proof of both conjectures with explicit bounds, which depend polynomially on the field K unless A/K admits a factor of good reduction everywhere. We also prove explicit bounds for elliptic curves over function fields of characteristic $p > 0$.

Over number fields, we prove both conjectures under a suitable high dimensional Szpiro conjecture (weaker than Hindry's version [Hin07, Conj. 3.4]) that we propose.

Contents

1. Introduction

The goal of this paper is to investigate and prove uniformity results in the number of rational torsion points and in the heights of the non-torsion rational points on abelian varieties. Our first motivation is the following *Uniform Boundedness Conjecture*:

Conjecture 1.1 (Uniform Boundedness Conjecture). *Let A be an abelian variety of dimension $g \geq 1$ defined over a number field K. Then $\#A(K)_{\mathrm{tor}} \leq c(g, K)$.*

Conjecture 1.1, if known for all $g \geq 1$, self-improves to $\#A(K)_{\mathrm{tor}} \leq c(g, [K:\mathbb{Q}])$ by Weil restriction. For $g = 1$, Frey [Fre89] deduced this conjecture from the *abc* Conjecture, while the degree-uniform bound was proved by Mazur [Maz77] for $K = \mathbb{Q}$, Kamienny and Mazur [Kam86, KM95] for K of small degrees, and Merel [Mer96] in general, with effective bounds by Parent [Par99]. For $g \geq 2$, Conjecture 1.1 is known for CM abelian varieties by Silverberg [Sil88] (see also the first-named author's [Gao17, Thm. 1.3]) and for ℓ-primary torsions over 1-parameter families by Cadoret–Tamagawa [CT12] and Ellenberg–Hall–Kowalski [EHK12] (see also Ji–Song–Xie [JSX26]). In general, Conjecture 1.1 remains widely open when $g \geq 2$.

The analogue of Conjecture 1.1 over function fields $k(B)$, *i.e.* the *Geometric Uniform Boundedness Conjecture*, also remained open; here B is a curve over k. In characteristic 0, it was proved for $g = 1$ by Levin [Lev68], Abramovich and Nguyen–Saito [Abr96, NS96] and for A with real multiplication by Bakker–Tsimerman [BT18]. In characteristic $p > 0$, it was proved for $g = 1$ by Cojocaru–Hall [CH05] and Poonen [Poo07].[1]

Remarkably, the Geometric Uniform Boundedness Conjecture is recently proved by Looper–Yap [LY26, Thm. 1.1] in characteristic 0 in its full generality. Here is the statement. Let k be an algebraically closed field. A *traceless* condition needs to be imposed.

Theorem 1.2 (Looper–Yap). *Assume* $\mathrm{char}(k) = 0$. *Let A be an abelian variety of dimension $g \geq 1$ defined over $K := k(B)$, where K is the function field of a smooth projective irreducible curve B defined over k. Assume* $\mathrm{Tr}_{K/k}(A) = 0$. *Then* $\#A(K)_{\mathrm{tor}} \leq c(g, K)$.

Theorem 1.2 also self-improves to $\#A(K)_{\mathrm{tor}} \leq c(g, [K : k(\mathbb{P}^1)])$ by Weil restriction. Looper–Yap's proof follows the strategy of Hindry–Silverman [HS88] on the Lang–Silverman Conjecture [Lan78, Sil84] for elliptic curves, and moreover uses Looper's dynamical basis [Loo24], equidistribution results *à la Fekete*, and ultrafilters. A key ingredient is Deligne's Arakelov Inequality [Del87]. Their bound is inexplicit. They also prove the Lang–Silverman Conjecture in this setting [LY26, Thm. 1.2] with an inexplicit bound, generalizing [HS88, Thm. 0.2] for elliptic curves whose bound is explicit.

1.1. **Brief account of our work.** In this paper, we *propose a different method* (see §1.4 for more details) to study both the Uniform Boundedness Conjecture and the Lang–Silverman Conjecture. Our method is inspired by Vojta's proof of the Mordell Conjecture [Voj91] and – already in the case of elliptic curves – is different from [HS88]. Also we do not use dynamical basis, equidistribution or ultrafilters. We achieve the following goals:

1. Over function fields of characteristic 0, we give a new proof of [LY26, Thm. 1.1 and Thm. 1.2] with *explicit* bounds, which furthermore depends *polynomially on the field* K unless A has a K-factor having good reduction everywhere; see Theorem 1.3 and Theorem 1.5. We also use Deligne's Arakelov Inequality.
2. Over function fields of characteristic $p > 0$, we prove an explicit version of both conjectures when $g = 1$; see Theorem 1.6.
3. Over number fields, in Theorem 1.9 we prove both conjectures for semi-stable A/K assuming a suitable generalization of Szpiro's Conjecture that we propose (Conjecture 1.7). Unconditionally, we prove explicit bounds on $\#A(K)_{\mathrm{tor}}$ and of Lang–Silverman type (Theorem 1.8), depending polynomially on $[K : \mathbb{Q}]$, the Szpiro ratio, and $\frac{h_{\mathrm{Fal}}(A)}{h_{\partial}(A)}$ (see (1.9) for $h_{\partial}(A)$; this last ratio is $\leq 6/\pi$ when $g = 1$).

[1] [Lev68, CH05] depend on the genus of B, [Abr96, NS96, BT18, Poo07] depend on the gonality of B.

1.2. **Over function fields.** We start with our results over function fields. *From now on, k is an algebraically closed field.* Let $K = k(B)$ be the function field of a smooth projective irreducible curve B defined over k. Denote by $\mathrm{gon}(B) = [K : k(\mathbb{P}^1)]$ the *gonality* of B and by $g(B)$ the genus of B.[2] Let $\gamma_g < e^{g^2/2}$ be the Minkowski constant from Lemma 2.1.

Theorem 1.3 (Main Theorem on rational torsion, function field)**.** *Assume* $\mathrm{char}(k) = 0$. *For the explicit constant* $c_0(g) = \left(512 \cdot 3^{12(8g)^2+2}(8g)^4(8g+2)\gamma_{8g}\right)^{8g} > 0$ *we have the following property. Any abelian variety A/K of dimension g, with* $\mathrm{Tr}_{K/k}(A) = 0$, *satisfies*

$$\#A(K)_{\mathrm{tor}} \leq c_0(g)\,(g(B)+1)^{16g} \cdot 2^{2^{2^{32g+2}(g(B)+3)^2}}. \tag{1.1}$$

Moreover, if no K-factor of A has good reduction everywhere, then we have a better bound

$$\#A(K)_{\mathrm{tor}} \leq c_0(g)\,(g(B)+1)^{16g}. \tag{1.2}$$

For $g = 2$ Looper [Loo26] independently proves such an explicit uniform bound in $g(B)$.

For A/K as in Theorem 1.3, the Weil restriction $\mathrm{Res}_{K/k(\mathbb{P}^1)}A$ is still traceless but has dimension $g{\cdot}\mathrm{gon}(B)$, and $A(K) = (\mathrm{Res}_{K/k(\mathbb{P}^1)}A)(k(\mathbb{P}^1))$. Hence (1.1), applied to $\mathrm{Res}_{K/k(\mathbb{P}^1)}A$ and the base curve $\mathbb{P}^1_k$ (genus 0), implies an explicit bound in terms of g and $\mathrm{gon}(B)$:

$$\#A(K)_{\mathrm{tor}} \leq c_0\big(g{\cdot}\mathrm{gon}(B)\big) \cdot 2^{2^{2^{32g\cdot\mathrm{gon}(B)+6}}} < e^{7200g^3\mathrm{gon}(B)^3} \cdot 2^{2^{2^{32g\cdot\mathrm{gon}(B)+6}}}. \tag{1.3}$$

The second term disappears if no K-factor of A has good reduction everywhere.

A standard application of Lefschetz Principle reduces Theorem 1.3 to $k = \mathbb{C}$. The core case of Theorem 1.3 is when *no* K-factor of A has good reduction everywhere; from this, it suffices to prove separately for A/K having good reduction everywhere with a rather short proof §11.1.2. Compared to Looper–Yap [LY26, Thm. 1.1], our bound is explicit and *has polynomial growth in $g(B)$ in the core case.* Moreover in the core case, the essential part is to treat the case where A/K furthermore has semi-stable reduction; this is done in §9, see Theorem 9.1 and below.

Remark 1.4. *When A/K is traceless and has good reduction everywhere, in an ongoing work [Gu26], the second-named author improves the bound to* $\#A(K)_{\mathrm{tor}} \leq \big(768\mathfrak{q}^5(\mathfrak{q}+2)\big)^{\mathfrak{q}}$, *where* $\mathfrak{q} = \frac{(8g+2)!3^{16g}}{2}g(B)$. *The proof uses methods and techniques of the current paper as well as Lefschetz pencils. Together with* (1.2), *this will improve the bound* (1.1) *to*

$$\#A(K)_{\mathrm{tor}} \leq c_0(g)\,(g(B)+1)^{16g}\,\big(768\mathfrak{q}^5(\mathfrak{q}+2)\big)^{\mathfrak{q}},$$

and (1.3) *accordingly (with the second term replaced by* $e^{e^{17g\cdot\mathrm{gon}(B)\log(10g\cdot\mathrm{gon}(B))}}$*).*

We also give a proof of the *Lang–Silverman Conjecture*, which is a uniform lower bound on the heights of non-torsion *rational* points, in this setting. Compared to [LY26, Thm. 1.2], our bound is explicit and the denominator depends polynomially on K.

Theorem 1.5 (Main Theorem on small rational points, function field)**.** *Assume* $\mathrm{char}(k) = 0$. *Set* $c_0'(g) := 2^{-1}\left(1024 \cdot 3^{12(8g)^2+2}(8g)^4(8g+2)\gamma_{8g}\right)^{-(2g+1)} > 0$. *Then for (i) any abelian variety A/K of dimension g such that* $A \neq \mathrm{Tr}_{K/k}(A) \otimes_k K$, *(ii) any symmetric ample line bundle L on A defined over K, (iii) any $P \in A(K)$ with* $\overline{\mathbb{Z}\cdot P}^{\mathrm{Zar}} = A$, *we have*

$$\widehat{h}_L(P) \geq c_0'(g)\frac{\max\{h_{\mathrm{Fal}}(A), 1\}}{\mathrm{gon}(B)\cdot(g(B)+1)^{4g+2}} \tag{1.4}$$

[2]Compared to number fields, $g(B)$ is the analogue of the discriminant of the field.

and, for the explicit constant $c_0(g)$ from Theorem 1.3,

$$\widehat{h}_L(P) \geq \frac{c_0'\big(g\cdot \mathrm{gon}(B)\big)}{\Big(c_0\big(g\cdot \mathrm{gon}(B)^2\big)\cdot 2^{2^{32g\cdot \mathrm{gon}(B)^2+6}}\Big)^2 \mathrm{gon}(B)} \max\{h_{\mathrm{Fal}}(A), 1\}. \tag{1.5}$$

Both bounds are explicit. The denominator of the right hand side of (1.4) depends polynomially on $g(B)$, while the right hand side of (1.5) depends only on g and $\mathrm{gon}(B)$. A standard application of Lefschetz Principle reduces Theorem 1.5 to $k = \mathbb{C}$. Lang–Silverman type bounds are of a rather different – in some sense orthogonal – nature from the *Lehmer-type bounds* of [Sil24, LS24, Loo24] [Mas86, GR25], which concern algebraic points of bounded degree but involve $h_{\mathrm{Fal}}(A)$ in the denominator of the lower bound.

For elliptic curves (*i.e.* $g = 1$), we can also prove uniform results when $\mathrm{char}(k) = p > 0$. A traceless semi-stable elliptic curve E defined over $K = k(B)$ induces a surjective modular map $j\colon B \to \mathbb{P}^1_k$. Denote by p^ν the inseparable degree of j. Let $L = O(0)$.

Theorem 1.6. (i) *$\#E(K)_{\mathrm{tor}} \leq 108 \cdot 144\,(2g(B) - 1)^4$.*
(ii) *For any non-torsion $P \in E(K)$, we have*

$$\widehat{h}_L(P) \geq \frac{\max\{h_{\mathrm{Fal}}(E), 1\}}{2^4 \cdot 3^{11}(p^\nu)^2 \mathrm{gon}(B)\,(2g(B)-1)^6} \max\left\{\frac{1}{(p^\nu)^4}, \frac{1}{2^{10}\cdot 3^5|2g(B)-1|^5}\right\}.$$

1.3. **Over number fields.** Let K be a number field. Let A be an abelian variety of dimension $g \geq 1$ defined over K. For convenience, by a *polarization on A*, we mean a *symmetric ample line bundle L on A defined over K* – this is a genuine abuse of notation.

Mazur, Kamienny, and Merel's proof of Conjecture 1.1 for $g = 1$ relies heavily on the theory of modular curves and their Jacobians. It is hence hard to be generalized to $g \geq 2$. Their bound is double exponential in $[K : \mathbb{Q}]$.

On the other hand, Petsche [Pet06] – using Hindry–Silverman's method [HS88] – proved a weaker bound for elliptic curves E/K, which involves the *Szpiro ratio* but is polynomial in $[K : \mathbb{Q}]$. The Szpiro ratio is conjectured to be bounded above solely in terms of K, according to the *Szpiro Conjecture* which is equivalent to some weak version of the famous *abc Conjecture*. A stronger conjecture is the *Height Conjecture* [Fre89, pp. 39, Conj. (H)] which claims that $h_{\mathrm{Fal}}(E)$ is bounded above solely in terms of K and the conductor of E; it immediately implies Szpiro's Conjecture by the Faltings–Silverman formula.

When $g \geq 2$, Hindry [Hin07, Conj. 3.4] proposed a generalized version of this Height Conjecture (he calls it the generalized Szpiro Conjecture), which claims that $h_{\mathrm{Fal}}(A)$ is bounded above solely in terms of g, K, and the conductor of A. In particular if [Hin07, Conj. 3.4] were true, then the archimedean contribution to $h_{\mathrm{Fal}}(A)$ is always negligible.

In this paper, we propose another generalization of Szpiro's Conjecture to $g \geq 2$ (Conjecture 1.7), which is closer to the original Szpiro Conjecture and does not claim to eliminate the archimedean contribution to $h_{\mathrm{Fal}}(A)$; it is hence weaker than [Hin07, Conj. 3.4]. We then show that Conjecture 1.7 implies Conjecture 1.1 and the Lang–Silverman Conjecture. We also prove some unconditional results, which for $g = 1$ is similar to [Pet06].

1.3.1. *Generalized Szpiro ratio and the boundary height.* Assume A/K has semi-stable reduction. For each v in the set of bad reduction places $\mathfrak{Bad}(A/K)$, write $A \pmod v$ for the reduction of the Néron model of A at v, and r_v for the toric rank of the identity component of $A \pmod v$. Let Φ_v be the group scheme of connected components of $A \pmod v$ defined over the residue field k_v, and write $\#\Phi_v$ for its order, *i.e.* $\#\Phi_v = \#\Phi_v(\overline{k}_v)$.

Let N_v be as in §2.1. Define the *generalized Szpiro ratio* to be

$$\mathfrak{s}_{A/K} := \frac{\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{1/r_v}\log N_v}{\sum_{v\in\mathfrak{Bad}(A/K)} r_v \log N_v} \tag{1.6}$$

if $\mathfrak{Bad}(A/K)\neq\emptyset$ and to be $1/g$ if A/K has good reduction everywhere. Since $r_v\leq g$ for all v, we have $\mathfrak{s}_{A/K}\geq 1/g$. When $g=1$, this is precisely the Szpiro ratio of A/K.

For general g, we also take the following *boundary height* into consideration. It is the sum of its non-archimedean and archimedean parts. The non-archimedean part is

$$h_{\partial,\mathrm{fin}}(A) := \frac{1}{[K:\mathbb{Q}]}\left(\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{1/r_v}\log N_v\right). \tag{1.7}$$

Our definition of the archimedean part is inspired by [Dav93, Paz13]. Let L be a polarization on A with induced isogeny $\phi_L\colon A\to A^\vee$. For any maximal isotropic subgroup H of $\ker(\phi_L)$ with respect to the Weil pairing, L descends to a principal polarization L_0 on A/H; cf. [Mum70, §23, pp.214-216]. At each archimedean place $v\in M_{K,\infty}$, L_0 gives rise to a *Siegel reduced* matrix $\tau_{v,H}$ such that $(A/H)_v(\mathbb{C})\simeq\mathbb{C}^g/(\mathbb{Z}^g+\tau_{v,H}\mathbb{Z}^g)$. Define

$$\alpha_v(A,L) := \max_{\substack{H\subseteq\ker(\phi_L)\\ \text{maximal isotropic}}}\mathrm{Tr}(\mathrm{Im}(\tau_{v,H}))\geq\frac{\sqrt{3}}{2}g; \tag{1.8}$$

then $\alpha_v(A,L)$ depends only on the class of $[L]$ in $\mathrm{NS}(A)$. Finally, set

$$h_\partial(A) := h_{\partial,\mathrm{fin}}(A)+\frac{1}{[K:\mathbb{Q}]}\min_{[L]}\left(\sum_{v\in M_{K,\infty}}\alpha_v(A,L)\log N_v\right)\geq\frac{\sqrt{3}}{2}g \tag{1.9}$$

where L runs over all classes of polarizations on A *of minimal degree.*

Merging [CX08, Conj. 22] and (a weaker version of) [Dav93, Conj. 1.7], we propose the following conjecture. It can be easily deduced from [Hin07, Conj. 3.4]; see Lemma 12.2.

Conjecture 1.7 (Generalized Szpiro Conjecture, weak version)**.** *There exist positive constants $\kappa_1=\kappa_1(g,K)>0$ and $\kappa_2=\kappa_2(g,K)>0$ such that*

$$\mathfrak{s}_{A/K}\leq\kappa_1 \tag{1.10}$$

and

$$h_{\mathrm{Fal}}(A)\leq\kappa_2 h_\partial(A). \tag{1.11}$$

When $g=1$, (1.11) holds true with $\kappa_2=6/\pi$ by the Faltings–Silverman Formula, so that Conjecture 1.7 becomes precisely Szpiro's Conjecture.

1.3.2. *Our main results.* Let K be a number field.

Theorem 1.8 (Main result over number field, unconditional)**.** *For any $g\in\mathbb{Z}_{\geq 1}$, there exist explicit constants $c_1(g), c_1'(g)>0$ (see* (12.6) *and* (12.9)*) such that: For any principally polarized semi-stable K-simple abelian variety $(A,L)/K$ of dimension g, we have*

$$\#A(K)_{\mathrm{tor}}\leq c_1(g)\mathfrak{s}_{A/K}^{2g}[K:\mathbb{Q}]^g\left(\log[K:\mathbb{Q}]+\max\left\{\frac{h_{\mathrm{Fal}}(A)}{h_\partial(A)},1\right\}\right)^g, \tag{1.12}$$

and, for any $P\in A(K)$ with $\overline{\mathbb{Z}\cdot P}^{\mathrm{Zar}}=A$,

$$\widehat{h}_L(P)\geq\frac{1}{c_1'(g)\max\left\{\frac{h_{\mathrm{Fal}}(A)}{h_\partial(A)},1\right\}}\cdot\frac{\max\{h_{\mathrm{Fal}}(A),1\}}{\mathfrak{s}_{A/K}^{2g+2}[K:\mathbb{Q}]^{2g+1}\left(\log[K:\mathbb{Q}]+\max\left\{\frac{h_{\mathrm{Fal}}(A)}{h_\partial(A)},1\right\}\right)^{2g}}. \tag{1.13}$$

When $g = 1$, we have $\frac{h_{\mathrm{Fal}}(A)}{h_\partial(A)} \leq 6/\pi$ by the Faltings–Silverman Formula, so we recover [Pet06] (which follows [Sil81, Sil84, HS88]) with better dependence on the Szpiro ratio but larger constants. For $g \geq 2$, some results towards (1.13) were obtained by David [Dav93], based on which Masser [Mas93] found a family of abelian varieties satisfying the Lang–Silverman Conjecture. See also Pazuki [Paz10, Paz12, Paz13] for some other cases.

Theorem 1.9 (Main result over number field, conditional)**.** *Assume Conjecture 1.7 holds true for K and all $g \geq 1$. Then for any A/K of semi-stable reduction, we have:*

(i) *the Uniform Boundedness Conjecture (Conjecture 1.1) holds true for A;*
(ii) *the Lang–Silverman Conjecture holds true for A, i.e. there exists a constant $\kappa = \kappa(\dim A, K) > 0$ such that for any polarization L on A, we have*

$$\widehat{h}_L(P) \geq \kappa \cdot \max\{h_{\mathrm{Fal}}(A), 1\}, \quad \textit{for any } P \in A(K) \textit{ with } \overline{\mathbb{Z} \cdot P}^{\mathrm{Zar}} = A.$$

By applying Weil restriction, κ can be shown to depend only on g and $[K : \mathbb{Q}]$ if we can remove the semi-stable assumption (see the beginning of §10 for an improvement on this by our method). The followings can be easily read off from the proof: For Conjecture 1.1, we need Conjecture 1.7 for K and all $g \in \{1, \ldots, \dim A\}$, and $\min_{[L]}$ in (1.9) can be relaxed to $\max_{[L]}$; for Lang–Silverman, we need Conjecture 1.7 for K and all $g \in \{1, \ldots, 8 \dim A\}$, which can be further relaxed if L defines a principal polarization; see §12.5.

1.4. **Strategy of our proof.** Overall speaking, our method is a *Diophantine approximation with perturbed metrics.*

Classical Diophantine approximation method of Liouville, Thue, Siegel, and Roth proceeds as follows: one constructs an auxiliary polynomial with small integral coefficients, uses *zero estimates* to find a point at which a suitable derivative does not vanish, bounds its value there, and compares this bound with the Liouville bound. Vojta [Voj91] implemented this method in an Arakelov-geometric setting to give a second proof of the Mordell Conjecture, in which auxiliary polynomials are replaced by *small sections* of Hermitian line bundles on an arithmetic variety. More generally, one can work with *adelic line bundles* defined by S. Zhang [Zha95b] and Yuan–Zhang [YZ26], which allow to treat archimedean places and places of bad reduction in a unified way.

The *main novelty* of our method, compared to Vojta, is that we construct nice bump functions at different places and use them to perturb the metrics of the adelic line bundle. Working with small sections of this perturbed adelic line bundle, we improve the bounds obtained in the evaluation step and get stronger results.

Let us explain in more details. Let L be a symmetric ample line bundle on our abelian variety A defined over K. It has a canonical adelic extension $\overline{L} = (L, \{\|\cdot\|_v\}_{v \in M_K})$, *i.e.* at each place v of K, the analytification $(L \otimes_K K_v)^{\mathrm{an}}$ is endowed with its canonical metric $\|\cdot\|_v$ and the associated height function $h_{\overline{L}}$ coincides with the Néron–Tate height. Also define local invariants $\widetilde{I}_v(A)$ to be $(\#\Phi_v)^{-1/r_v}$ (for v a non-archimedean place at which A has bad reduction) and to be $\alpha_v(A, L)$ from (1.8) (for v archimedean places).

For simplicity assume A is simple and let us focus on rational torsion points $A(K)_{\mathrm{tor}}$.

Assume A/K has semi-stable reduction. At each $v \in M_K$, we construct a bump function

$$f_v \colon (A \otimes_K K_v)^{\mathrm{an}} \longrightarrow \mathbb{R}_{\geq 0},$$

with $f_v \equiv 0$ at places of good reduction, and perturb the canonical metric to $\|\cdot\|'_v := N_v^{-\epsilon f_v} \|\cdot\|_v$; here $\epsilon > 0$ and N_v is as in §2.1. The required properties of f_v will be explained later. These metrics together yield a new adelic line bundle $\overline{L}(\epsilon \mathbf{f}) := (L, \{\|\cdot\|'_v\}_v)$.

It is with this $\overline{L}(\epsilon\mathbf{f})$ on A that we run the Diophantine approximation process.

The first step is to find a small section s of $\overline{L}(\epsilon\mathbf{f})^{\otimes m}$ for all $m \gg 1$. For this, we prove the arithmetic bigness of $\overline{L}(\epsilon\mathbf{f})$ using arithmetic Hilbert–Samuel (see [YZ26, Thm. 5.2.2] for the most general version), for which we need $\overline{L}(\epsilon\mathbf{f})$ to be nef. Nefness can be checked place by place, and this is where enters *a first required property for f_v*. At archimedean places v, f_v is smooth and we need the curvature form $c_1(L, \|\cdot\|_v') = c_1(L, \|\cdot\|_v) + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\epsilon f_v)$ to be a positive $(1,1)$-form. At a non-archimedean place v, we want $\|\cdot\|_v'$ to be *relatively semipositive* (*i.e.* can be uniformly approximated by nef model metrics), which is already true when A has good reduction at v. When A has semi-stable bad reduction at v, *in our particular case* this can be checked using superforms on non-archimedean Berkovich spaces. More precisely, the tropicalization maps $(A \otimes_K K_v)^{\mathrm{an}}$ to its *skeleton* Σ_v which is a real torus, and f_v is defined to be the pullback of some smooth function on Σ_v. Lagerberg [Lag12] developed a theory of superforms on Σ_v, which Chambert-Loir–Ducros [CLD12] generalized to non-archimedean Berkovich spaces. For our purpose we use the variant of Gubler–Künnemann [GK17] and obtain a $(1,1)$-superform $c_1(L, \|\cdot\|_v')$. *In our specific case*, by the semipositivity criterion [GS23, Thm. 4.10] it suffices to show that $c_1(L, \|\cdot\|_v')$ is a positive $(1,1)$-superform, for which a simple computation is enough.

Next, we apply Nakamaye's zero estimate [Nak07] to find an $x \in A(K)_{\mathrm{tor}}$, such that the vanishing order ℓ of s at x is bounded in terms of g and $\#A(K)_{\mathrm{tor}}$. If A is not simple, the bound is more complicated and we will do induction after our Diophantine approximation.

Then, we evaluate the non-zero element $\mathrm{jet}^{\ell} s(x) \in J := \mathrm{Sym}^{\ell}((\mathrm{Lie}\, A)^{\vee}) \otimes L_x^{\otimes m}$. Our computation is inspired by Gaudron–Rémond [GR25, §6] (while much simpler, *e.g.* no interpolation is needed). In this step, *we require two other properties for f_v*: it vanishes on a suitable region, and its integral with respect to the standard normalized measure is bounded below linearly by our local invariant $\widetilde{I}_v(A)$. Now the canonical adelic extension $\overline{L}$ of L and the canonical adelic structure on $\mathrm{Lie}\, A$ make J into an adelic vector space, whose metrics are denoted by $\{\|\cdot\|_{J,v}\}_{v \in M_K}$. The properties of our f_v allow to bound $-\log \|\mathrm{jet}^{\ell} s(x)\|_{J,v}$ from below linearly by $\widetilde{I}_v(A) \log N_v$ for each v (with a slight modification at archimedean places). Compare it with the Liouville inequality for adelic vector spaces and Gaudron's bound on the maximal slope of the adelic cotangent space [Gau19], we get an upper bound of $\#A(K)_{\mathrm{tor}}$ (roughly) in terms of $\frac{h_{\mathrm{Fal}}(A)}{\sum_v \widetilde{I}_v(A) \log N_v}$.

Here are some further remarks on this evaluation step. For v non-archimedean, f_v is required to vanish on the classical points $A(K_v)$ which lie in the inverse image of a lattice in Σ_v under the tropicalization, and we have an easy comparison $\|\mathrm{jet}^{\ell} s(x)\|_{J,v} \leq \|s(x)\|_v$ of Gaudron–Rémond [GR14b, §6.7.1]. Then desired bound on $-\log \|\mathrm{jet}^{\ell} s(x)\|_{J,v}$ follows since s is a small section for the perturbed metric and by our condition on the integral of f_v. For v archimedean, we divide $(A \otimes_K K_v)^{\mathrm{an}}$ into several strips and require f_v to vanish on some of them, and the evaluation of $\|\mathrm{jet}^{\ell} s(x)\|_{J,v}$ uses computation with theta series in [Gau19] and Cauchy's estimate in complex analysis.

This finishes our *Diophantine approximation with perturbed metrics*. To conclude for uniformity, we then use the boundary height and invoke Deligne's Arakelov Inequality (in the function field case) and Conjecture 1.7 (in the number field case) to bound $\frac{h_{\mathrm{Fal}}(A)}{\sum_v \widetilde{I}_v(A) \log N_v}$ uniformly from above. When A/K is not simple, an induction is also needed.

Removing the semi-stable assumption requires adding local contributions to the denominator $\sum_v \widetilde{I}_v(A) \log N_v$ at each bad but potentially good place v. We do not construct bump functions at such v. Instead, we exploit the fact that K-points lie in a *proper* infinitesimal neighborhood after passing to the semi-stable extension K' of K. More precisely for a place v' of K' lying over v and $\mathcal{N}$ (resp. $\mathcal{A}'$) the Néron model of $A \otimes_K K_v$ (resp. of $A \otimes_K K'_{v'}$), the section associated to every K-point lies in a proper subscheme of $\mathcal{A}'$, namely the image of the natural map $\mathcal{N} \otimes_{\mathcal{O}_{K_v}} \mathcal{O}_{K'_{v'}} \to \mathcal{A}'$. Consider the special fiber N_0 of this subscheme and its infinitesimal neighborhoods N_n, and modify the metric of $\overline{L}(\epsilon \mathbf{f})$ accordingly at this v. We then obtain an estimate of $\|\mathrm{jet}^\ell s(x)\|_{J,v}$ at this v for most sections $s \in H^0(A, L^{\otimes m})$. In this process, we need to compare $h^0(N_n, \mathcal{L}^{\otimes m})$ and $h^0(A, L^{\otimes m})$, where $\mathcal{L}$ is the extension of L to $\mathcal{A}'$, via their comparisons with $h^0(N_0, \mathcal{L}^{\otimes m})$. A subtlety here is that tame ramification is required at v' because we need Edixhoven's [Edi92] to compare $h^0(A, L^{\otimes m})$ and $h^0(N_0, \mathcal{L}^{\otimes m})$, so we restrict §10 to the function field case $K = \mathbb{C}(B)$ for simplicity.

Finally, the sketch above also allows to count the number of rational points of small heights. Then we can prove the Lang–Silverman Conjecture by considering the multiples of the non-torsion point $P \in A(K)$ in question when L defines a principle polarization on A, and by using Zarhin's trick and *Zarhin's box* for general L.

In the end, we point out that our method differs from [HS88], which starts with a local-global decomposition of the Néron–Tate height $\widehat{h}_L$ and uses Elkies's lower bound crucially at archimedean places. Our proof does not use either one of these ingredients.

1.5. **Organization of the paper.** In §2, we collect some notation and basic background knowledge throughout the whole paper. In particular, we define the Faltings height in §2.3 and include a summary on adelic geometry over functions field in §2.5.

The constructions of our bump functions are carried in §3-5: at archimedean places in §3, and at non-archimedean places in §5. The construction at non-archimedean places requires knowledge on the tropicalization of an abelian variety over a non-archimedean local field and the associated monodromy pairing, for which we recall in §4.

The core of our proof is in §6-8, and we organize as follows. In §6, we state a first quantitative bound for the number of rational torsion points as well as its analogue for rational points of small height, in terms of $h_{\mathrm{Fal}}(A)$ and the local invariants $\widetilde{I}_v(A)$ (we separate the statements for function fields and for number fields, because the latter requires extra subtlety). These bounds are then proved in §7, following the *Diophantine approximation with perturbed metrics* strategy discussed in §1.4. In §8, we start relating $\widetilde{I}_v(A)$ to global arithmetic invariants, by comparing the boundary height and $h_{\mathrm{Fal}}(A)$.

Our main results over function fields are then proved in §9-11, where Deligne's Arakelov Inequality is used. In §9, we prove the main results in the essential case, *i.e.* A/K has semi-stable reduction. In §10, we explain how to remove the semi-stable assumption. In §11 we finish the proofs of Theorem 1.3 and Theorem 1.5.

In §12 we turn to number fields. We first discuss our Generalized Szpiro Conjecture. Then we prove our main results Theorem 1.8 and Theorem 1.9, using the bounds in §6.

Finally, §13 specializes the arguments to elliptic curves and proves Theorem 1.6. We also include an Appendix A in which we prove a bound on the maximal slope of the adelic cotangent space of A/K in the function field case, and an Appendix B to prove a line bundle version of Zarhin's trick.

Readers who are interested only in the function field case can skip all discussions at archimedean places (§3, §6.2-6.3, §7.3.2, §7.6; §12). The proof becomes shorter.

Acknowledgements. We would like to thank Jit Wu Yap and Xinyi Yuan for numerous discussions. We would like to thank Éric Gaudron and Gaël Rémond for explanation on their paper [GR25] and Junyi Xie for discussions on Berkovich spaces. We would like to thank Vesselin Dimitrov, Chenxin Huang, Marc Hindry, Klaus Künnemann, Nicole Looper, Wenbin Luo, Hector Pasten, Yinchong Song for relevant discussions. We would like to thank Pascal Autissier and Joseph Silverman for comments on an earlier draft.

2. Some notations and preliminaries

2.1. **Some notation regarding K.** We will unify some notations in the number field and the function field cases. Denote by

$$M_K = M_K^0 \sqcup M_{K,\infty}$$

the set of places of K, with M_K^0 the subset of non-archimedean places and $M_{K,\infty}$ the subset of archimedean places. When $K = \mathbb{C}(B)$ is the function field of an irreducible smooth projective curve B, M_K^0 is in bijection with points in $B(\mathbb{C})$ and $M_{K,\infty} = \emptyset$.

When K is a number field, for each place $v \in M_K$ define

$$N_v := \begin{cases} \#(\mathcal{O}_K/\mathfrak{p}) & \text{if } v \text{ corresponds to the maximal ideal } \mathfrak{p} \text{ of } \mathcal{O}_K, \\ e & \text{if } v \text{ is a real place,} \\ e^2 & \text{if } v \text{ is a complex place.} \end{cases}$$

The *residue field* k_v for $v \in M_K^0$ is defined to be $\mathcal{O}_K/\mathfrak{p}$, where v corresponds to $\mathfrak{p}$.

When $K = \mathbb{C}(B)$ is a function field, define

$$N_v := e \qquad \text{for each } v \in M_K.$$

The *residue field* k_v for each $v \in M_K$ is $\mathbb{C}$.

2.2. **Basic notation regarding abelian varieties.** Throughout the whole paper, A/K is an abelian variety of dimension g and L is a symmetric ample line bundle on A defined over K.

2.2.1. *Over number fields.* Assume K is a number field. Let $\mathcal{A}$ be the identity component of the Néron model of A. Write $f\colon \mathcal{A} \to \mathrm{Spec}\mathcal{O}_K$ for the structural morphism and let $\underline{0}$ be the zero section.

For each $v \in M_K^0$, denote by $\mathcal{A}_v$ the pullback of f along the natural morphism $\mathrm{Spec}(k_v) \to \mathrm{Spec}\mathcal{O}_K$. Then $\mathcal{A}_v$ is an abelian variety of dimension g for all but at most finitely many $v \in M_K^0$.

The (finite) set of *bad reduction places* of A/K is defined to be

$$\mathfrak{Bad}(A/K) := \{v \in M_K^0 : \mathcal{A}_v \text{ is not an abelian variety of dimension } g\}.$$

We say that A has *semi-stable reduction at v* if $\mathcal{A}_v$ is a semi-abelian variety. In this case, denote the toric rank by $r_v \geq 0$.

Assume A/K has semi-stable reduction, *i.e.* A has semi-stable reduction at all $v \in M_K^0$. Then $v \in \mathfrak{Bad}(A/K)$ if and only if $r_v \geq 1$. The *conductor* of A/K is defined to be

$$\mathfrak{f}_{A/K} := \prod_{v \in M_K^0} N_v^{r_v} = \prod_{v \in \mathfrak{Bad}(A/K)} N_v^{r_v}. \tag{2.1}$$

This is compatible with [Hin07, (3.17)].

2.2.2. *Over function fields.* Assume $K = \mathbb{C}(B)$ is the function field of an irreducible smooth projective curve B. Use the bijection M_K and $B(\mathbb{C})$.

Let $\mathcal{A}$ be identity component of the Néron model of A. Write $f\colon \mathcal{A} \to B$ for the structural morphism and let $\underline{0}$ be the zero section. Then $\mathcal{A}_v = f^{-1}(v)$ is an abelian variety of dimension g for all but at most finitely many $v \in B(\mathbb{C})$.

The (finite) set of *bad reduction places* of A/K is defined to be

$$\mathfrak{Bad}(A/K) := \{v \in M_K : \mathcal{A}_v \text{ is not an abelian variety of dimension } g\}.$$

We say that A has *semi-stable reduction at v* if $\mathcal{A}_v$ is a semi-abelian variety. In this case, denote the toric rank by $r_v \geq 0$. We say that A/K has semi-stable reduction if $\mathcal{A}/B$ is an abelian scheme, or equivalently A has semi-stable reduction at all $v \in M_K$.

2.3. **Faltings height.** We now recall the definition of the *stable Faltings height* $h_{\mathrm{Fal}}(A)$.

2.3.1. *Over number fields.* Assume K is a number field. Use the notations from §2.2.1.

We start with the case where A/K has semi-stable reduction. Consider the *(determinant) Hodge bundle*

$$\omega_{\mathcal{A}} := \underline{0}^*\Omega^{\wedge g}_{\mathcal{A}/\mathcal{O}_K} = f_*\Omega^{\wedge g}_{\mathcal{A}/\mathcal{O}_K}$$

which is a line bundle on $\mathrm{Spec}\mathcal{O}_K$. For each $\sigma\colon K \hookrightarrow \mathbb{C}$, endow the metric $\|\cdot\|_\sigma$ on $\omega_{\mathcal{A}} \otimes_\sigma \mathbb{C}$ by

$$\|\alpha\|_\sigma^2 = \frac{1}{(2\pi)^g}\left|\int_{A_\sigma(\mathbb{C})} \alpha \wedge \overline{\alpha}\right| \qquad \text{for any } \alpha \in \omega_{\mathcal{A}} \otimes_\sigma \mathbb{C}.$$

Then we get a Hermitian line bundle $\overline{\omega}_{\mathcal{A}} := (\omega_{\mathcal{A}}, \{\|\cdot\|_\sigma\}_\sigma)$. The *(stable) Faltings height* of A is

$$h_{\mathrm{Fal}}(A) := \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}(\overline{\omega}_{\mathcal{A}}), \tag{2.2}$$

where $\widehat{\deg}(\overline{\omega}_{\mathcal{A}})$ on the right hand side is the Arakelov degree, *i.e.* equals $\log \#(\omega_{\mathcal{A}}/s\mathcal{O}_K) - \sum_{\sigma\colon K\hookrightarrow\mathbb{C}} \log \|s\|_\sigma$ for any non-zero $s \in H^0(\mathcal{A}, \omega_{\mathcal{A}})$.

For general A/K, there exists a finite extension K' of K such that $A_{K'}$ has semi-stable reduction over K'. Then we define

$$h_{\mathrm{Fal}}(A) := h_{\mathrm{Fal}}(A_{K'}).$$

It is known that this definition does not depend on the choice of K'.

2.3.2. *Over function fields.* Assume $K = \mathbb{C}(B)$ is the function field of an irreducible smooth projective curve B. Use the notations from §2.2.2.

Again we start with the case where A/K has semi-stable reduction. Consider the *(determinant) Hodge bundle*

$$\omega_{\mathcal{A}} := \underline{0}^*\Omega^{\wedge g}_{\mathcal{A}/B} = f_*\Omega^{\wedge g}_{\mathcal{A}/B}$$

which is a line bundle on B. Define the *(stable) Faltings height* of A to be

$$h_{\mathrm{Fal}}(A) := \frac{1}{[K:\mathbb{C}(\mathbb{P}^1)]}\deg(\omega_{\mathcal{A}}). \tag{2.3}$$

For general A/K, there exists a finite extension K' of K such that $A_{K'}$ has semi-stable reduction over K'. Then we define

$$h_{\mathrm{Fal}}(A) := h_{\mathrm{Fal}}(A_{K'}).$$

This definition does not depend on the choice of K'.

2.4. **Minkowski reduction.** Consider the lattice $\mathbb{Z}^g$ in the real vector space $\mathbb{R}^g$, and denote by $e_1, \ldots, e_g$ the basis vectors.

A positive-definite symmetric matrix B is called *Minkowski-reduced* if, for the quadratic form $q(x) := x^{\mathrm{t}}Bx$ on $\mathbb{R}^g$, we have $q(e_j) \leq q(w)$ for all $j \in \{1, \ldots, g\}$ and $w = (w_1, \ldots, w_g) \in \mathbb{Z}^g$ with $\gcd(w_1, \ldots, w_g) = 1$. Here are some basic properties of Minkowski-reduced matrices.

Lemma 2.1. *There exists a number $\gamma_g \in [1, e^{g^2/2})$, depending only on g and called the Minkowski constant, such that any Minkowski reduced matrix $B = (b_{ij})_{1 \leq i,j \leq g}$ satisfies:*

(i) $0 \leq b_{11} \leq \ldots \leq b_{gg}$,
(ii) $|b_{ij}| \leq \frac{1}{2}|b_{ii}|$ *for all* i, j,
(iii) $\det B \leq \prod_{j=1}^g b_{jj} \leq \gamma_g \det B$,
(iv) *the matrix* $B - \frac{1}{e\gamma_g}\operatorname{diag}(b_{11}, \ldots, b_{gg})$ *is positive semi-definite.*

Proof. Parts (i)-(iii) are standard results from reduction theory except the (crude) bound $\gamma_g < e^{g^2/2}$, which itself is an easy computation by a result of van der Waerden (cf. [NRR23, Thm. 5.16]) and Minkowski's Second Theorem. Notice that we can take $\gamma_1 = 1$.

Let us prove (iv). Set for simplicity $D := \operatorname{diag}(b_{11}, \ldots, b_{gg})$. Then $D^{-1/2}BD^{-1/2}$ is a positive-definite symmetric matrix. So it suffices to prove that the minimal eigenvalue λ_1 of $D^{-1/2}BD^{-1/2}$ is $> 1/(e\gamma_g)$.

When $g = 1$ then it is clearly 1 and we can take $\gamma_g = 1$. Assume $g \geq 2$. Use $\lambda_2, \ldots, \lambda_g$ to denote the other eigenvalues of $D^{-1/2}BD^{-1/2}$. Notice that the diagonal entries of $D^{-1/2}BD^{-1/2}$ are 1. So $\lambda_1 + \sum_{j=2}^g \lambda_j = \operatorname{Tr}(D^{-1/2}BD^{-1/2}) = g$. Hence

$$\det(D^{-1/2}BD^{-1/2}) = \lambda_1 \prod_{j=2}^g \lambda_j \leq \lambda_1 \left(\frac{\sum_{j=2}^g \lambda_j}{g-1}\right)^{g-1} = \lambda_1 \left(\frac{g-\lambda_1}{g-1}\right)^{g-1} = \lambda_1 \left(1 + \frac{1-\lambda_1}{g-1}\right)^{g-1}.$$

But the left hand side equals $(\det B)/\prod_{j=1}^g b_{jj}$, which is $\geq 1/\gamma_g$ by part (iii). The right hand side $< \lambda_1 e^{1-\lambda_1}$ because$(1+\frac{t}{n})^n$ is increasing in n and has limit e^t when $n \to \infty$. So $\lambda_1 e^{1-\lambda_1} \geq 1/\gamma_g$. But $e^{1-\lambda_1} < e$ since $\lambda_1 > 0$. Hence we have $e\lambda_1 > 1/\gamma_g$. We are done. $\square$

2.5. **Adelic geometry over function fields.** In this subsection, we recall the theory of adelic vector spaces and adelic line bundles over function fields $K = \mathbb{C}(B)$. The discussion applies to arbitrary $k = \overline{k}$. For $v \in B$, let K_v be the local field and $\mathcal{O}_v$ its (discrete) valuation ring. Take $|\varpi_v|_v = e^{-1}$ for compatibility with N_v in §2.1, where ϖ_v is a uniformizer of K_v.

2.5.1. *Adelical(ly metrized) vector spaces.* Let $\mathcal{F}$ be a vector bundle on B, and set $F := \mathcal{F}_K$ to be the generic fiber of $\mathcal{F}$. Then F is a finite dimensional K-vector space. Define the following *model metric* $\|\cdot\|_{v,\mathrm{mod}}$ on $F_v := F \otimes_K K_v$ to be: for any $f \in F_v$, set

$$\|f\|_{v,\mathrm{mod}} := \inf_{\xi \in K_v} \{|\xi|_v : f \in \xi \cdot (\mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{O}_v)\} \in e^{\mathbb{Z}}.$$

The unit ball $\{f \in F_v : \|f\|_{v,\mathrm{mod}} \leq 1\}$ is an $\mathcal{O}_v$-lattice, and the model metric is the lattice norm induced by this unit ball. Hence any $\mathcal{O}_v$-basis $\{f_1, \ldots, f_r\}$ of the unit ball is an orthonormal basis of F_v, *i.e.* $\|\sum_{i=1}^r x_i f_i\|_{v,\mathrm{mod}} = \max_{i=1}^r |x_i|_v$ for any $x_1, \ldots, x_r \in K_v$.

Definition 2.2. *Let E be a finite dimensional K-vector space. An adelic metrization is a collection $\{\|\cdot\|_v : E_v \to \mathbb{R}_{\geq 0}\}_{v \in M_K}$ satisfying:*

- *At each place v, the metric is multiplicative for scalar ($|a|_v\|f\|_v = \|af\|_v$ for any $a \in K_v$ and $f \in E_v$) and satisfies the ultrametric triangular inequality ($\|f_1+f_2\|_v \leq \max\{\|f_1\|_v, \|f_2\|_v\}$ for any $f_1, f_2 \in E_v$).*
- *(Coherence) There exists a vector bundle $\mathcal{E}$ on a Zariski open dense subset $U \subseteq B$ such that $\|\cdot\|_v$ is induced by the model metric defined by $\mathcal{E}$ at each $v \in U$.*

Such a pair $\overline{E} := (E, \{\|\cdot\|\}_{v\in M_K})$ is called an adelic vector space. If the adelic metric on $\overline{E}$ coincides with a model metric, we say that $\overline{E}$ is a model adelic vector space.

Let $\overline{E} := (E, \{\|\cdot\|\}_{v\in M_K})$ be an adelic space over K. The set of effective sections is $\widehat{H}^0(\overline{E}) := \{f \in E : \|f\|_v \leq 1, \forall v\}$. If $\overline{E}$ is a model adelic vector space, *i.e.* in the coherence condition we can take $U = B$, then the set of effective sections $\widehat{H}^0(\overline{E})$ coincides with the usual global sections $H^0(B, \mathcal{E})$ of vector bundle.

Denote by $r := \dim E$. The *determinant* $\det \overline{E} := \bigwedge^r \overline{E}$ is defined as follows. For each $v \in M_K$ and the ultrametric $\|\cdot\|_v$, there exist a K_v-basis $\{e_{v,1}, \ldots, e_{v,r}\}$ of E_v and real numbers $a_{v,1}, \ldots, a_{v,r} \in \mathbb{R}$ such that $\|\sum_{i=1}^r x_i e_{v,i}\|_v = \max_{1\leq i\leq r} \{|x_i|_v e^{-a_{v,i}}\}$.[3] Then the metric $\|\cdot\|_{\det \overline{E},v}$ is defined by

$$\|e_1 \wedge \cdots \wedge e_r\|_{\det \overline{E},v} = \prod_{i=1}^r \|e_i\|_v = \prod_{i=1}^r e^{-a_{v,i}} = e^{-\sum_{i=1}^r a_{v,i}}. \tag{2.4}$$

The arithmetic degree of $\overline{E}$ is defined to be

$$\widehat{\deg}(\overline{E}) := -\sum_{v\in M_K} \log \|s\|_{\det \overline{E},v}, \quad \text{for any non-zero } s \in \det E. \tag{2.5}$$

The definition does not depend on the choice of s by the Product Formula.

The following *Minkowski Theorem* over function fields is surely known to experts. We include a proof here since we did not find an explicit reference.

Theorem 2.3. *Let $\overline{E} = (E, \{\|\cdot\|_v\}_{v\in M_K})$ be an adelic vector space over K of dimension r. Assume there exists a vector bundle $\mathcal{E}$ on a Zariski open dense subset $U \subseteq B$ such that $\|\cdot\|_v$ is induced by the model metric defined by $\mathcal{E}$ at each $v \in U$. Then*

$$\widehat{\deg}(\overline{E}) > r(g(B) - 1 + \#(B \setminus U)) \Longrightarrow \widehat{H}^0(\overline{E}) \neq 0.$$

Proof. Let $v \in M_K$. Let the K_v-basis $\{e_{v,1}, \ldots, e_{v,r}\}$ of E_v and the real numbers $a_{v,1}, \ldots, a_{v,r} \in \mathbb{R}$ be as above (2.4). We may and do take $a_{v,1} = \cdots = a_{v,r} = 0$ for all $v \in U$; see above Definition 2.2.

Let us modify the metric on E_v at each $v \in B \setminus U$ as follows. For each $v \in B$, set

$$b_{v,i} := \lfloor a_{v,i} \rfloor$$

(notice that $b_{v,i} = a_{v,i} = 0$ when $v \in U$) and define the modified metric by

$$\|\sum_{i=1}^r x_i e_{v,i}\|_v^- := \max_{1\leq i\leq r} \left\{|x_i|_v e^{-b_{v,i}}\right\}.$$

Notice that $\|\cdot\|_v^-$ is precisely the lattice norm induced by the $\mathcal{O}_v$-lattice

$$\Lambda_v^- := \bigoplus_{i=1}^r \mathcal{O}_v \cdot \varpi_v^{-b_{v,i}} e_{v,i}.$$

[3]Consider the lattice filtration $\Lambda_{v,t} := \{x \in E_v : \|x\|_v \leq e^{-t}\}$ with $t \in \mathbb{R}$. It satisfies $\Lambda_{v,t+1} = \varpi_v \Lambda_{v,t}$. So we are done after choosing a basis adapted to the finitely many jumps of this filtration modulo $\mathbb{Z}$.

We claim that Λ_v^- is the unit ball of the original norm $\|\cdot\|_v$. Indeed, we have

$$\begin{aligned}\|\textstyle\sum_i x_i e_{v,i}\|_v \leq 1 &\Longleftrightarrow \mathrm{ord}_v(x_i) + a_{v,i} \geq 0 \text{ for all } i \\ &\Longleftrightarrow \mathrm{ord}_v(x_i) + b_{v,i} \geq 0 \text{ for all } i, \quad \text{since } \mathrm{ord}_v(x_i) \in \mathbb{Z} \text{ and } b_{v,i} = \lfloor a_{v,i} \rfloor \\ &\Longleftrightarrow \textstyle\sum_i x_i e_{v,i} \in \Lambda_v^-,\end{aligned}$$

and hence $\{x \in E_v : \|x\|_v \leq 1\} = \Lambda_v^-$.

The Beauville–Laszlo Theorem [BL95] glues $\mathcal{E}$ and the lattices $\{\Lambda_v^-\}_{v \notin U}$ into a vector bundle $\mathcal{E}^-$ on B with generic fiber E. Indeed for each $v \notin U$, the $\mathcal{O}_v$-lattice $\Lambda_v^- \subseteq E_v$ defines a vector bundle on $\mathrm{Spec}\mathcal{O}_v$, whose restriction to $\mathrm{Spec}K_v$ is $E_v = \Lambda_v^- \otimes_{\mathcal{O}_v} K_v$ and coincides with the pullback of $\mathcal{E}$ along $\mathrm{Spec}K_v \to U$. Then the Beauville–Laszlo Theorem glues $\mathcal{E}$ and Λ_v^-. This glueing process terminates within finitely many steps since $\#(B \setminus U) < \infty$. Moreover

$$\begin{aligned}H^0(B, \mathcal{E}^-) &= H^0(U, \mathcal{E}) \times_{\prod_{v \notin U} E_v} \prod_{v \notin U} \Lambda_v^- \\ &= \{s \in E : \|s\|_v \leq 1 \text{ for all } v \in M_K\} = \widehat{H}^0(\overline{E}).\end{aligned}$$

We have, by (2.4),

$$\|e_{v,1} \wedge \cdots \wedge e_{v,r}\|_{\det \overline{E}, v} = e^{-\sum_i a_{v,i}} \quad \text{and} \quad \|e_{v,1} \wedge \cdots \wedge e_{v,r}\|^-_{\det \overline{E}^-, v} = e^{-\sum_i b_{v,i}}.$$

It follows that

$$\widehat{\deg}(\overline{E}) - \deg(\mathcal{E}^-) = \sum_{v \notin U} \sum_{i=1}^r (a_{v,i} - b_{v,i}),$$

which is then $\in [0, r\#(B \setminus U))$ because $0 \leq a_{v,i} - b_{v,i} < 1$. Therefore

$$\widehat{\deg}(\overline{E}) > r(g(B) - 1 + \#(B \setminus U)) \Longrightarrow \deg(\mathcal{E}^-) > r\,(g(B) - 1)\,,$$

which furthermore implies $H^0(B, \mathcal{E}^-) \neq 0$ by Riemann–Roch on B. Hence we are done because $\widehat{H}^0(B, \overline{E}) = H^0(B, \mathcal{E}^-)$. □

2.5.2. *Adelic line bundles.* Let X/K be a projective variety and let L be a line bundle on X defined over K. For each $v \in M_K$, a projective model $(\mathcal{X}_v, \mathcal{L}_v)$ of $(X_{K_v}, L_{K_v}^{\otimes n})$ over $\mathcal{O}_v$ induces a *model metric* $\|\cdot\|_{\mathcal{L}_v}$ on $L_v := L \otimes_K K_v$ as follows: Any $x \in X(K_v)$ extends to

$$\overline{x}\colon \mathrm{Spec}(\mathcal{O}_v) \longrightarrow \mathcal{X}.$$

For $f \in x^*L$, define

$$\|f\|_{\mathcal{L}_v} := \inf_{\xi \in K_v} \{|\xi|_v^{1/n} : f^n \in \xi \overline{x}^* \mathcal{L}_v\}.$$

A metric $\|\cdot\|_v$ on L_v is continuous and bounded if $\log \frac{\|\cdot\|_v}{\|\cdot\|_{\mathcal{L}_v}}$ is continuous and bounded for some model metric $\|\cdot\|_{\mathcal{L}_v}$. An *adelic metric* on L is a collection $\{\|\cdot\|_v\}_{v \in M_K}$ of continuous and bounded metrics $\|\cdot\|_v$ on L_v for all $v \in B$ satisfying the following property: there exists a projective model $(\mathcal{X}, \mathcal{L})$ of $(X|_U, L|_U^{\otimes n})$ for some $n \geq 1$ and a Zariski open subset $U \subseteq B$ such that $\|\cdot\|_v$ is the model $\|\cdot\|_{\mathcal{L}_v}$ for all $v \in U$ – such a model is said to be *relatively semipositive* if $\mathcal{L}$ is nef on special fibers of $\mathcal{X}$.

An *adelic line bundle* is a pair $\overline{L} = (L, \{\|\cdot\|_v\}_{v \in M_K})$ consisting of a line bundle L and an adelic metric $\|\cdot\| := \{\|\cdot\|_v\}_{v \in M_K}$ on L.

To define *nefness* of adelic line bundles, we need to following notion. A sequence $\{\|\cdot\|_n : n = 1, 2, \ldots\}$ of adelic metrics *converges to* an adelic metric $\|\cdot\|$ if there exists a Zariski open subset $U \subseteq B$ such that $\|\cdot\|_{n,v} = \|\cdot\|_v$ for all n, and that $\log \frac{\|\cdot\|_{n,v}}{\|\cdot\|_v}$ converges to 0 uniformly on $X(K)$. Finally, an adelic line bundle $\overline{L} = (L, \{\|\cdot\|_v\}_{v\in M_K})$ is called *nef* if the underlying adelic metric is the limit of a sequence of adelic metrics induced by relatively semipositive models.

Let $\overline{L}$ be an adelic line bundle on X. Define the *height function* associated with $\overline{L}$ as

$$h_{\overline{L}}\colon X(\overline{K}) \to \mathbb{R}, \qquad x \mapsto -\frac{1}{[K(x):\mathbb{C}(\mathbb{P}^1)]}\sum_{v\in M_K}\sum_{z\in \mathrm{Gal}(\overline{K}/K)x\times_K K_v} \log\|s(z)\|_v^{\deg_{K_v}(z)}.$$

For each $s \in H^0(X,L)$ and $v \in M_K$, define $\|s\|_{v,\sup} := \sup_{x\in X(K_v)}\|s(x)\|_v$. Define

$$\widehat{H}^0(X,\overline{L}) := \{s \in H^0(X,L) : \|s\|_{v,\sup} \leq 1\}.$$

It is known that $\widehat{H}^0(X,\overline{L})$ is a $\mathbb{C}$-vector space. If $\overline{L}$ is induced from $\mathcal{L}$ on some projective model $\mathcal{X}$ of X over B, then $\widehat{H}^0(X,\overline{L}) = H^0(\mathcal{X},\mathcal{L})$.

Finally, the *volume* of $\overline{L}$ is defined to be

$$\mathrm{vol}(\overline{L}) = \lim_{m\to\infty}\frac{(\dim X+1)!}{m^{\dim X+1}}\dim \widehat{H}^0(X, m\overline{L}),$$

where we write $m\overline{L}$ for $\overline{L}^{\otimes m}$. This limit always exists. And $\overline{L}$ is said to be *big* if $\mathrm{vol}(\overline{L}) > 0$. For a nef adelic line bundle $\overline{L}$, one can use its top self-intersection (see for example [YZ26, Prop. 4.1.1]) for definition) to compute its volume.

Theorem 2.4 (Arithmetic Hilbert–Samuel)**.** *For a nef adelic line bundle $\overline{L}$ on X, we have* $\mathrm{vol}(\overline{L}) = \overline{L}^{\dim X+1}$.

2.5.3. *Link.* Let X/K be a projective variety and let $\overline{L}$ be an adelic line bundle on X defined over K. Then

$$\overline{E}_m := (H^0(X,mL), \{\|\cdot\|_{m\overline{L},v,\sup}\}_{v\in M_K})$$

is an adelic vector space over K of dimension $h^0(X,mL)$, and $\widehat{H}^0(\overline{E}_m) = \widehat{H}^0(X,m\overline{L})$. In this language, Arithmetic Hilbert–Samuel can be expressed as: For $\overline{L}$ nef, we have

$$\overline{L}^{\dim X+1} = \lim_{m\to\infty}\frac{(\dim X+1)!}{m^{\dim X+1}}\widehat{\deg}(\overline{E}_m). \tag{2.6}$$

3. Bump functions at archimedean places

The goal of this section is to construct the desired bump functions at archimedean places. Take $\sigma\colon K \hookrightarrow \mathbb{C}$. For the curvature from $c_1(L,\|\cdot\|_\sigma)$, its associated volume form $\mathrm{d}\mu_\sigma := \frac{1}{\deg_L(A)}c_1(L,\|\cdot\|_\sigma)^{\wedge g}$ satisfies

$$\int_{A_\sigma(\mathbb{C})}\mathrm{d}\mu_\sigma = 1.$$

We need *(multi-)strips* on $A_\sigma(\mathbb{C})$, for which we do the following set up. Take an isogeny

$$\rho_\sigma\colon A_\sigma \longrightarrow A_{0,\sigma} \tag{3.1}$$

such that L_σ descends to a principal polarization $L_{0,\sigma}$ on $A_{0,\sigma}$, *i.e.* $L_\sigma \simeq \rho_\sigma^* L_{0,\sigma}$, and $L_{0,\sigma}$ has a cubist metric $\|\cdot\|_{0,\sigma}$ such that $\|\cdot\|_\sigma = \rho_\sigma^*\|\cdot\|_{0,\sigma}$.[4] Then L_0 gives rise to a *Siegel reduced matrix* $\tau_{0,\sigma}$ in the Siegel upper half space such that $A_{0,\sigma}(\mathbb{C}) \simeq \mathbb{C}^g/(\mathbb{Z}^g+\tau_{0,\sigma}\mathbb{Z}^g)$; see [Igu72, Chap. V, §4, above Lemma 15] for definition and Lemma 3.3 for the properties we need. Then we can identify $\mathrm{Lie}\,A_{0,\sigma}$ with $\mathbb{C}^g$.

[4]The key case of the construction is when L is a principal polarization on A. Then we take $\rho_\sigma = \mathrm{id}$.

We can furthermore identify $\mathrm{Lie}\, A_\sigma = \mathrm{Lie}\, A_{0,\sigma} = \mathbb{C}^g$ via ρ_σ. Then $\ker(\exp_{A_\sigma})$ is a lattice Λ in $\mathbb{Z}^g + \tau_{0,\sigma}\mathbb{Z}^g$. Denote by

$$(3.2) \qquad u_{0,\sigma} \colon \mathbb{C}^g \longrightarrow A_{0,\sigma}(\mathbb{C}) = \mathbb{C}^g/(\mathbb{Z}^g + \tau_{0,\sigma}\mathbb{Z}^g), \qquad u_\sigma \colon \mathbb{C}^g \longrightarrow A_\sigma(\mathbb{C}) = \mathbb{C}^g/\Lambda$$

the uniformizations (then $u_{0,\sigma} = \rho_\sigma \circ u_\sigma$), and consider the Betti coordinates

$$(3.3) \qquad \beta_\sigma \colon \mathbb{R}^g \times \mathbb{R}^g \longrightarrow \mathbb{C}^g, \qquad (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} + \tau_{0,\sigma}\mathbf{b}.$$

For our purpose, we need to consider all the translates of $[0, \frac{1}{g+2})^g$ chosen as follows. Consider the vectors in $[0,1)^g \subseteq \mathbb{R}^g$ whose coordinates are integer multiples of $\frac{1}{2(g+2)}$. There are $N_g := 2^g(g+2)^g$ such vectors, which we call $\mathbf{e}'_1, \ldots, \mathbf{e}'_{N_g}$. We thus have N_g translates $[0, \frac{1}{g+2})^g + \mathbf{e}'_q$ for $q \in \{1, \ldots, N_g\}$, and hence N_g strips

$$(3.4) \qquad \Box'_{\sigma,q} := (u_{0,\sigma} \circ \beta_\sigma)\left([0,1]^g \times \left([0, \frac{1}{g+2})^g + \mathbf{e}'_q\right)\right) \subseteq A_{0,\sigma}(\mathbb{C})$$

for all $k \in \{1, \ldots, N_g\}$. Each $\rho_\sigma^{-1}(\Box'_{\sigma,q})$ is a multi-stripe in $A_\sigma(\mathbb{C})$.

We can be more precise. There are $h^0(A, L) = \deg(\rho_\sigma)$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{h^0(A,L)}$ in $\mathbb{C}^g$ such that the disjoint union of the translates $\bigsqcup_{l=1}^{h^0(A,L)} (\mathbf{v}_l + \beta_\sigma([0,1)^{2g}))$ is a fundamental domain for u_σ. Then (denote for simplicity by $I_q := [0,1]^g \times \left([0, \frac{1}{g+2})^g + \mathbf{e}'_q\right)$)

$$(3.5) \qquad \rho_\sigma^{-1}(\Box'_{\sigma,q}) = u_\sigma\left(\bigsqcup_{l=1}^{h^0(A,L)} (\mathbf{v}_l + \beta_\sigma(I_q))\right) \subseteq A_\sigma(\mathbb{C}).$$

In practice we need to consider the union of at most $g+1$ such multi-strips.

Proposition 3.1. *Let $\Box'_\sigma$ be the union of $\leq (g+1)$ such $\Box'_{\sigma,q}$'s. Then there exists a smooth function*

$$f_\sigma \colon A_\sigma(\mathbb{C}) \to \mathbb{R}_{\geq 0}$$

satisfying the following properties:

(i) *f_σ vanishes on $\rho_\sigma^{-1}(\Box'_\sigma)$,*
(ii) *$c_1(L, e^{-\epsilon f_\sigma}\|\cdot\|_\sigma) \geq 0$ for any $\epsilon \in [-1,1]$,*
(iii) *For the Minkowski constant $\gamma_g \geq 1$ from Lemma 2.1, we have*

$$(3.6) \qquad \int_{A_\sigma(\mathbb{C})} f_\sigma \mathrm{d}\mu_\sigma \geq \frac{\pi}{8e(g+1)^3(g+2)^3(100g+274)\gamma_g} \mathrm{Tr}(\mathrm{Im}\tau_{0,\sigma}).$$

Remark 3.2. *An equivalent way to state Proposition 3.1.(ii) is: We have*

$$-c_1(L, \|\cdot\|_\sigma) \leq \epsilon \mathrm{dd}^{\mathrm{c}} f_\sigma \leq c_1(L, \|\cdot\|_\sigma) \quad \textit{for all } \epsilon \in [-1,1].$$

3.1. **Preparation and notation.** For simplicity, we will denote by

$$\tau_{0,\sigma} = X_\sigma + \sqrt{-1}Y_\sigma,$$

and $Y_\sigma = (y_{jj'})_{1 \leq j,j' \leq g}$. We will use the following properties for $\tau_{0,\sigma}$ which is Siegel reduced. See [Igu72, Chap. V, §4], and more precisely pp. 194 and pp. 192.

Lemma 3.3. (i) *All entries of X_σ have absolute value $\leq 1/2$,*
(ii) *Y_σ is a Minkowski-reduced matrix, with $\frac{\sqrt{3}}{2} \leq y_{11} \leq \ldots \leq y_{gg}$.*

For any smooth function F on $\mathbb{C}^g$, the composite $F \circ \rho$ is a smooth function on $\mathbb{R}^{2g}$ for the change of variables ρ defined in (3.3). Let

$$\begin{bmatrix} F_{\mathbf{aa}} & F_{\mathbf{ab}} \\ F_{\mathbf{ba}} & F_{\mathbf{bb}} \end{bmatrix} \tag{3.7}$$

be the Hessian of $F \circ \rho$. Denote by $\mathbf{w} = (w_1, \ldots, w_g)$ the standard coordiantes on $\mathbb{C}^g$. Then (3.3) says that $\mathbf{w} = (\mathbf{a} + X_\sigma \mathbf{b}) + \sqrt{-1} Y_\sigma \mathbf{b}$. So one can compute

$$\partial_{\mathbf{w}} \overline{\partial}_{\mathbf{w}} F = \frac{1}{4} (\mathrm{d}\mathbf{w})^{\mathrm{t}} \left(Y_\sigma^{-1} \begin{bmatrix} \tau_{0,\sigma} & -I \end{bmatrix} \begin{bmatrix} F_{\mathbf{aa}} & F_{\mathbf{ab}} \\ F_{\mathbf{ba}} & F_{\mathbf{bb}} \end{bmatrix} \begin{bmatrix} \overline{\tau_{0,\sigma}} \\ -I \end{bmatrix} Y_\sigma^{-1} \right) \mathrm{d}\overline{\mathbf{w}}. \tag{3.8}$$

3.2. **Auxiliary smooth function on $\mathbb{R}$.** Define the smooth function $\eta \colon \mathbb{R} \to \mathbb{R}$ by

$$\eta(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0 \end{cases}$$

and the smooth function $\theta \colon \mathbb{R} \to \mathbb{R}$ by

$$\theta(t) = \frac{\eta(t)}{\eta(t) + \eta(1-t)}.$$

Then $\theta(t) = 0$ for $t \le 0$ and $\theta(t) = 1$ for $t \ge 1$, $|\theta'|_\infty \le 5$ and $|\theta''|_\infty \le 112$.

Consider the 1-periodic smooth function $\psi \colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\psi(s) = \theta\left(\frac{\{s\} - \frac{1}{g+2}}{\delta} \right) \theta\left(\frac{1 - \{s\}}{\delta} \right), \qquad \text{with } \delta := \frac{1}{4(g+1)(g+2)}, \tag{3.9}$$

where $\{s\} := s - \lfloor s \rfloor$. Then ψ vanishes precisely on $[0, \frac{1}{g+2}] + \mathbb{Z}$. On $[\frac{1}{g+2} + \delta, 1 - \delta]$, we have $\psi(s) = 1$. Direct computation shows $|\psi|_\infty \le 1$, $|\psi'|_\infty \le 40(g+1)(g+2)$ and $|\psi''|_\infty \le 2(112 + 5^2) \cdot 16(g+1)^2(g+2)^2 = 16 \cdot 274 (g+1)^2(g+2)^2$. Moreover

$$\int_0^1 \psi(s) \mathrm{d}s \ge 1 - \frac{1}{g+2} - 2\delta \ge \frac{7}{12} \ge \frac{1}{2}. \tag{3.10}$$

3.3. **Proof of Proposition 3.1.** Up to renumbering we may assume $\square'_\sigma \subseteq \bigcup_{k=1}^{g+1} \square'_{\sigma,q}$. We will write $\mathbf{e}'_q = (e_q^{(1)}, \ldots, e_q^{(g)})$. Let

$$\widetilde{F} \colon \mathbb{R}^{2g} = \mathbb{R}^g \times \mathbb{R}^g \longrightarrow \mathbb{R}_{\ge 0}, \qquad (\mathbf{a}, \mathbf{b}) \mapsto \frac{4\pi}{16e(g+1)^3(g+2)^2(100g+274)\gamma_g} \sum_{j=1}^g y_{jj} \Psi(b_j)$$

where $\Psi(b_j) = \prod_{q=1}^{g+1} \psi(b_j - e_q^{(j)})$. Here we still denote by $\mathbf{b} = (b_1, \ldots, b_g)$. Then $\widetilde{F}$ is smooth, $\mathbb{Z}^{2g}$-periodic, and vanishes on $\bigcup_{q=1}^{g+1} \left([0,1]^g \times \left([0, \frac{1}{g+2})^g + \mathbf{e}'_q \right) \right)$. So the composite $\widetilde{F} \circ \beta_\sigma^{-1} \colon \mathbb{C}^g \to \mathbb{R}^{2g} \to \mathbb{R}$ descends to $A_{0,\sigma}(\mathbb{C})$, *i.e.* there exists a smooth function

$$f_{0,\sigma} \colon A_{0,\sigma}(\mathbb{C}) \longrightarrow \mathbb{R}_{\ge 0}$$

such that $\widetilde{F} \circ \beta_\sigma^{-1} = f_{0,\sigma} \circ u_{0,\sigma}$ for the uniformization $u_{0,\sigma} \colon \mathbb{C}^g \to A_{0,\sigma}(\mathbb{C})$. This $f_{0,\sigma}$ clearly vanishes on $\square'_\sigma$.

Set

$$f_\sigma = f_{0,\sigma} \circ \rho_\sigma \colon A_\sigma(\mathbb{C}) \longrightarrow \mathbb{R}_{\ge 0}.$$

Then f_σ clearly satisfies (i).

The variable $\mathbf{a}$ does not appear in the definition $\widetilde{F}$. So by (3.8) applied to $\widetilde{F}\circ\beta_\sigma^{-1}$, we have

$$\partial_{\mathbf{w}}\overline{\partial}_{\mathbf{w}}\widetilde{F} = \frac{1}{4}(\mathrm{d}\mathbf{w})^{\mathrm{t}}Y_\sigma^{-1}(\widetilde{F}\circ\beta_\sigma^{-1})_{\mathbf{bb}}Y_\sigma^{-1}\mathrm{d}\overline{\mathbf{w}}.$$

Notice that $(\widetilde{F}\circ\beta_\sigma^{-1})_{\mathbf{bb}}$ is the diagonal matrix with diagonal entries $4\pi y_{jj}\Psi''(b_j)/16e(g+1)^3(g+2)^2(100g+274)\gamma_g$. Direct computation shows that $|\Psi''|_\infty \leq (g+1)|\psi''|_\infty + g(g+1)|\psi'|_\infty^2 \leq 16(g+1)^3(g+2)^2(100g+274)$, using the bounds for $|\psi|_\infty$ and $|\psi'|_\infty$ and $|\psi''|_\infty$. So Lemma 2.1.(iv) (applied to $B=Y_\sigma$) implies that the matrix $Y_\sigma + \frac{\epsilon}{4\pi}(\widetilde{F}\circ\beta_\sigma^{-1})_{\mathbf{bb}}$ is positive definite for all $\epsilon\in[-1,1]$. Hence

$$\begin{aligned} u_{0,\sigma}^* c_1(L_0, e^{-\epsilon f_{0,\sigma}}\|\cdot\|_{0,\sigma}) &= u_{0,\sigma}^* c_1(L_0,\|\cdot\|_{0,\sigma}) + \frac{\sqrt{-1}}{2\pi}\partial_{\mathbf{w}}\overline{\partial}_{\mathbf{w}}(\epsilon f_{0,\sigma}\circ u_{0,\sigma}) \\ &= \frac{\sqrt{-1}}{2}(\mathrm{d}\mathbf{w})^{\mathrm{t}}\wedge Y_\sigma^{-1}\mathrm{d}\overline{\mathbf{w}} + \frac{\sqrt{-1}}{2\pi}\partial_{\mathbf{w}}\overline{\partial}_{\mathbf{w}}\left(\epsilon\widetilde{F}\circ\beta_\sigma^{-1}\right) \geq 0. \end{aligned}$$

So $c_1(L, e^{-\epsilon f_\sigma}\|\cdot\|_\sigma) = \rho_\sigma^* c_1(L_0, e^{-\epsilon f_{0,\sigma}}\|\cdot\|_{0,\sigma}) \geq 0$. This establishes (ii).

Finally, let us bound $\int_0^1 \Psi(b_j)\mathrm{d}b_j$ from below. We claim that

$$\int_0^1 \Psi(b_j)\mathrm{d}b_j \geq \frac{1}{2(g+2)}. \tag{3.11}$$

Indeed, notice that Ψ is the product of $(g+1)$ translates of ψ and, on $[0,1)$, ψ equals 1 on an interval of length $1-\frac{1}{g+2}-\frac{1}{2(g+1)(g+2)}$. Thus ψ is not 1 on a subset of $[0,1)$ of Lebesgue measure $\leq \frac{1}{g+2}+\frac{1}{2(g+1)(g+2)}$. Hence on $[0,1)$, Ψ is not 1 on a subset of Lebesgue measure $\leq (g+1)\left(\frac{1}{g+2}+\frac{1}{2(g+1)(g+2)}\right)$. Thus $\int_0^1\Psi(b_j)\mathrm{d}b_j \geq 1-(g+1)\left(\frac{1}{g+2}+\frac{1}{2(g+1)(g+2)}\right) = \frac{1}{2(g+2)}$.

Denote by $\mathrm{d}\mu_{0,\sigma} := \frac{1}{\deg_{L_{0,\sigma}}(A_{0,\sigma})}c_1(L_0,\|\cdot\|_{0,\sigma})^{\wedge g}$. Then $\mathrm{d}\mu_{0,\sigma} = (\rho_\sigma)_*\mathrm{d}\mu_\sigma$. So we can conclude for (iii) by

$$\begin{aligned} \int_{A_\sigma(\mathbb{C})} f_\sigma \mathrm{d}\mu_\sigma = \int_{A_{0,\sigma}(\mathbb{C})} f_{0,\sigma}\mathrm{d}\mu_{0,\sigma} &= \int_{[0,1]^{2g}} \widetilde{F}\mathrm{d}a_1\wedge\cdots\mathrm{d}a_g\wedge\mathrm{d}b_1\wedge\cdots\wedge\mathrm{d}b_g \\ &= \frac{4\pi}{16e(g+1)^3(g+2)^2(100g+274)\gamma_g}\sum_{j=1}^g y_{jj}\int_0^1\Psi(b_j)\mathrm{d}b_j \\ &\geq \frac{\pi}{8e(g+1)^3(g+2)^3(100g+274)\gamma_g}\mathrm{Tr}(Y_\sigma) \text{ by (3.11).} \end{aligned}$$ □

3.4. **A technical lemma.** Later on we need to compare wedge powers of $c_1(L,\|\cdot\|_\sigma)$ and $\mathrm{dd}^{\mathrm{c}}f_\sigma$, which are related by Remark 3.2. We prove:

Lemma 3.4. *Let α and β be two $(1,1)$-forms on $\mathbb{C}^g$, with α positive everywhere. Assume $-\alpha\leq\beta\leq\alpha$. Then for any $k\in\{0,\ldots,g\}$, we have*

$$-\alpha^{\wedge g} \leq \alpha^{\wedge(g-k)}\wedge\beta^{\wedge k} \leq \alpha^{\wedge g}.$$

Proof. Since α is positive everywhere, pointwise there exists a basis $\{w_1,\ldots,w_g\}$ of $\mathbb{C}^g$ such that $\alpha = \sum_{j=1}^g \mathrm{d}w_j\wedge\mathrm{d}\overline{w}_j$ and $\beta=\sum_{j=1}^g\lambda_j\mathrm{d}w_j\wedge\mathrm{d}\overline{w}_j$. The assumption $-\alpha\leq\beta\leq\alpha$ implies that $|\lambda_j|\leq 1$ for all j. Then $\alpha^{\wedge g} = g!\mathrm{d}w_1\wedge\mathrm{d}\overline{w}_1\wedge\cdots\wedge\mathrm{d}w_g\wedge\mathrm{d}\overline{w}_g$ and

$$\alpha^{\wedge(g-k)}\wedge\beta^{\wedge k} = (g-k)!k!\sum_{1\leq i_1<\cdots<i_k\leq g}(\lambda_{i_1}\cdots\lambda_{i_k})\mathrm{d}w_1\wedge\mathrm{d}\overline{w}_1\wedge\cdots\wedge\mathrm{d}w_g\wedge\mathrm{d}\overline{w}_g.$$

Now we are done since
$$\left|\frac{(g-k)!k!\sum_{1\le i_1<\cdots<i_k\le g}(\lambda_{i_1}\cdots\lambda_{i_k})}{g!}\right| \le \frac{1}{\binom{g}{k}}\sum_{1\le i_1<\cdots<i_k\le g}|\lambda_{i_1}|\cdots|\lambda_{i_k}| \le 1. \qquad \square$$

4. Revision on tropicalization at non-archimedean places

In this section, we review the theory of uniformization of abelian varieties at non-archimedean places and the tropicalization. There are other nice reviews, *e.g* [FRSS18, §3], [dJS22, §6-7.1], *etc.*

Take $v \in M_K^0$. For simplicity denote by $F := K_v$ and by R the ring of integers of F.

Assume A has semi-stable reduction at v. By abuse of notation and only in this section, we will use A (resp. use L) to denote $A \otimes_K K_v$ (resp. denote $L \otimes_K K_v$). Then A is an abelian variety defined over F of dimension g, with polarization given by the ample line bundle L. In the whole section, by $(\cdot)^{\mathrm{an}}$ we mean the Berkovich analytification.

The identity component of the Néron model $\mathcal{A}$ of A is a semi-abelian scheme over $\mathrm{Spec}(R)$, whose special fiber has toric rank $r \ge 0$. To simplify the situation, we also assume that *the toric part of the special fiber of $\mathcal{A}$ is split over the residue field k_v.* Two extreme cases are: (i) $r = 0$, in which case A has good reduction; (ii) $r = g$, in which case the special fiber of $\mathcal{A}$ is $(\mathbb{G}_{\mathrm{m}})^g$.

A particular example to keep in mind is the Tate curve $F^\times/\mathfrak{q}^{\mathbb{Z}}$, in which case $r = g = 1$.

4.1. **Analytic uniformization and Raynaud cross.** The analytic uniformization $p\colon E^{\mathrm{an}} \to A^{\mathrm{an}}$ fits into the *Raynaud cross*

$$\begin{tikzcd} & Y \arrow[d, "\mathfrak{v}"'] \arrow[dr, "\varphi"] & \\ T^{\mathrm{an}} \arrow[r] & E^{\mathrm{an}} \arrow[r, "q"'] \arrow[d, "p"] & B^{\mathrm{an}} \\ & A^{\mathrm{an}} & \end{tikzcd} \tag{4.1}$$

Let us explain how this diagram is constructed. Before moving on, we point out that in the case where A is a Tate curve $F^\times/\mathfrak{q}^{\mathbb{Z}}$, we have $Y = \mathbb{Z}$, $T = E = \mathbb{G}_{\mathrm{m}}$, B is a point, and the map $Y = \mathbb{Z} \to E^{\mathrm{an}} = F$ is $a \mapsto \mathfrak{q}^a$.

4.1.1. *Horizontal.* The space E^{an} is the analytification of a semi-abelian variety E (constructed below) defined over F

$$1 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 1 \tag{4.2}$$

whose toric part T has rank r and abelian part B has good reduction. More precisely, the horizontal line of the Raynaud cross (4.1) is the analytification of (4.2). Moreover, T is an F-split torus by our assumption.

Here is a brief construction of E. The identity component of the Néron model $\mathcal{A}$ of A is a semi-abelian scheme over $S := \mathrm{Spec}(R)$. Write $\mathfrak{m}$ for the maximal ideal of R. Set

$$S_i := \mathrm{Spec}(R/\mathfrak{m}^i), \qquad \mathcal{A}_i := \mathcal{A} \times_S S_i$$
$$\widehat{S} := \lim S_i, \qquad \widehat{\mathcal{A}} := \lim \mathcal{A}_i.$$

Then $\widehat{\mathcal{A}}$ is a formal semi-abelian scheme over $\widehat{S}$; it is the formal completion of $\mathcal{A}$ along its closed fiber. Moreover $\widehat{\mathcal{A}}$ is algebraizable, *i.e.* there exists a semi-abelian scheme $\widetilde{\mathcal{A}}$ over S such that $\widehat{\mathcal{A}}$ is the associated formal scheme. Our E is the generic fiber of $\widetilde{\mathcal{A}}$.

To construct the vertical maps of (4.1), we also need to apply the discussion above to the dual of A, which we denote by A^{t}, to get

$$1 \longrightarrow T^{\mathrm{t}} \longrightarrow E^{\mathrm{t}} \longrightarrow B^{\mathrm{t}} \longrightarrow 1 \tag{4.3}$$

where B^{t} is the dual abelian variety of B. The toric part T^{t} also has rank r and is F-split.

4.1.2. *Vertical.* The period lattice $Y := \mathrm{Hom}(T^{\mathrm{t}}, \mathbb{G}_{\mathrm{m}})$ in (4.1) is the group of characters of T^{t}, and hence is isomorphic to $\mathbb{Z}^r$.

We first construct the group homomorphism $\varphi\colon Y \to B(F)$, using (4.3). For each $u' \in Y = \mathrm{Hom}(T^{\mathrm{t}}, \mathbb{G}_{\mathrm{m}})$, we have a pushout diagram in the category of algebraic groups

$$\begin{tikzcd} 1 \arrow[r] & T^{\mathrm{t}} \arrow[r] \arrow[d, "u'"] & E^{\mathrm{t}} \arrow[r] \arrow[d, "e_{u'}"] & B^{\mathrm{t}} \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & \mathbb{G}_{\mathrm{m}} \arrow[r] & (E^{\mathrm{t}})_{u'} \arrow[r] & B^{\mathrm{t}} \arrow[r] & 1. \end{tikzcd} \tag{4.4}$$

Now $(E^{\mathrm{t}})_{u'}$ is a semi-abelian variety which is an extension of B^{t} by $\mathbb{G}_{\mathrm{m}}$, and all such extensions are parametrized by the F-points of the dual of B^{t}, *i.e.* by $B(F)$. Hence we obtain a map $\varphi\colon Y \to B(F)$, which is easily checked to be a group homomorphism.

Next we brielfy explain the construction of $p\colon E^{\mathrm{an}} \to A^{\mathrm{an}}$. Denote by η the generic point of $S = \mathrm{Spec}(R)$. Then $\mathcal{A}_\eta = A$. General knowledge of rigid analytic geometry says that there is an open immersion

$$i_{\mathcal{A}}\colon \widehat{\mathcal{A}}_\eta \longrightarrow (\mathcal{A}_\eta)^{\mathrm{an}} = A^{\mathrm{an}},$$

where $\widehat{\mathcal{A}}_\eta$ is the generic fiber of $\widehat{\mathcal{A}}$ viewed as a rigid analytic space (often denoted by $(\widehat{\mathcal{A}})^{\mathrm{rig}}$). But E is the generic fiber of the algebraization of $\widehat{\mathcal{A}}$. So there is another open immersion $\widehat{\mathcal{A}}_\eta \to E^{\mathrm{an}}$. By [BL91, Thm. 1.2], $i_{\mathcal{A}}$ uniquely extends to a surjective group morphism $p\colon E^{\mathrm{an}} \to A^{\mathrm{an}}$ whose kernel is a lattice in $E(F)$ of rank r (*i.e.* $\simeq \mathbb{Z}^r$); the proof uses local trivializations. In particular, Y and $\ker(p)$ are isomorphic as groups.

The homomorphism $\mathfrak{v}$ is an isomorphism from Y to $\ker(p)$ which lifts φ. We will explain how to construct this $\mathfrak{v}$ by fixing a trivialization of a certain $\mathbb{G}_{\mathrm{m}}$-biextension above (4.12).

4.1.3. *Duality and Poincaré line bundle.* Later on we shall also use the Raynaud cross associated with the dual abelian variety A^{t} which is the diagram below on the right. Its horizontal line is the analytification of (4.3), and $X := \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$ is the group of characters of T and hence is isomorphic to $\mathbb{Z}^r$.

$$\begin{tikzcd} & Y \arrow[d, "\mathfrak{v}"] \arrow[dr, "\varphi"] & \\ T^{\mathrm{an}} \arrow[r] & E^{\mathrm{an}} \arrow[r, "q"'] \arrow[d, "p"] & B^{\mathrm{an}} \\ & A^{\mathrm{an}} & \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[d, "\mathfrak{v}'"] \arrow[dr, "\varphi'"] & \\ (T^{\mathrm{t}})^{\mathrm{an}} \arrow[r] & (E^{\mathrm{t}})^{\mathrm{an}} \arrow[r, "q'"'] \arrow[d, "p'"] & (B^{\mathrm{t}})^{\mathrm{an}} \\ & (A^{\mathrm{t}})^{\mathrm{an}} & \end{tikzcd} \tag{4.5}$$

The horizontal lines of the two Raynaud crosses are related by the polarization $\lambda_{A,L}\colon A \to A^{\mathrm{t}}$ as follows. This polarization induces an isogeny $\mathcal{A} \to \mathcal{A}^{\mathrm{t}}$ between the identity components of their Néron models, and hence the following commutative diagram whose vertical maps are isogenies (and are isomorphisms if L is a principal polarization)

$$\begin{tikzcd} 1 \arrow[r] & T \arrow[r] \arrow[d, "\lambda_{T,L}"] & E \arrow[r] \arrow[d, "\lambda_{E,L}"] & B \arrow[r] \arrow[d, "\lambda_{B,L}"] & 1 \\ 1 \arrow[r] & T^{\mathrm{t}} \arrow[r] & E^{\mathrm{t}} \arrow[r] & B^{\mathrm{t}} \arrow[r] & 1. \end{tikzcd} \tag{4.6}$$

Notice that $\lambda_{B,L}$ is a polarization on B, which is furthermore defined by an ample line bundle M on B such that

$$q^*M^{\mathrm{an}} = p^*L^{\mathrm{an}}. \tag{4.7}$$

This is because, roughly speaking, the Poincaré line bundle on $A \times A^{\mathrm{t}}$ extends to the rigidified Poincaré line bundle on $\mathcal{A} \times_S \mathcal{A}^{\mathrm{t}}$ and hence to $E \times E^{\mathrm{t}}$, and furthermore descends to the Poincaré line bundle on $B \times B^{\mathrm{t}}$. We refer to [FC10, Chap. II, §2] for more details. The *Poincaré line bundle* P on $B \times B^{\mathrm{t}}$ can be canonically identified with

$$(1, \lambda_{B,L})^*P = m_B^*M \otimes p_1^*M^{\otimes -1} \otimes p_2^*M^{\otimes -1}. \tag{4.8}$$

Observe that $\lambda_{T,L}$ induces $\lambda_{T,L}^*\colon Y \to X$ and fits into the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\lambda_{T,L}^*"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi'"] \\ B \arrow[r, "\lambda_{B,L}"] & B^{\mathrm{t}}. \end{tikzcd} \tag{4.9}$$

The horizontal maps are isomorphisms if L is a principal polarization.

The following construction is important to defining the tropicalization of A^{an} later on. For each $u \in X$, we have a pushout diagram in the category of algebraic groups

$$\begin{tikzcd} 1 \arrow[r] & T \arrow[r] \arrow[d, "u"] & E \arrow[r] \arrow[d, "e_u"] & B \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & \mathbb{G}_{\mathrm{m}} \arrow[r] & E_u \arrow[r] & B \arrow[r] & 1, \end{tikzcd} \tag{4.10}$$

and E_u is a semi-abelian variety which is an extension of B by $\mathbb{G}_{\mathrm{m}}$. Notice that E_u can also be viewed as a $\mathbb{G}_{\mathrm{m}}$-torsor on B, whose associated line bundle is $P|_{B\times\{\varphi'(u)\}}$ (*i.e.* E_u is $P|_{B\times\{\varphi'(u)\}}$ with the zero section removed). Moreover, the homomorphism $\varphi\colon X \to B^{\mathrm{t}}(F)$ is defined using (4.10), by sending $u \in X$ to the point of $B^{\mathrm{t}}(F)$ parametrizing E_u.

Finally, let us take a closer look at the construction of the homomorphisms $\mathfrak{v}\colon Y \to E(F)$ and $\mathfrak{v}'\colon X \to E^{\mathrm{t}}(F)$, in which we see how the maps e_u from (4.10) and $e_{u'}$ from (4.4) are related. Denote by $P^\times$ the $\mathbb{G}_{\mathrm{m}}$-torsor on $B \times B^{\mathrm{t}}$ corresponding to P. Then for any $u \in X$ and $u' \in Y$, we have

$$P^\times|_{\{\varphi(u')\}\times B^{\mathrm{t}}} = (E^{\mathrm{t}})_{u'} \qquad \text{and} \qquad P^\times|_{B\times\{\varphi'(u)\}} = E_u.$$

For $(\varphi, \varphi')\colon Y \times X \to B \times B^{\mathrm{t}}$, the pullback $(\varphi, \varphi')^*P^\times$ is a $\mathbb{G}_{\mathrm{m}}$-biextension over $Y \times X$, which is trivial. Taking a trivialization of $(\varphi, \varphi')^*P^\times$ amounts to taking a bilinear map

$$t\colon Y \times X \to P^\times \tag{4.11}$$

lifting (φ, φ'), *i.e.* $t(u', u) \in P^\times_{\varphi(u'),\varphi'(u)}$ for any $u' \in Y$ and $u \in X$.

From now on, fix such a t as in [FC10, Chap. II, Thm. 5.1].

To lift φ to $\mathfrak{v}\colon Y \to E(F)$, it suffices to define $e_u \circ \mathfrak{v}\colon Y \to E_u(F)$ for all $u \in X$. Notice that $P^\times_{\varphi(u'),\varphi'(u)} \subseteq P^\times|_{B\times\{\varphi'(u)\}} = E_u$. Now we define $(e_u \circ \mathfrak{v})(u') := t(u', u)$ for each $u' \in Y$, and this gives rise to the desired lift of φ to $\mathfrak{v}$.

The homomorphism $\mathfrak{v}'\colon X \to E^{\mathrm{t}}(F)$ is defined in a similar way. By construction,

$$t(u', u) = e_u(\mathfrak{v}(u')) = e_{u'}(\mathfrak{v}'(u)) \qquad \text{for all } u' \in Y, u \in X. \tag{4.12}$$

4.2. **Tropicalization, skeleton, and Néron components.** Let $X^* := \mathrm{Hom}(X, \mathbb{Z})$ be the dual of X, and $X^*_{\mathbb{R}} := X^* \otimes \mathbb{R}$. Let

$$\langle , \rangle \colon X \times X^*_{\mathbb{R}} \longrightarrow \mathbb{R}$$

be the natural evaluation pairing.

In the following, we will take log with respect to the base N_v defined in §2.1.

4.2.1. *Tropicalization of E^{an}.* The *tropicalization* $\mathrm{trop}\colon T^{\mathrm{an}} \to X^*_{\mathbb{R}}$ is defined by the rule

$$\langle u, \mathrm{trop}(z)\rangle = -\log |u(z)|, \qquad \text{for all } u \in X \text{ and } z \in T^{\mathrm{an}}.$$

Indeed, the function $-\log|\cdot|$ here is the valuation. This map extends in a natural way to a surjective homomorphism

$$\mathrm{trop}\colon E^{\mathrm{an}} \longrightarrow X^*_{\mathbb{R}} \tag{4.13}$$

as follows. Let $\|\cdot\|_M$ be the cubist model metric on M (it exists because B has good reduction), and let $\|\cdot\|_P$ be the induced metric on P by (4.8). For any $u \in X$, consider the pushout (4.10) and the associated line bundle $P|_{B\times\{\varphi'(u)\}}$ on B. Let $\|\cdot\|_{E_u}$ be the metric on $P|_{B\times\{\varphi'(u)\}}$ induced by $\|\cdot\|_P$. Now define (4.13) by the rule

$$\langle u, \mathrm{trop}(z)\rangle = -\log \|e_u(z)\|_{E_u}, \qquad \text{for all } u \in X \text{ and } z \in E^{\mathrm{an}}.$$

This is a desired extension since $e_u(z) = u(z)$ for all $z \in T^{\mathrm{an}}$.

4.2.2. *Skeleton and tropicalization of A^{an}.* We have a $\mathbb{Z}$-valued non-degenerate bilinear map

$$[\cdot,\cdot]\colon Y \times X \longrightarrow \mathbb{Z}, \qquad (u', u) \mapsto -\log \|t(u', u)\|_P \tag{4.14}$$

where t is from (4.11). It realizes Y as a subgroup of X^* of finite index, by sending each $u' \in Y$ to $[u', \cdot] \in X^*$. From now on, we view Y as a subset of X^* under this realization.

By (4.12), the inclusion $Y \subseteq X^*_{\mathbb{R}}$ equals the composite $\mathrm{trop}\circ\mathfrak{v}\colon Y \to E^{\mathrm{an}} \to X^*_{\mathbb{R}}$. Define the *skeleton* of A^{an} to be the real torus of rank r

$$\Sigma := X^*_{\mathbb{R}}/Y.$$

This gives rise to the following commutative diagram with surjective vertical maps

$$\begin{tikzcd} 1 \arrow[r] & Y \arrow[r] \arrow[d, "="] & E^{\mathrm{an}} \arrow[r] \arrow[d, "\mathrm{trop}"] & A^{\mathrm{an}} \arrow[r] \arrow[d, "\overline{\mathrm{trop}}"] & 1 \\ 1 \arrow[r] & Y \arrow[r] & X^*_{\mathbb{R}} \arrow[r] & \Sigma \arrow[r] & 1. \end{tikzcd} \tag{4.15}$$

Moreover, $\overline{\mathrm{trop}}$ admits a splitting, *i.e.* there exists an injective morphism $\iota\colon \Sigma \to A^{\mathrm{an}}$ such that $\overline{\mathrm{trop}}\circ\iota = \mathrm{id}_\Sigma$. So $A^{\mathrm{an}} \to \Sigma$ can be seen as a contraction of A^{an} to the subset Σ.

Since Y is a lattice in $X^*_{\mathbb{R}}$, the bilinear map $Y \times Y \to \mathbb{Z}$, $(u'_1, u'_2) \mapsto [u'_1, \lambda^*_{T,L}(u'_2)]$ extends linearly to a bilinear map

$$b\colon X^*_{\mathbb{R}} \times X^*_{\mathbb{R}} \to \mathbb{R}. \tag{4.16}$$

The associated quadratic form $q_0\colon X^*_{\mathbb{R}} \to \mathbb{R}$ is positive definite since L is ample.

4.2.3. *Néron components.* In practice, we are interested in the classical points $A(F)$, which form a subgroup of A^{an}. The group of components of the special fiber of the Néron model of A can be naturally identified with $\Phi := A(F)/\mathcal{A}(R)$. It is known that the natural group homomorphism $A(F) \subseteq A^{\mathrm{an}} \xrightarrow{\overline{\mathrm{trop}}} \Sigma = X^*_{\mathbb{R}}/Y$ has image X^*/Y and kernel $\mathcal{A}(R)$ (see *e.g.* [dJS22, Lem. 7.1]), and hence induces a group isomorphism

$$\Phi \simeq X^*/Y. \tag{4.17}$$

On the other hand, denote by $\det_Y(q_0)$ the determinant of the matrix of q_0 under a $\mathbb{Z}$-basis of Y; this determinant is independent of the choice of the basis. Then it is known that

$$\det_Y(q_0) = \#\Phi \cdot [X : \lambda^*_{T,L}(Y)] \tag{4.18}$$

where $\lambda^*_{T,L}$ is the map from (4.9) defined be the polarization L on A. When L is a principal polarization, we have $[X : \lambda^*_{T,L}(Y)] = 1$.

In the case of Tate curve $F^\times/\mathfrak{q}^{\mathbb{Z}}$ with the canonical principal polarization, we have $\Phi = \mathrm{ord}_v(\mathfrak{q})$ and the skeleton is a circle $\simeq \mathbb{R}/\mathbb{Z}$ of circumference $\sqrt{\mathrm{ord}_v(\mathfrak{q})}$. If we identify Y with $\mathbb{Z}$ and $X^*_{\mathbb{R}}$ with $\mathbb{R}$ (with standard coordinate x), then $X^* = \frac{1}{\mathrm{ord}_v(\mathfrak{q})}\mathbb{Z}$ and the quadratic form $q_0(x) = \mathrm{ord}_v(\mathfrak{q}) \cdot x^2$.

5. Bump functions at non-archimedean places

The goal of this section is to construct the desired bump functions at non-archimedean places. Let $v \in M_K^0$ be a non-archimedean place at which A has semi-stable reduction. The construction is void if A has good reduction at v.

Here is the setup. Assume the toric part of $A \pmod v$ is split over the residue field k_v[5] until §5.4. Then we can apply the discussion of §4 to $A_v := A \otimes_K K_v$ and use the same notation (but we add a subscript v to each object). In particular we have a tropicalization

$$\overline{\mathrm{trop}}_v \colon A_v^{\mathrm{an}} \longrightarrow \Sigma_v := X^*_{v,\mathbb{R}}/Y_v$$

to the skeleton Σ_v which is a real torus of rank r_v; see (4.15). Moreover, $\overline{\mathrm{trop}}_v(A(K_v))$ is contained in $\Phi_v = X_v^*/Y_v$, which is a finite subgroup of Σ_v corresponding to the Néron components of A_v; see (4.17).

Let N_v be from §2.1. Recall that in §4.2, all the log's are taken with base N_v.

Finally, let $\mathrm{d}\mu_v$ be the canonical volume form on Σ_v, normalized such that $\int_{\Sigma_v} \mathrm{d}\mu_v = 1$. It is the descent of the standard Lebesgue measure on $X^*_{v,\mathbb{R}}$ divided by $[X_v^* : Y_v] = \#\Phi_v$.

Here is the main result of this section, which we prove in §5.1-5.3. In §5.4, we summarize our construction in terms of $f_v = f_{0,v} \circ \overline{\mathrm{trop}}_v \colon A_v^{\mathrm{an}} \longrightarrow \mathbb{R}$ and we also include the case where the toric part of $A \pmod v$ is not necessarily split over k_v.

Proposition 5.1. *There exists a smooth function*

$$f_{0,v} \colon \Sigma_v = X^*_{v,\mathbb{R}}/Y_v \longrightarrow \mathbb{R}$$

satisfying the following properties:

(i) $f_{0,v}|_{\Phi_v} = 0$*, and in particular* $(f_{0,v} \circ \overline{\mathrm{trop}}_v)(A(K_v)) = 0$*;*
(ii) *the metric* $N_v^{-\epsilon f_{0,v} \circ \overline{\mathrm{trop}}_v} \| \cdot \|_v$ *on* L_v^{an} *is relatively semipositive for all* $\epsilon \in [-1, 1]$*;*
(iii) *for the Minkowski constant* $\gamma_{r_v} \geq 1$ *from Lemma 2.1, we have*

$$\int_{\Sigma_v} f_{0,v} \mathrm{d}\mu_v \geq \frac{r_v}{64 e \gamma_{r_v}} (\#\Phi_v)^{-1/r_v}.$$

[5]This is always the case when $K = \mathbb{C}(B)$.

Moreover, $f_{0,v}$ *is* X_v^**-periodic, i.e.* $f_{0,v}(\overline{x}+\overline{u}) = f_{0,v}(\overline{x})$ *for any* $\overline{u} \in X_v^*/Y_v$ *and* $\overline{x} \in \Sigma_v$.

Remark 5.2. *Let us give an explanation on part (iii). We take* N_v *as the base of the exponential since all the* log*'s are taken with this base in §4.2. We use S. Zhang's [Zha95b] definition of relatively semipositive metrics on* L_v^{an}*: it is a uniform limit of nef model metrics. In the proof, we use the equivalent description in terms of superforms, which is reviewed in §5.2.*[6] *In this terminology, condition (iii) is equivalent to:*

$$-c_1(L_v^{\mathrm{an}}, \|\cdot\|_v) \leq \epsilon \mathrm{d}'\mathrm{d}''(f_{0,v} \circ \overline{\mathrm{trop}}) \leq c_1(L_v^{\mathrm{an}}, \|\cdot\|_v), \qquad \textit{for all } \epsilon \in [-1,1]. \tag{5.1}$$

In the proof of Proposition 5.1, we shall only work with the place v. So in §5.1-5.3, to ease notation we will use the notation from §4 without adding the subscript v.

5.1. **Construction of $f_{0,v}$, proofs of (i) and (iii).** Recall from (4.16) the bilinear form b on $X_{\mathbb{R}}^*$ and the associated quadratic form q_0 which is positive-definite. An (ordered) integral basis $e_1, \ldots, e_r$ of X^* is called *Minkowski-reduced* with respect to q_0 if the following holds: For each $j \in \{1, \ldots, r\}$ and any $w = \sum_{j=1}^r w_j e_j \in X^*$ with $\gcd(w_1, \ldots, w_r) = 1$, we have we have $q_0(e_j) \leq q_0(w)$.

General reduction theory guarantees the existence of a Minkowski-reduced basis $e_1, \ldots, e_r$ of X^*. Then the matrix of q_0 under this basis is an $r \times r$ Minkowski matrix $B = (b_{ij})_{1\leq i,j\leq r}$. Let $\gamma_r \geq 1$ be the Minkowski constant from Lemma 2.1.

Now define the function

$$F_0\colon X_{\mathbb{R}}^* \longrightarrow \mathbb{R}, \qquad x = \sum_{j=1}^r x_j e_j \longmapsto \frac{1}{e\gamma_r} \sum_{j=1}^r b_{jj}\psi_0(x_j),$$

where $\psi_0\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto \frac{1}{64}(1-\cos(2\pi t))$. In particular, ψ_0 is smooth and $\mathbb{Z}$-periodic, $\psi_0(0) = 0$ and $|\psi_0''|_\infty \leq 1$. Hence F_0 is X^*-periodic, and so descends to a smooth function

$$f_{0,v}\colon \Sigma_v = X_{\mathbb{R}}^*/Y \longrightarrow \mathbb{R}$$

satisfying (i) and the "Moreover" part of Proposition 5.1. Chasing the diagram (4.15) yields

$$f_{0,v} \circ \overline{\mathrm{trop}} \circ p = F_0 \circ \mathrm{trop} \tag{5.2}$$

for the uniformization $p\colon E^{\mathrm{an}} \to A_v^{\mathrm{an}}$.

Let us turn to (iii). The standard Lebesgue measure on $X_{\mathbb{R}}^*$ is $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_r$ for the usual differential operator d. By the definition of $\mathrm{d}\mu_v$ (see above Proposition 5.1) and the definition of $f_{0,v}$, we have

$$\begin{aligned}
\int_{\Sigma_v} f_{0,v}\mathrm{d}\mu_v &= \int_{[0,1]^r} F_0 \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_r \\
&= \frac{1}{e\gamma_r} \sum_{j=1}^r \int_0^1 b_{jj}\psi_0(x_j)\mathrm{d}x_j \\
&= \frac{1}{e\gamma_r} \left(\int_0^1 \psi_0(s)\mathrm{d}s\right) \sum_{j=1}^r b_{jj} \geq \frac{1}{64e\gamma_r} \cdot r \left(\prod_{j=1}^r b_{jj}\right)^{1/r} \qquad \text{by (3.10).}
\end{aligned}$$

Hence Lemma 2.1.(iii) yields

$$\int_{\Sigma_v} f_{0,v}\mathrm{d}\mu_v \geq \frac{r}{64e\gamma_r} (\det B)^{1/r}.$$

[6]To our knowledge, the (semi-)positivity of superforms is not known to imply Zhang's semipositivity in general. In our particular case, they are equivalent by Gubler–Stadlöder [GS23, Thm. 4.10].

The matrix B is the matrix of q_0 under a $\mathbb{Z}$-basis of X^*. So $\det B = \frac{1}{[X^*:Y]^2}\det_Y(q_0) = \frac{\det_Y(q_0)}{(\#\Phi_v)^2}$, and hence $\det B \geq (\#\Phi_v)^{-1}$ by (4.18). This establishes (iii). $\square$

5.2. **A quick review on superforms.** Our proof of Proposition 5.1.(ii) relies on Lagerberg's superforms [Lag12] on $\mathbb{R}^n$ and its generalization to non-archimedean Berkovich spaces by Chambert-Loir–Ducros [CLD12]; for our purpose we use the variant of Gubler–Künnemann [GK17]. Let us briefly recall this theory with a focus on our application.

For any $p, q \geq 0$, the space of *(p,q)-superforms* on $X^*_{\mathbb{R}}$, denoted by $A^{p,q}(X^*_{\mathbb{R}})$, is the vector space $C^\infty(X^*_{\mathbb{R}}) \otimes (\bigwedge^p X^*_{\mathbb{R}} \otimes \bigwedge^q X^*_{\mathbb{R}})$. For any $x \in X^*_{\mathbb{R}}$, we have a $(1,0)$-superform $\mathrm{d}'x := x \otimes 1$ and a $(0,1)$-form $\mathrm{d}''x := 1 \otimes x$.

Use $(x_1, \ldots, x_r)$ to denote the coordinate functions under the basis $\{e_1, \ldots, e_r\}$ of $X^*_{\mathbb{R}}$. Then a basis of $A^{p,q}(X^*_{\mathbb{R}})$ is

$$\left\{\mathrm{d}'x_{i_1} \wedge \cdots \wedge \mathrm{d}'x_{i_p} \wedge \mathrm{d}''x_{j_1} \wedge \cdots \wedge \mathrm{d}''x_{j_q}\right\}_{1 \leq i_1 < \ldots < i_p \leq r,\ 1 \leq j_1 < \ldots < j_q \leq r}.$$

We have two natural operators

$$\mathrm{d}' \colon A^{p,q}(X^*_{\mathbb{R}}) \to A^{p+1,q}(X^*_{\mathbb{R}}), \qquad \mathrm{d}'' \colon A^{p,q}(X^*_{\mathbb{R}}) \to A^{p,q+1}(X^*_{\mathbb{R}})$$

which are analogues of ∂ and $\overline{\partial}$ in the complex setting. In particular for any smooth function $h \colon X^*_{\mathbb{R}} \to \mathbb{R}$, we have

$$\mathrm{d}'\mathrm{d}''h = \sum_{1 \leq i,j \leq r} \frac{\partial^2 h}{\partial x_i \partial x_j} \mathrm{d}'x_i \wedge \mathrm{d}''x_j. \tag{5.3}$$

There exists an automorphism of algebras

$$J \colon \bigoplus_{p,q} A^{p,q}(X^*_{\mathbb{R}}) \longrightarrow \bigoplus_{p,q} A^{p,q}(X^*_{\mathbb{R}})$$

defined by $J(\mathrm{d}'x) = \mathrm{d}''x$ and $J(\mathrm{d}''x) = -\mathrm{d}'x$ for all $x \in X^*_{\mathbb{R}}$. In particular, J sends (p,q)-superforms to (q,p)-superforms. A (p,p)-superform $\alpha \in A^{p,p}(X^*_{\mathbb{R}})$ is called *symmetric* (resp. *anti-symmetric*) if $J(\alpha) = (-1)^p\alpha$ (resp. if $J(\alpha) = (-1)^{p+1}\alpha$).

An (r,r)-superform $f\mathrm{d}'x_1 \wedge \cdots \wedge \mathrm{d}'x_r \wedge \mathrm{d}''x_1 \wedge \cdots \wedge \mathrm{d}''x_r$ is said to be *positive* if f is a non-negative function on $X^*_{\mathbb{R}}$. A symmetric $(1,1)$-superform $\alpha \in A^{1,1}(X^*_{\mathbb{R}})$ is called *positive* if one of the following equivalent conditions holds (see [Lag12, Lem. 2.2 and Prop. 2.1]):

(a) $\alpha \wedge \alpha_1 \wedge J(\alpha_1) \wedge \cdots \wedge \alpha_{r-1} \wedge J(\alpha_{r-1})$ is a positive (r,r)-superform for all $(1,0)$-superforms $\alpha_1, \ldots, \alpha_{r-1}$,

(b) $(-1)^{(r-1)(r-2)/2}\alpha \wedge \beta \wedge J(\beta)$ is a positive (r,r)-superform for every $(r-1,0)$-superform β,

(c) $\alpha = \sum_{j=1}^r f_j \mathrm{d}'y_j \wedge \mathrm{d}''y_j$ for a basis $\{y_1, \ldots, y_r\}$ of $X^*_{\mathbb{R}}$ and smooth non-negative function f_j.

(c') $\alpha = \sum_{1 \leq i,j \leq r} f_{ij}\mathrm{d}'x_i \wedge \mathrm{d}''x_j$ such that the symmetric matrix of smooth functions $(f_{ij})_{1 \leq i,j \leq r}$ is semi-positive definite pointwise.

For any $\alpha, \beta \in A^{1,1}(X^*_{\mathbb{R}})$ (or $\alpha, \beta \in A^{r,r}(X^*_{\mathbb{R}})$), we will use $\alpha \geq \beta$ to denote that $\alpha - \beta$ is positive. In particular, $\alpha \geq 0$ means that α is positive.

The discussion above is generalized to A_v^{an} [CLD12]. In our paper we use the variant [GK17]. We point out the following facts here. The polarization L on A defines a positive $(1,1)$-superform $c_1(L, \|\cdot\|_v)$ on A_v^{an}, and $c_1(L, \|\cdot\|_v)^{\wedge g}$ is a Chambert-Loir measure on A_v^{an} [Gub10]. In particular,

$$\int_{A_v^{\mathrm{an}}} (f_{0,v} \circ \overline{\mathrm{trop}}) \frac{1}{\deg_L(A)} c_1(L_v^{\mathrm{an}}, \|\cdot\|_v)^{\wedge g} = \int_{\Sigma_v} f_{0,v} \mathrm{d}\mu_v;$$

see for example [CLD12, Cor. 19.2.8]. Hence Proposition 5.1.(iii) can be rewritten as

$$\int_{A_v^{\mathrm{an}}} (f_{0,v} \circ \overline{\mathrm{trop}}) \frac{1}{\deg_L(A)} c_1(L_v^{\mathrm{an}}, \|\cdot\|_v)^{\wedge g} \geq \frac{r_v}{64e\gamma_{r_v}} (\#\Phi_v)^{-1/r_v}. \tag{5.4}$$

We end this recall with the following lemma, whose proof is local and hence is almost a verbalized copy of Lemma 3.4.

Lemma 5.3. *Let α and β be two $(1,1)$-superforms on A_v^{an}, with α positive everywhere. Assume $-\alpha \leq \beta \leq \alpha$. Then for any $k \in \{0, \ldots, g\}$, we have*

$$-\alpha^{\wedge g} \leq \alpha^{\wedge(g-k)} \wedge \beta^{\wedge k} \leq \alpha^{\wedge g}.$$

5.3. **Proof of Proposition 5.1.(ii).** Write $\mathbf{x} = (x_1, \ldots, x_r)$ for the coordinates on $X^*_{\mathbb{R}}$ under our Minkowski-reduced basis $\{e_1, \ldots, e_r\}$ of X^*. Recall $q_0 \colon X^*_{\mathbb{R}} \to \mathbb{R}$, $\mathbf{x} \mapsto \mathbf{x}^{\mathrm{t}} B\mathbf{x}$.

The $(1,1)$-superform $c_1(L, \|\cdot\|_v)$ was computed by Gubler–Künnemann [GK17, Example 8.15] as follows. The pullback $p^*\|\cdot\|_v$ under the uniformization $p \colon E^{\mathrm{an}} \to A_v^{\mathrm{an}}$ from (4.1) satisfies

$$p^*\|\cdot\|_v = e^{-q_0 \circ \mathrm{trop}} \|\cdot\|_{\mathrm{sm}}$$

for a smooth metric $\|\cdot\|_{\mathrm{sm}}$. More precisely, $\|\cdot\|_{\mathrm{sm}} = q^*\|\cdot\|_M$ for the natural isomorphism $p^*L^{\mathrm{an}} = q^*M^{\mathrm{an}}$ from (4.7). Here M is an ample line bundle on B, which is an abelian variety over K_v with good reduction, and $\|\cdot\|_M$ is the model metric. In particular, $c_1(M^{\mathrm{an}}, \|\cdot\|_M) \geq 0$. Therefore $c_1(p^*L^{\mathrm{an}}, \|\cdot\|_{\mathrm{sm}}) \geq 0$. So

$$p^*c_1(L^{\mathrm{an}}, \|\cdot\|_v) = \mathrm{trop}^*\big(\mathrm{d}'\mathrm{d}''(\mathbf{x}^{\mathrm{t}} B\mathbf{x})\big) + c_1(p^*L^{\mathrm{an}}, \|\cdot\|_{\mathrm{sm}}) \geq \mathrm{trop}^*\big((\mathrm{d}'\mathbf{x})^{\mathrm{t}} \wedge B\mathrm{d}''\mathbf{x}\big).$$

Therefore

$$\begin{aligned}
p^*c_1(L^{\mathrm{an}}, N_v^{-\epsilon f_{0,v}\circ\overline{\mathrm{trop}}}\|\cdot\|_v) &= p^*c_1(L^{\mathrm{an}}, \|\cdot\|_v) + \mathrm{trop}^*\left(\epsilon \mathrm{d}'\mathrm{d}''F\right) \qquad \text{by (5.2)}\\
&\geq \mathrm{trop}^*\left((\mathrm{d}'\mathbf{x})^{\mathrm{t}} \wedge B\mathrm{d}''\mathbf{x} + \epsilon \frac{1}{e\gamma_r}\sum_{j=1}^r b_{jj}\psi_0''(x_j)\mathrm{d}'x_j \wedge \mathrm{d}''x_j\right)\\
&\geq \mathrm{trop}^*\left((\mathrm{d}'\mathbf{x})^{\mathrm{t}} \wedge \left(B + \epsilon\frac{1}{e\gamma_r}\mathrm{diag}(b_{11}, \ldots, b_{rr})\right)\mathrm{d}''\mathbf{x}\right),
\end{aligned}$$

where the last inequality holds true because $|\psi_0''|_\infty \leq 1$. So $c_1(L^{\mathrm{an}}, N_v^{-\epsilon f_{0,v}\circ\overline{\mathrm{trop}}}\|\cdot\|_v) \geq 0$ for all $\epsilon \in [-1,1]$ by Lemma 2.1.(ii). This establishes Proposition 5.1.(ii) by the semipositivity criterion [GS23, Thm. 4.10]. Notice that this also induces (5.6). □

5.4. **General semi-stable reduction.** Assume A has semi-stable reduction at v. Now we rewrite Proposition 5.1 and at the same time without assuming the toric part of A (mod v) to be split over k_v.

Let F' be an unramified finite Galois extension of $F := K_v$ such that the toric part of A (mod v) is split over the residue field of F'. General theory of Berkovich spaces asserts that $A_v^{\mathrm{an}} = (A/F)^{\mathrm{an}} = \frac{(A/F')^{\mathrm{an}}}{\mathrm{Gal}(F'/F)}$. The whole discussion in §4 applies to $A_{F'}$, and we will write all data with subscript F'. For the function $f_{0,F'}$ constructed in Proposition 5.1 for $A_{F'}$, the average of $f_{0,F'} \circ \overline{\mathrm{trop}}_{F'} \colon (A/F')^{\mathrm{an}} \to \mathbb{R}$ under $\mathrm{Gal}(F'/F)$ descends to a function

$$f_v \colon A_v^{\mathrm{an}} = (A/F)^{\mathrm{an}} \longrightarrow \mathbb{R}. \tag{5.5}$$

Recall our notation that $\#\Phi_v := \Phi_v(\overline{k}_v)$ is the order of the group scheme Φ_v over k_v. So Proposition 5.1 and the discussion in §5.2 (in particular (5.4)) imply that:

Proposition 5.1′. *The function $f_v \colon A_v^{\mathrm{an}} \longrightarrow \mathbb{R}$ satisfies:*

(i) $f_v(A(K_v)) = 0$,

(ii) *the metric* $N_v^{-\epsilon f_v}\|\cdot\|_v$ *on* L_v^{an} *is relatively semipositive for all* $\epsilon \in [-1,1]$, *and*

$$-c_1(L_v^{\mathrm{an}}, \|\cdot\|_v) \leq \epsilon \mathrm{d}'\mathrm{d}'' f_v \leq c_1(L_v^{\mathrm{an}}, \|\cdot\|_v), \qquad \text{for all } \epsilon \in [-1,1], \tag{5.6}$$

(iii) *for the Minkowski constant* $\gamma_{r_v} \geq 1$ *from Lemma 2.1, we have*

$$\int_{A_v^{\mathrm{an}}} f_v \frac{1}{\deg_L(A)} c_1(L_v^{\mathrm{an}}, \|\cdot\|_v)^{\wedge g} \geq \frac{r_v}{64e\gamma_{r_v}} (\#\Phi_v)^{-1/r_v}. \tag{5.7}$$

6. Statement of a first bound on the number of rational torsion points

Assume A/K have semi-stable reduction. The goal of this section is the state a bound on the number of rational torsion points $A(K)_{\mathrm{tor}} = \{x \in A(K) : \widehat{h}_L(x) = 0\}$ (resp. of the number of small rational points). The proof will be executed in the next section.

6.1. **Statements in the function field case.** These bounds are easier to state in the function field case, which we do now. Let $K = \mathbb{C}(B)$ be a function field. *Assume furthermore A/K does not have good reduction everywhere.*

Theorem 6.1. *There exists a proper abelian subvariety A' of A defined over K such that*

$$\left(\#\left(\frac{A(K)_{\mathrm{tor}} + A'}{A'}\right) \cdot \frac{h^0(A',L)}{h^0(A,L)}\right)^{\frac{1}{g-\dim A'}} \leq 512e \cdot g^2(g+2)\gamma_g \frac{[K:\mathbb{C}(\mathbb{P}^1)] \cdot h_{\mathrm{Fal}}(A)}{\sum_{v \in \mathfrak{Bad}(A/K)} (\#\Phi_v)^{-1/r_v}}. \tag{6.1}$$

In practice, we will prove the following equivalent version of Theorem 6.1. For each abelian subvariety A' of $A_{\overline{K}}$, define $\mathfrak{N}(A')$ to be the number on the left hand side of (6.1). Gaudron–Rémond [GR25, Lem. 6.1] proved[7]

$$\min_{A' \text{ proper abelian subvarieties of } A \text{ defined over } K} \mathfrak{N}(A') = \min_{A' \text{ proper abelian subvarieties of } A_{\overline{K}}} \mathfrak{N}(A').$$

Call this number $\mathfrak{N}$. Then Theorem 6.1 is equivalent to

$$\mathfrak{N} \leq 512e \cdot g^2(g+2)\gamma_g \frac{[K:\mathbb{C}(\mathbb{P}^1)] \cdot h_{\mathrm{Fal}}(A)}{\sum_{v \in \mathfrak{Bad}(A/K)} (\#\Phi_v)^{-1/r_v}}. \tag{6.2}$$

We can also handle small rational points. Indeed, let

$$\Gamma' := \left\{ x \in A(K) : \widehat{h}_L(x) \leq \frac{1}{2024e \cdot g^2\gamma_g[K:\mathbb{C}(\mathbb{P}^1)]} \sum_{v \in \mathfrak{Bad}(A/K)} (\#\Phi_v)^{-1/r_v} \right\}, \tag{6.3}$$

Theorem 6.2. *There exists a proper abelian subvariety A' of A defined over K such that*

$$\left(\#\left(\frac{\Gamma' + A'}{A'}\right) \cdot \frac{h^0(A',L)}{h^0(A,L)}\right)^{\frac{1}{g-\dim A'}} \leq 1024e \cdot g^2(g+2)\gamma_g \frac{[K:\mathbb{C}(\mathbb{P}^1)] \cdot h_{\mathrm{Fal}}(A)}{\sum_{v \in \mathfrak{Bad}(A/K)} (\#\Phi_v)^{-1/r_v}}.$$

As for Theorem 6.1, we have an equivalent version of Theorem 6.2: Define

$$\mathfrak{N}' := \min_{A' \text{ proper abelian subvarieties of } A_{\overline{K}}} \left(\#\left(\frac{\Gamma' + A'}{A'}\right) \cdot \frac{h^0(A',L)}{h^0(A,L)}\right)^{\frac{1}{g-\dim A'}}.$$

Then in defining $\mathfrak{N}'$, it suffices to let A' run over all proper abelian subvarieties of A defined over K by Gaudron–Rémond [GR25, Lem. 6.1]. So Theorem 6.2 is equivalent to

$$\mathfrak{N}' \leq 1024e \cdot g^2(g+2)\gamma_g \frac{[K:\mathbb{C}(\mathbb{P}^1)] \cdot h_{\mathrm{Fal}}(A)}{\sum_{v \in \mathfrak{Bad}(A/K)} (\#\Phi_v)^{-1/r_v}}. \tag{6.4}$$

[7]Their proof is executed when K is a number field, but holds verbally for $K = \mathbb{C}(B)$.

6.2. **Division of $A(K)_{\mathrm{tor}}$ in the number field case.** Let K be a number field. The statements of the corresponding upper bounds are then more complicated. We need to decompose $A(K)_{\mathrm{tor}}$ into $(2g)^g(g+2)^g$ subsets and will prove a bound for each one of them. Let us explain this division now.

Use the notation from §3. In particular for each $\sigma'\colon K \hookrightarrow \mathbb{C}$ and each isogeny $\rho_{\sigma'}\colon A_{\sigma'} \to A_{0,\sigma'}$ to a principally polarized abelian variety $(A_{0,\sigma'}, L_{0,\sigma'})$ with $\rho_{\sigma'}^* L_{0,\sigma'} \simeq L_{\sigma'}$, we have a Siegel reduced matrix $\tau_{0,\sigma'}$ which is the period for $A_{0,\sigma'}$ given by $L_{0,\sigma'}$.

Fix a pair (σ, ρ_σ) such that $\mathrm{Tr}(\mathrm{Im}\tau_{0,\sigma})$ is maximal among all embeddings $K \hookrightarrow \mathbb{C}$ and all such $A_{\sigma'} \to A_{0,\sigma'}$; in particular $\alpha_\sigma(A,L) \le \mathrm{Tr}(\mathrm{Im}\tau_{0,\sigma})$ by the definition (1.8) of $\alpha_\sigma(A,L)$. We have the uniformization $u_{0,\sigma}\colon \mathbb{C}^g \to A_{0,\sigma}(\mathbb{C})$, and the Betti coordinates $\beta_\sigma\colon \mathbb{R}^g \times \mathbb{R}^g \to \mathbb{C}^g$ sending $(\mathbf{a},\mathbf{b}) \mapsto \mathbf{a} + \tau_{0,\sigma}\mathbf{b}$; see (3.1), (3.2) and (3.3).

Consider all the vectors in $[0,1)^g$ whose coordinates are integer multiples of $1/2g(g+2)$. There are $N_g := (2g)^g(g+2)^g$ such vectors, which we name by $\mathbf{e}_1, \ldots, \mathbf{e}_{N_g}$. Then $A_{0,\sigma}(\mathbb{C})$ can be decomposed into the disjoint union of

$$\Box_j := (u_{0,\sigma} \circ \beta_\sigma)\left([0,1]^g \times \left([\frac{1}{4g(g+2)}, \frac{3}{4g(g+2)})^g + \mathbf{e}_j\right)\right) \subseteq A_{0,\sigma}(\mathbb{C}). \tag{6.5}$$

Later on we need to relate $\Box_j$ to the wider strips defined in (3.4). For any $i \in \{0, \ldots, g\}$, it is not hard to check that the sum (by convention, the sum of 0-copies of $\Box_j$ is $\{0\}$)

$$\underbrace{\Box_j + \cdots + \Box_j}_{i\text{-copies}} := \{x_1 + \cdots + x_i : x_1, \ldots, x_i \in \Box_j\}$$

is contained in a subset $\Box'_{\sigma,q}$ in $A_{0,\sigma}(\mathbb{C})$ of the form (3.4). We thus obtain at most $g+1$ strips of $A_{0,\sigma}(\mathbb{C})$ of the form (3.4). Let their union be $\Box'_\sigma$.

But in fact, we can do better. It is not hard to show that

$$\underbrace{\Box_j + \cdots + \Box_j}_{i\text{-copies}} \subseteq (u_{0,\sigma} \circ \beta_\sigma)\left([0,1]^g \times \left([\frac{1}{4(g+2)}, \frac{3}{4(g+2)})^g + \mathbf{e}'_k\right)\right) \subsetneq \Box'_{\sigma,q}. \tag{6.6}$$

This finer observation will be needed for the archimedean estimates later.

6.3. **Statements of the bounds in the number field case.** Now we are ready to state the bounds for the number field case. Below let K be a number field. Use the notation N_v from §2.1. Let

$$M'_K := \mathfrak{Bad}(A/K) \bigcup \{\sigma\},$$

where $\sigma\colon K \hookrightarrow \mathbb{C}$ is from §6.2. Define

$$\widetilde{I}_v(A) := \begin{cases} (\#\Phi_v)^{-1/r_v} & \text{if } v \in \mathfrak{Bad}(A/K) \\ \mathrm{Tr}(\mathrm{Im}\tau_{0,\sigma}) \ge \alpha_\sigma(A,L) & \text{if } v = \sigma. \end{cases} \tag{6.7}$$

6.3.1. *On rational torsion points.* For each $j \in \{1, \ldots, (2g)^g(g+2)^g\}$, define

$$\Gamma_j := \{x \in A(K)_{\mathrm{tor}} : x_\sigma \in \rho_\sigma^{-1}(\Box_j)\}, \tag{6.8}$$

where we use the division and notation from §6.2.

Theorem 6.1′. *For any $g \in \mathbb{Z}_{\ge 1}$, there exists an explicit constant $C_1 = C_1(g) > 0$ defined in* (7.26) *such that: For each $j \in \{1, \ldots, (2g)^g(g+2)^g\}$, there exists a proper abelian subvariety A' of A, defined over K, such that*

$$\left(\#\left(\frac{\Gamma_j + A'}{A'}\right) \cdot \frac{h^0(A',L)}{h^0(A,L)}\right)^{\frac{1}{g - \dim A'}} \le C_1 \frac{\max\{h_{\mathrm{Fal}}(A), 1\} + \log h^0(A,L) + \log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v \in M'_K} \widetilde{I}_v(A) \log N_v}. \tag{6.9}$$

For the convenience of the proof, we will make use of the following notations. For each abelian subvariety A' of $A_{\overline{K}}$, define $\mathfrak{N}_j(A')$ to be the number on the left hand side of (6.9). Gaudron–Rémond [GR25, Lem. 6.1] proved

$$\min_{A' \text{ proper abelian subvarieties of } A \text{ defined over } K} \mathfrak{N}_j(A') = \min_{A' \text{ proper abelian subvarieties of } A_{\overline{K}}} \mathfrak{N}_j(A').$$

Call this number $\mathfrak{N}_j$. Then the conclusion of Theorem 6.1′ is equivalent to: For each $j \in \{1, \ldots, (2g)^g(g+2)^g\}$, we have

$$\mathfrak{N}_j \leq C_1 \frac{\max\{h_{\mathrm{Fal}}(A), 1\} + \log h^0(A, L) + \log[K : \mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M'_K} \widetilde{I}_v(A) \log N_v}. \tag{6.10}$$

6.3.2. *On small rational points.* We can also handle small rational points. Indeed, for each $j \in \{1, \ldots, (2g)^g(g+2)^g\}$, let

$$\Gamma'_j := \left\{ x \in A(K) : x_\sigma \in \rho_\sigma^{-1}(\square_j),\ \widehat{h}_L(x) \leq \frac{\pi}{32 \cdot 16e \cdot g^3(g+1)^3(g+2)^3(100g+274)\gamma_g} \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M'_K} \widetilde{I}_v(A) \log N_v \right\}. \tag{6.11}$$

We will prove that Theorem 6.1′ holds true verbally with Γ_j replaced by Γ'_j, up to modifying the constants.

Theorem 6.2′. *For any $g \in \mathbb{Z}_{\geq 1}$, there exists a constant $C'_1 = C'_1(g) > 0$ such that: For each $j \in \{1, \ldots, (2g)^g(g+2)^g\}$, there exists a proper abelian subvariety A' of A, defined over K, such that*

$$\left(\#\left(\frac{\Gamma'_j + A'}{A'} \right) \cdot \frac{h^0(A', L)}{h^0(A, L)} \right)^{\frac{1}{g - \dim A'}} \leq C'_1 \frac{\max\{h_{\mathrm{Fal}}(A), 1\} + \log h^0(A, L) + \log[K : \mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M'_K} \widetilde{I}_v(A) \log N_v}. \tag{6.12}$$

In fact we can take $C'_1(g) = 3C_1(g)$, with $C_1(g)$ from Theorem 6.1′. As for Theorem 6.1′, for each abelian subvariety B of $A_{\overline{K}}$, define $\mathfrak{N}'_j(A')$ to be the number on the left hand side of (6.12). Then

$$\min_{A' \text{ proper abelian subvarieties of } A \text{ defined over } K} \mathfrak{N}'_j(A') = \min_{A' \text{ proper abelian subvarieties of } A_{\overline{K}}} \mathfrak{N}'_j(A')$$

by Gaudron–Rémond [GR25, Lem. 6.1]. Call this number $\mathfrak{N}'_j$. Then the conclusion of Theorem 6.2′ is equivalent to: For each $j \in \{1, \ldots, (2g)^g(g+2)^g\}$, we have

$$\mathfrak{N}'_j \leq C'_1 \frac{\max\{h_{\mathrm{Fal}}(A), 1\} + \log h^0(A, L) + \log[K : \mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M'_K} \widetilde{I}_v(A) \log N_v}. \tag{6.13}$$

7. Proofs of the bounds in §6

The goal of this section is to prove the bounds in §6. We will focus on rational torsion points (*i.e.* Theorem 6.1 and Theorem 6.1′), and point out how to modify the proof to handle rational points of small height (*i.e.* Theorem 6.2 and Theorem 6.2′). Through the whole section, we make the following assumption:

`(Hyp)` A/K has semi-stable reduction, and not everywhere good reduction if $K = \mathbb{C}(B)$.

A large part of the proof works both in the number field and in the function field case. To ease the writing, we will use some unified notations. Let $\overline{L}$ be the canonical adelic extension of L. Then $\overline{L}$ is a metrized line bundle on the Berkovich analytification of A, and write $\|\cdot\|_v$ for the metric at v for each $v \in M_K$. Moreover, $\widehat{h}_L = h_{\overline{L}}$.

When K is a function field $\mathbb{C}(B)$, denote by $M'_K := \mathfrak{Bad}(A/K)$ and by $\widetilde{I}_v(A) := (\#\Phi_v)^{-1/r_v}$ for $v \in M'_K$. Also for each $v \in M'_K$, recall the convention $N_v = e$ from §2.1.

When K is a number field, recall that $M'_K = \mathfrak{Bad}(A/K) \bigcup \{\sigma\}$ for the embedding $\sigma\colon K \hookrightarrow \mathbb{C}$ chosen in §6.2 and recall the local invariants $\widetilde{I}_v(A)$ for each $v \in M'_K$ defined in (6.7). For each $v \in M'_K$, the number N_v is defined in §2.1.

For each $j \in \{1, \ldots, (2g)^g(g+2)^g\}$, denote by

$$\Gamma_j := \begin{cases} A(K)_{\mathrm{tor}} & \text{if } K \text{ is a function field} \\ \text{the subset (6.8) of } A(K)_{\mathrm{tor}} & \text{if } K \text{ is a number field.} \end{cases} \tag{7.1}$$

This set gives rise to a number $\mathfrak{N}$ defined below Theorem 6.1 (in the function field case) and a number $\mathfrak{N}_j$ defined below Theorem 6.1′ (in the number field case). To unify notation, we shall use $\mathfrak{N}_j$ to denote this number in both cases. Then proving Theorem 6.1 and Theorem 6.1′ is equivalent to proving a suitable upper bound of $\mathfrak{N}_j$; see (6.2) and (6.10).

When handling rational points of small height (*i.e.* Theorem 6.2 and Theorem 6.2′), we use the following unified notations: For each $j \in \{1, \ldots, (2g)^g(g+2)^g\}$, we use (i) Γ'_j denotes the set (6.3) when K is a function field and denotes the set (6.11) when K is a number field; (ii) $\mathfrak{N}'_j$ to denote the number $\mathfrak{N}'$ defined above (6.4) when K is a function field and to denote the number $\mathfrak{N}'_j$ defined above (6.13) when K is a number field.

7.1. **Perturbed adelic line bundle and existence of small sections.** Recall (Hyp) that A has semi-stable reduction over K. Consider the tuple of functions $(f_v)_{v \in M'_K}$, where

- For $v = \sigma$, set f_v to be the function f_σ from Proposition 3.1, with $\Box'_\sigma$ defined above (6.6);
- For $v \in \mathfrak{Bad}(A/K)$, take the function f_v defined in (5.5).

Next let us normalize the tuple to be

$$\mathbf{f} := (f_v - \widetilde{J}_v(A))_{v \in M'_K}, \qquad \text{with } \widetilde{J}_v(A) := \frac{1}{2} \int_{A_v^{\mathrm{an}}} f_v \cdot \frac{1}{\deg_L(A)} c_1(L, \|\cdot\|_v)^{\wedge g}. \tag{7.2}$$

Below we shall work with the following *twist* of $\overline{L}$. For any $\epsilon \in \mathbb{R}$, we have the adelic line bundle

$$\overline{L}(\epsilon \mathbf{f})$$

whose metric at $v \in \mathfrak{Bad}(A/K)$ is $N_v^{-\epsilon(f_v - \widetilde{J}_v(A))}\|\cdot\|_v$, at σ (and $\overline{\sigma}$) is $e^{-\epsilon(f_\sigma - \widetilde{J}_\sigma(A))}\|\cdot\|_\sigma$, and at all the other v's are $\|\cdot\|_v$.

Before moving on, let us point out the following fact. By definition (6.7) of $\widetilde{I}_v(A)$, the bounds (3.6) for $v = \sigma$ and (5.7) for $v \in \mathfrak{Bad}(A/K)$, we have

$$\widetilde{J}_v(A) \geq \frac{\pi}{16e \cdot (g+1)^3(g+2)^3(100g+274)\gamma_g} \widetilde{I}_v(A) > 0 \qquad \text{for all } v \in M'_K. \tag{7.3}$$

In the function field case, we have the better bound

$$\widetilde{J}_v(A) \geq \frac{g}{128e\gamma_g} \widetilde{I}_v(A) > 0 \qquad \text{for all } v \in \mathfrak{Bad}(A/K). \tag{7.4}$$

Proposition 7.1. *Let $\epsilon = 1/(8g)$. Then the adelic line bundle $\overline{L}(\epsilon\mathbf{f})$ is big.*

As a consequence, there exists a constant $m_0 > 0$ with the following property: there exists a non-zero small global section $s \in \widehat{H}^0(A, m\overline{L}(\epsilon\mathbf{f}))$ for all $m \geq m_0$.

By "small", we mean $\|s\|_{m\overline{L}(\epsilon\mathbf{f}),v,\sup} \leq 1$ for all $v \in M_K$. The constant m_0 depends on many choices. But this will not cause a problem, because eventually we will let $m \to \infty$.

Proof. To ease notation, we denote $\mathbf{f}_0 = (f_v)_{v\in M'_K}$, and define the adelic line bundle $\overline{L}(\epsilon\mathbf{f}_0)$ similarly. By Proposition 5.1′.(iii) and Corollary 3.1.(ii), $\overline{L}(\epsilon\mathbf{f})$ is nef. So we can apply Arithmetic Hilbert–Samuel to $\overline{L}(\epsilon\mathbf{f})$ and get

$$\begin{aligned}\mathrm{vol}(\overline{L}(\epsilon\mathbf{f})) = \overline{L}(\epsilon\mathbf{f})^{g+1} &= \overline{L}(\epsilon\mathbf{f}_0)^{g+1} - \sum_{v\in M'_K}\int_{A_v^{\mathrm{an}}}(g+1)\epsilon\widetilde{J}_v(A)c_1(\overline{L}(\epsilon\mathbf{f}_0))^{\wedge g} \\ &= \overline{L}(\epsilon\mathbf{f}_0)^{g+1} - (g+1)\epsilon\sum_{v\in M'_K}\widetilde{J}_v(A)\int_{A_v^{\mathrm{an}}}c_1(\overline{L}(\epsilon\mathbf{f}_0))^{\wedge g} \qquad (7.5)\\ &= \overline{L}(\epsilon\mathbf{f}_0)^{g+1} - (g+1)\epsilon\deg_L(A)\sum_{v\in M'_K}\widetilde{J}_v(A) \quad \text{by Chern's integration formula.}\end{aligned}$$

We can further expand

$$\overline{L}(\epsilon\mathbf{f}_0)^{g+1} = \overline{L}^{g+1} + \sum_{k=0}^{g}\binom{g+1}{k+1}\overline{L}^{g-k}\cdot\overline{\mathcal{O}}(\epsilon\mathbf{f}_0)^{k+1} = \sum_{k=0}^{g}\binom{g+1}{k+1}\overline{L}^{g-k}\cdot\overline{\mathcal{O}}(\epsilon\mathbf{f}_0)^{k+1}.$$

Next let us compute the sum on the right hand side. Notice that $\binom{g+1}{k+1} = \frac{g+1}{k+1}\binom{g}{k}$. In the computation below, we use the uniform notation $\mathrm{dd^c}$ for both archimedean and non-archimedean places, which means $\mathrm{d'd''}$ for non-archimedean places. So

$$\begin{aligned}\overline{L}(\epsilon\mathbf{f}_0)^{g+1} &= (g+1)\epsilon\sum_{k=0}^{g}\sum_{v\in M'_K}\frac{1}{k+1}\binom{g}{k}\int_{A_v^{\mathrm{an}}}f_vc_1(L,\|\cdot\|_v)^{\wedge(g-k)}\wedge(\epsilon\mathrm{dd^c}f_v)^{\wedge k}\\ &= (g+1)\epsilon\sum_{v\in M'_K}\Bigg(\sum_{k=0}^{g}\binom{g}{k}\int_{A_v^{\mathrm{an}}}f_vc_1(L,\|\cdot\|_v)^{\wedge(g-k)}\wedge(\epsilon\mathrm{dd^c}f_v)^{\wedge k}\\ &\qquad -\sum_{k=1}^{g}\frac{k}{k+1}\binom{g}{k}\int_{A_v^{\mathrm{an}}}f_vc_1(L,\|\cdot\|_v)^{\wedge(g-k)}\wedge(\epsilon\mathrm{dd^c}f_v)^{\wedge k}\Bigg)\\ &= (g+1)\epsilon\sum_{v\in M'_K}\Bigg(\int_{A_v^{\mathrm{an}}}f_v\Big(c_1(L,\|\cdot\|_v)+\epsilon\mathrm{dd^c}f_v\Big)^{\wedge g}\\ &\qquad -\sum_{k=1}^{g}\frac{k\epsilon^k}{k+1}\binom{g}{k}\int_{A_v^{\mathrm{an}}}f_vc_1(L,\|\cdot\|_v)^{\wedge(g-k)}\wedge(\mathrm{dd^c}f_v)^{\wedge k}\Bigg).\end{aligned}$$

Let us estimate the two terms in the last expression.

For the first term, we use $-\epsilon c_1(L,\|\cdot\|_v) \le \epsilon\mathrm{dd^c}f_v \le \epsilon c_1(L,\|\cdot\|_v)$ by Remark 3.2 when v is archimedean and by (5.6) when v is non-archimedean. So $(1+\epsilon)c_1(L,\|\cdot\|_v) \ge c_1(L,\|\cdot\|_v)+\epsilon\mathrm{dd^c}f_v \ge (1-\epsilon)c_1(L,\|\cdot\|_v) \ge 0$. Moreover $f_v \ge 0$. So

$$\int_{A_v^{\mathrm{an}}}f_v\Big(c_1(L,\|\cdot\|_v)+\epsilon\mathrm{dd^c}f_v\Big)^{\wedge g} \ge (1-\epsilon)^g\int_{A_v^{\mathrm{an}}}f_vc_1(L,\|\cdot\|_v)^{\wedge g} = 2(1-\epsilon)^g\deg_L(A)\widetilde{J}_v(A). \qquad (7.6)$$

For the second term, by Lemma 5.3 in the non-archimedean case and Lemma 3.4 in the archimedean case, we have $c_1(L,\|\cdot\|_v)^{\wedge(g-k)}\wedge(\mathrm{dd^c}f_v)^{\wedge k} \le c_1(L,\|\cdot\|_v)^{\wedge g}$. Hence (because $\epsilon\in(0,1)$, $f_v\ge 0$, and $\frac{k}{k+1}<1$)

$$\begin{aligned}\sum_{k=1}^{g}\frac{k\epsilon^k}{k+1}\binom{g}{k}\int_{A_v^{\mathrm{an}}}f_vc_1(L,\|\cdot\|_v)^{\wedge(g-k)}\wedge(\mathrm{dd^c}f_v)^{\wedge k} &\le \sum_{k=1}^{g}\epsilon^k\binom{g}{k}\int_{A_v^{\mathrm{an}}}f_vc_1(L,\|\cdot\|_v)^{\wedge g} \qquad (7.7)\\ &= 2((1+\epsilon)^g-1)\deg_L(A)\widetilde{J}_v(A).\end{aligned}$$

Taking (7.6) - (7.7), we get a lower bound for $\overline{L}(\epsilon\mathbf{f}_0)$. Then we get from (7.5)

$$\mathrm{vol}(\overline{L}(\epsilon\mathbf{f})) = \overline{L}(\epsilon\mathbf{f})^{g+1} \geq (g+1)\epsilon \deg_L(A)\big(2(1-\epsilon)^g - 2(1+\epsilon)^g + 1\big) \sum_{v\in M'_K} \widetilde{J}_v(A). \tag{7.8}$$

By our choice $\epsilon = \frac{1}{8g}$, we have $2(1-\epsilon)^g - 2(1+\epsilon)^g + 1 > 0$. Moreover $M'_K \neq \emptyset$ by assumption (either K is a number field, or A/K does not have good reduction everywhere). So $\sum_{v\in M'_K} \widetilde{J}_v(A) > 0$ by (7.3). We are done. $\square$

7.2. **Choice of an auxiliary point.** We need the following result on zero estimates proved by Nakamaye [Nak07]. Recall the notations Γ_j and $\mathfrak{N}_j$ from the paragraph around (7.1). For each $i \in \{1,\ldots,g\}$, set $\Gamma_j[i] := \{x_1 + \cdots + x_i : x_1,\ldots,x_i \in \Gamma_j$ and set $\Gamma_j[0] := \{0\}$. Then $\Gamma_j[i] \subseteq A(K)_{\mathrm{tor}}$ because $\Gamma_j \subseteq A(K)_{\mathrm{tor}}$.

Proposition 7.2. *Assume $m > \mathfrak{N}_j/g$ and $s \in H^0(A, m\overline{L}(\epsilon\mathbf{f}))$ with $\epsilon = 1/(8g)$. Then there exists a point $x \in \bigcup_{i=0}^g \Gamma_j[i]$ such that the vanishing order ℓ of s at x satisfies*

$$\ell \leq g\left(\frac{gm}{\mathfrak{N}_j} + 1\right).$$

Here, the vanishing order ℓ is characterized by: $s \in H^0(A, \mathcal{I}_x^\ell + mL)$ but $s \notin H^0(A, \mathcal{I}_x^{\ell+1} + mL)$, where $\mathcal{I}_x$ is the ideal sheaf of $\{x\}$.

Proof. Assume that the conclusion does not hold true. Apply the main result of [Nak07] to the set Γ_j (so $G = A_{\overline{K}}$, $d = g$, $\Lambda = \mathrm{Lie}\, A$, and the line bundle mL). Then there exists a proper abelian subvariety B of $A_{\overline{K}}$ such that

$$\left(\frac{gm}{\mathfrak{N}_j} + 1\right)^{g-\dim A_0} \#\left(\frac{\Sigma + A_0}{A_0}\right) \deg_{mL} A_0 \leq \deg_{mL} A.$$

But $\deg_{mL}(A_0) = (\dim A_0)! m^{\dim A_0} h^0(A_0, L)$, $\deg_{mL} A = g! m^g h^0(A, L)$, and $\mathfrak{N}_j(A_0)^{g-\dim A_0} = \#((\Sigma + A_0)/A_0) \cdot h^0(A_0, L)/h^0(A, L)$. So

$$\left(\frac{gm}{\mathfrak{N}_j} + 1\right)^{g-\dim A_0} \left(\frac{\mathfrak{N}_j(A_0)}{m}\right)^{g-\dim A_0} (\dim A_0)! \leq g!.$$

But $\mathfrak{N}_j \leq \mathfrak{N}_j(A_0)$ by definition. So $g^{g-\dim A_0}(\dim A_0)! < g!$, which is impossible. Hence we get a contradiction, and we are done. $\square$

Remark 7.3. *If we work with the set Γ'_j from above §7.1, then the conclusion becomes: we can find a point $x \in \bigcup_{i=0}^g \Gamma'_j[i]$ such that the vanishing order of s at x is $\leq g^2 m/\mathfrak{N}'_j + g$, with $\Gamma'_j[i] := \{x_1 + \cdots + x_i : x_1,\ldots,x_i \in \Gamma'_j\}$.*

7.3. **Jet estimates.** From now on, fix $\epsilon = 1/(8g)$. For each $m > \max\{m_0, \mathfrak{N}_j/g\}$ (with m_0 from Proposition 7.1), take a non-zero small global section $s \in H^0(A, m\overline{L}(\epsilon\mathbf{f}))$ from Proposition 7.1 and then an auxiliary point $x \in \bigcup_{i=0}^g \Gamma_j[i]$ from Proposition 7.2. When K is a number field, the discussion around (6.6) implies (see (3.1) for the definition of ρ_σ)

$$\rho_\sigma(x_\sigma) \in \square'_\sigma,$$

and better, $\rho_\sigma(x_\sigma)$ is in one of the subsets from (6.6); here x_σ is $x \in A(K) \hookrightarrow A_\sigma(\mathbb{C})$.

Now set

$$J := \mathrm{Sym}^\ell((\mathrm{Lie}\, A)^\vee) \otimes L_x^{\otimes m} = x^*\mathrm{Sym}^\ell((\mathrm{Lie}\, A)^\vee) \otimes L_x^{\otimes m} \tag{7.9}$$

which is a K-vector space. We shall endow J with the structure of adelic space, *i.e.* a metric on $J \otimes_K K_v$ at each $v \in M_K$; it suffices to do this for $\mathrm{Lie}\, A$ and for L_x.

The adelic line bundle $\overline{L}$ makes $L_x = x^*L$ into a natural adelic space $\overline{L}_x$. The structure of adelic space $\overline{\mathrm{Lie}\, A}$ on $\mathrm{Lie}\, A$ is given as follows. At $v \in M_{K,\infty}$ corresponding to $\sigma\colon K \hookrightarrow \mathbb{C}$, define $\|z\|_\sigma^2 := H(z,z)$ for any $z \in \mathrm{Lie}\, A_\sigma$ where H is the Riemann form of L_σ. At $v \in M_K^0$, consider the Néron model $\mathcal{A}/\mathcal{O}_K$ of A with L extended to $\mathcal{L}$ on $\mathcal{A}$. Then for any $s \in \mathrm{Lie}\, A$ define $\|s\|_v := \min\{|\lambda|_v : \lambda \in K_v \setminus \{0\},\ s/\lambda \in \mathrm{Lie}(\mathcal{A}/\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{O}_v\}$;

The *ℓ-th jet* of s at x, denoted by $\mathrm{jet}^\ell s(x)$, is the under of s under the composition

$$H^0(A, \mathcal{I}_x^\ell + mL) \to H^0(\mathrm{Spec}\overline{K}, x^*\Big(\frac{\mathcal{I}_x^\ell}{\mathcal{I}_x^{\ell+1}} + mL\Big)) \to H^0(\mathrm{Spec}\overline{K}, x^*\mathrm{Sym}^\ell(\Omega_A + mL)).$$

Then $\mathrm{jet}^\ell s(x)$ is an element in J, which is non-zero by the definition of vanishing order. Let us estimate $\log \|\mathrm{jet}^\ell s(x)\|_v$ in terms of the number $\widetilde{J}_v(A)$ (see below (7.2)) at $v \in M_K$, where by abuse of notation $\|\cdot\|_v$ denotes the metric on $J \otimes_K K_v$.

7.3.1. *Non-archimedean places.*

Lemma 7.4. *Assume v is a non-archimedean place of K. Then*

$$\log \|\mathrm{jet}^\ell s(x)\|_v \leq -m\epsilon \widetilde{J}_v(A) \log N_v.$$

Proof. We have

$$\|\mathrm{jet}^\ell s(x)\|_v \leq \|s(x)\|_v$$

by [GR14b, §6.7.1]. For the sake of completeness we give a sketch here. Consider the Néron model $\mathcal{A}/\mathcal{O}_K$ of A and extend L to $\mathcal{L}$ on $\mathcal{A}$. Let $\underline{x}$ be the Zariski closure of $\{x\}$ in $\mathcal{A}$; then $\underline{x}$ is a section of $\mathcal{A}/\mathcal{O}_K$. At the non-archimedean place v of K, the section $s \in H^0(A, mL) \hookrightarrow H^0(A, mL) \otimes_K K_v$ can be written as $\lambda \cdot s_0$ with $s_0 \in H^0(\mathcal{A}, m\mathcal{L}) \otimes_{\mathcal{O}_K} \mathcal{O}_v$ of norm 1. So $\mathrm{jet}^\ell s(x)$ equals $\lambda \cdot \mathrm{jet}^\ell s_0(\underline{x}_v)$. Therefore by definition of the adelic structure on J, we have $\|\mathrm{jet}^\ell s(x)\|_v \leq |\lambda|_v = \|s(x)\|_v$.

But $x \in A(K_v)$, so $f_v(x) = 0$ by Proposition 5.1′.(i). Since s is a small section of $m\overline{L}(\epsilon \mathbf{f})$, we have $\|s(x)\|_v N_v^{-m\epsilon(f_v(x) - \widetilde{J}_v(A))} \leq 1$. So

$$\|s(x)\|_v \leq N_v^{-m\epsilon \widetilde{J}_v(A)}.$$

We are done by combining the two inequalities above. □

7.3.2. *Archimedean places.* If K is a number field, we also need estimates at $v \in M_{K,\infty}$.

Lemma 7.5. *Let $\sigma'\colon K \hookrightarrow \mathbb{C}$ be an archimedean place of K which is not σ or $\overline{\sigma}$. Then*

$$\log \|\mathrm{jet}^\ell s(x)\|_{\sigma'} \leq \frac{g^2 m}{2\mathfrak{N}_j}\left(1 + \log^+ \frac{\pi \mathfrak{N}_j}{2g^2}\right) + o(m).$$

Proof. We follow the treatment of [GR25, Prop. 6.8]. By [Gau19, Rmk. 3.2.4], we have

$$\|\mathrm{jet}^\ell s(x)\|_{\sigma'} \leq \|s\|_{\sigma', L^2} \left(\frac{e\pi m}{\ell}\right)^{\ell/2} e^{o(m)}.$$

The trivial bound $\|s\|_{\sigma', L^2} \leq \|s\|_{\sigma', \sup}$ implies $\|s\|_{\sigma', L^2} \leq 1$. The function $x \mapsto (e\pi m/x)^x$ takes its maximum at $x = \pi m$. Hence we are done because $\|s\|_{\sigma', L^2} \leq \|s\|_{\sigma', \sup}$, which is ≤ 1 by our choice of s. The result follows from Proposition 7.2. □

It remains to do the estimates at σ and $\overline{\sigma}$. Below we do it for σ, and the conclusion at $\overline{\sigma}$ is the same. Let $\gamma_g \geq 1$ be the Minkowski constant from Lemma 2.1. Set

$$r_\sigma := \min\left\{\frac{1}{8e\gamma_g(g+2)}, \frac{2}{\pi}\left(\frac{\epsilon\widetilde{J}_\sigma(A)}{2}\right)^{1/2}\right\} > 0. \tag{7.10}$$

Use the notation from §3, which we reformulate now. For a Siegel reduced matrix in the Siegel upper half space $\tau_{0,\sigma}$, we fine a sub-lattice $\Lambda \subseteq \mathbb{Z}^g + \tau_{0,\sigma}\mathbb{Z}^g = \beta_\sigma(\mathbb{Z}^{2g})$ such that $A_\sigma(\mathbb{C}) = \mathbb{C}^g/\Lambda$. We then have an isogeny $\rho_\sigma \colon A_\sigma \to A_{0,\sigma} = \mathbb{C}^g/(\mathbb{Z}^g + \tau_{0,\sigma}\mathbb{Z}^g)$. Moreover, there exists a line bundle L_0 on $A_{0,\sigma}$ inducing principal polarization, and $(L_\sigma, \|\cdot\|_\sigma) = \rho_\sigma^*(L_{0,\sigma}, \|\cdot\|_{0,\sigma})$. Thus the Hermitian form associated with $c_1(L_\sigma, \|\cdot\|_\sigma)$ is

$$H \colon \mathbb{C}^g \times \mathbb{C}^g \longrightarrow \mathbb{C}, \quad (z, z') \longmapsto \overline{z}^{\mathrm{t}}\mathrm{Im}(\tau_{0,\sigma})^{-1}z'.$$

For the uniformizations from (3.2), we have

$$u_{0,\sigma} \colon \mathbb{C}^g \xrightarrow{u_\sigma} A_\sigma(\mathbb{C}) \xrightarrow{\rho_\sigma} A_{0,\sigma}(\mathbb{C}).$$

We have chosen a fundamental domain $\bigsqcup_{l=1}^{h^0(A,L)}(\mathbf{v}_l + \beta_\sigma([0,1)^{2g})) \subseteq \mathbb{C}^g$ for u_σ above (3.5). For our auxiliary point x_σ (which is $x \in A(K) \hookrightarrow A_\sigma(\mathbb{C})$), let $z_\sigma \in u_\sigma^{-1}(x_\sigma)$ be in this fundamental domain. Then $z_\sigma \in \mathbf{v}_l + \beta_\sigma([0,1)^{2g})$ for a fixed $l \in \{1, \ldots, h^0(A, L)\}$.

Better, recall that $\rho_\sigma(x_\sigma) \in A_{0,\sigma}(\mathbb{C})$ is in one of the subsets from (6.6) (see above (7.9)), and hence is in $\square'_{\sigma,q}$ for some $q \in \{1, \ldots, N_g = 2^g(g+2)^g\}$. The definition (3.4) of $\square'_{\sigma,q}$ then yields

$$z_\sigma \in \mathbf{v}_l + \beta_\sigma\Big([0,1]^g \times \big([\frac{1}{4(g+2)}, \frac{3}{4(g+2)})^g + \mathbf{e}_q\big)\Big)$$

By Proposition 3.1.(i) and (3.5), $f_\sigma \circ u_\sigma$ vanishes on (the wider)

$$\mathbf{v}_l + \beta_\sigma\Big([0,1]^g \times \big([0, \frac{1}{g+2}]^g + \mathbf{e}_q\big)\Big).$$

Denote by $Y_\sigma := \mathrm{Im}(\tau_{0,\sigma})$. We claim that

$$(f_\sigma \circ u_\sigma)\big(z_\sigma + Y_\sigma^{1/2}\zeta\big) = 0 \qquad \text{for all } |\zeta| \leq r_\sigma. \tag{7.11}$$

Indeed, it suffices to show: each of the last g coordinates of $\beta_\sigma^{-1}(Y_\sigma^{1/2}\zeta) = Y_\sigma^{-1/2}\mathrm{Im}(\zeta)$ is $< \frac{1}{4(g+2)}$ if $|\zeta| \leq r_\sigma$, for the Betti coordinate map $\beta_\sigma \colon \mathbb{R}^g \times \mathbb{R}^g \to \mathbb{C}^g$ sending $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} + \tau_{0,\sigma}\mathbf{b}$. The smallest eigenvalue $\lambda_{1,\sigma}$ of Y_σ satisfies $\lambda_{1,\sigma} \geq \frac{\sqrt{3}}{2e\gamma_g}$ (by Lemma 2.1.(iv), applied to $B = Y_\sigma$, together with Lemma 3.3.(ii)). So for all $|\zeta| \leq r_\sigma$,

$$\left|Y_\sigma^{-1/2}\mathrm{Im}(\zeta)\right|_\infty \leq \frac{|\zeta|}{\sqrt{\lambda_{1,\sigma}}} \leq \frac{r_\sigma}{\sqrt{\lambda_{1,\sigma}}} < \frac{1}{4(g+2)}$$

by our choice of r_σ, as desired.

Proposition 7.6. $\log\|\mathrm{jet}^\ell s(x)\|_\sigma \leq -\frac{m\epsilon}{2}\widetilde{J}_\sigma(A) + \frac{10g^4m}{\mathfrak{N}_\mathrm{j}} + o(m)$.

Proof. Classical theory of complex abelian varieties says that the global section s of mL corresponds to a theta function $\vartheta \colon \mathbb{C}^g \to \mathbb{C}$. By [Gau19, §2.1], the metric $\|\cdot\|_\sigma$ on mL (induced by the cubist metric on $\overline{L}$) satisfies

$$\|s(u_\sigma(z))\|_\sigma = |\vartheta(z)| \exp\Big(-\frac{\pi}{2}mH(z,z)\Big). \tag{7.12}$$

Next we want to express $\|\mathrm{jet}^\ell s(x)\|_\sigma^2$, for which we need an H-orthonormal basis of $\mathbb{C}^g$. Indeed, for the standard basis $\{e_1, \ldots, e_g\}$ of $\mathbb{C}^g$, we have a natural choice of H-orthonormal basis $\{Y_\sigma^{1/2} e_1, \ldots, Y_\sigma^{1/2} e_g\}$. For a multi-index $\tau = (\tau_1, \ldots, \tau_g) \in \mathbb{N}^g$, denote by $D^\tau := \prod_{j=1}^g \left(\partial/\partial_{Y_\sigma^{1/2} e_j}\right)^{\tau_j}$. Then [Gau19, §2.2] gives

$$\|\mathrm{jet}^\ell s(x)\|_\sigma^2 = \sum_{\tau \in \mathbb{N}^g,\ |\tau|=\ell} \frac{\tau!}{\ell!} \left| \frac{1}{\tau!} D^\tau \vartheta(z_\sigma) \exp\left(-\frac{\pi m}{2} H(z_\sigma, z_\sigma)\right) \right|^2. \tag{7.13}$$

For our purpose, we shall use a translated version centered at z_σ under the H-orthonormal basis as follows: define

$$\vartheta_{z_\sigma} \colon \mathbb{C}^g \to \mathbb{C}, \quad w \mapsto \vartheta(z_\sigma + Y_\sigma^{1/2} w) \exp\left(-\pi m H(z_\sigma, Y_\sigma^{1/2} w) - \frac{\pi m}{2} H(z_\sigma, z_\sigma)\right).$$

This function is holomorphic in w since $H(z_\sigma, w)$ is linear in w. Using $H(z_\sigma + w, z_\sigma + w) = H(z_\sigma, z_\sigma) + H(w, w) + 2\mathrm{Re}H(z_\sigma, w)$ and $H(Y_\sigma^{1/2} w, Y_\sigma^{1/2} w) = |w|^2$, one gets from (7.12)

$$\|s(u_\sigma(z + Y_\sigma^{1/2} w))\|_\sigma = |\vartheta_{z_\sigma}(w)| \exp\left(-\frac{\pi m}{2} |w|^2\right). \tag{7.14}$$

The section s vanishes at x to order ℓ. So $D^{\tau'} \vartheta(z_\sigma) = 0$ for all $|\tau'| < \ell$. Thus

$$D^\tau \vartheta_{z_\sigma}(0) = \exp\left(-\frac{\pi m}{2} H(z_\sigma, z_\sigma)\right) D^\tau \vartheta(z_\sigma) \quad \text{for } |\tau| = \ell.$$

So (7.13) becomes

$$\|\mathrm{jet}^\ell s(x)\|_\sigma^2 = \sum_{\tau \in \mathbb{N}^g,\ |\tau|=\ell} \frac{\tau!}{\ell!} \left| \frac{1}{\tau!} D^\tau \vartheta_{z_\sigma}(0) \right|^2. \tag{7.15}$$

Cauchy's estimate gives, for $|\tau| = \ell$,

$$\left| \frac{1}{\tau!} D^\tau \vartheta_{z_\sigma}(0) \right| \le \left(\frac{\ell^\ell}{\tau^\tau} \right)^{1/2} \frac{\sup_{|\zeta|=r_\sigma} |\vartheta_{z_\sigma}(\zeta)|}{r_\sigma^\ell},$$

with the convention $0^0 = 1$. (Robbins') Stirling's Formula implies $\frac{\tau!}{\ell!} \frac{\ell^\ell}{\tau^\tau} \le (2\pi\ell)^{g/2}$. Hence

$$\|\,\mathrm{jet}^\ell\, s(x)\|_\sigma \le \sqrt{(\ell+1)^g (2\pi\ell)^{g/2}} \frac{\sup_{|\zeta|=r_\sigma} |\vartheta_{z_\sigma}(\zeta)|}{r_\sigma^\ell} \le (\ell+1)^{3g/4} (2\pi)^{g/2} \frac{\sup_{|\zeta|=r_\sigma} |\vartheta_{z_\sigma}(\zeta)|}{r_\sigma^\ell}. \tag{7.16}$$

Let us bound $\sup_{|\zeta|=r_\sigma} |\vartheta_{z_\sigma}(\zeta)|$. Our s is a small section for the perturbed metric. So $|s(y) e^{-m\epsilon(f_\sigma(y) - \widetilde{J}_\sigma(A))}| \le 1$ for all $y \in A_\sigma(\mathbb{C})$. Apply this to $y = u_\sigma(z_\sigma + Y_\sigma^{1/2} \zeta)$ with $|\zeta| = r_\sigma$, we get $|s(y)| \le \exp(-m\epsilon \widetilde{J}_\sigma(A))$ from (7.11). Thus (7.14) implies

$$|\vartheta_{z_\sigma}(\zeta)| \le \exp\left(-m\epsilon \widetilde{J}_\sigma(A) + \frac{\pi m}{2} r_\sigma^2\right) \quad \text{for all } |\zeta| = r_\sigma.$$

Combining this with (7.16) and the upper bound of ℓ given by Proposition 7.2, we get

$$\begin{aligned}
\log \|\mathrm{jet}^\ell s(x)\|_\sigma &\le -m\epsilon \widetilde{J}_\sigma(A) + \left(\frac{g^2 m}{\mathfrak{N}_j} + 1 \right) \log \frac{1}{r_\sigma} + \frac{\pi m}{2} r_\sigma^2 + \frac{3g}{4} \log\left(\frac{g^2 m}{\mathfrak{N}_j} + 2\right) + \log(2\pi)^{g/2} \\
&\le -\frac{m\epsilon}{2} \widetilde{J}_\sigma(A) + \left(\frac{g^2 m}{\mathfrak{N}_j} + 1 \right) \log r_\sigma^{-1} + o(m).
\end{aligned}$$

Here the second inequality holds true since $\frac{\pi r_\sigma^2}{2} \le \frac{\epsilon \widetilde{J}_\sigma(A)}{2}$ by choice (7.10) of r_σ and, because $\mathfrak{N}_j \ge h^0(A, L)^{-1}$ by definition, $\frac{3g}{4} \log(\frac{g^2 m}{\mathfrak{N}_j} + 2) + \log(2\pi)^{g/2} \le \frac{3g}{4} \log(g^2 m h^0(A, L) + 2) + \log(2\pi)^{g/2} = o(m)$.

It remains to handle $\log \frac{1}{r_\sigma}$. A lower bound on $\widetilde{J}_\sigma(A)$ is given by (7.3) and the fact that $\widetilde{I}_\sigma(A) \geq \sqrt{3}/2$. So we get a crude bound (recall $\epsilon = \frac{1}{8g}$)

$$\log r_\sigma^{-1} \leq \log \gamma_g + 9\log(g+2) \leq 10g^2.$$

This gives our desired bound. $\square$

7.4. **Liouville inequality for J.** Consider now the adelic space $\overline{J} := (J, \|\cdot\|_v)$ defined over K from (7.9) and below. Any non-zero K-subspace $F \subseteq J$ has a natural structure of adelic subspace $\overline{F}$ of $\overline{J}$ by considering the restriction of $\|\cdot\|_v$ to $F \otimes_K K_v$ at each place $v \in M_K$. The *normalized degree* of $\overline{F}$ is defined to be (here $K_0 = \mathbb{Q}$ if K is a number field and $K_0 = \mathbb{C}(\mathbb{P}^1)$ if $K = \mathbb{C}(B)$)

$$\widehat{\deg}_{\mathrm{n}}(\overline{F}) := -\frac{1}{[K:K_0]} \sum_{v \in M_K} \log \|s'\|_v \quad \text{for any non-zero } s' \in \bigwedge^{\dim F} F$$

and the *slope* of $\overline{F}$ is defined to be

$$\widehat{\mu}(\overline{F}) := \frac{\widehat{\deg}_{\mathrm{n}}(F)}{\dim F}.$$

See [Gau08, Lem. 4.4 and 4.5]. Then the *maximal slope* of $\overline{J}$ is

$$\widehat{\mu}_{\max}(\overline{J}) := \max\{\widehat{\mu}(\overline{F}) : 0 \neq \overline{F} \subseteq \overline{J}\}.$$

Now the non-zero element $\mathrm{jet}^\ell s(x) \in J$ defines a line $K \cdot \mathrm{jet}^\ell s(x)$ in J, and hence an adelic subspace $\overline{K \cdot \mathrm{jet}^\ell s(x)}$ of J. By definition of the maximal slope, we have

$$\widehat{\mu}(\overline{K \cdot \mathrm{jet}^\ell s(x)}) \leq \mu_{\max}(\overline{J}). \tag{7.17}$$

Let us analyze both sides. For the left hand side, by definition we have

$$\widehat{\mu}(\overline{K \cdot \mathrm{jet}^\ell s(x)}) = -\frac{1}{[K:K_0]} \sum_{v \in M_K} \log \|\mathrm{jet}^\ell s(x)\|_v. \tag{7.18}$$

For the right hand side, we have

$$\begin{aligned} \widehat{\mu}_{\max}(\overline{J}) &= m\widehat{\mu}(\overline{L}_x) + \widehat{\mu}_{\max}(\mathrm{Sym}^\ell(\overline{\mathrm{Lie}\, A}^\vee)) \quad \text{by [Gau08, Prop. 5.7.(1)]} \\ &= mh_{\overline{L}}(x) + \widehat{\mu}_{\max}(\mathrm{Sym}^\ell(\overline{\mathrm{Lie}\, A}^\vee)) \quad \text{by direct computation.} \end{aligned}$$

Now we separate the function field and the number field case.

$\boxed{\text{Case: } K = \mathbb{C}(B) \text{ is a function field}}$ By [Gau08, Thm. 7.1.(1)], we then have

$$\widehat{\mu}_{\max}(\overline{J}) = mh_{\overline{L}}(x) + \ell \cdot \widehat{\mu}_{\max}(\overline{\mathrm{Lie}\, A}^\vee).$$

Therefore by (7.17) and (7.18) (and Theorem A.1) we have

$$-\frac{1}{[K:\mathbb{C}(\mathbb{P}^1)]} \sum_{v \in M_K} \log \|\mathrm{jet}^\ell s(x)\|_v \leq m\widehat{h}_L(x) + \ell\left(\frac{g}{2}+1\right) h_{\mathrm{Fal}}(A). \tag{7.19}$$

An upper bound of ℓ is given by Proposition 7.2.

$\boxed{\text{Case: } K \text{ is a number field}}$ By [Gau08, Thm. 7.1.(2)], we then have

$$\widehat{\mu}_{\max}(\overline{J}) \leq mh_{\overline{L}}(x) + \ell\left(\widehat{\mu}_{\max}(\overline{\mathrm{Lie}\, A}^\vee) + 2g\log g\right).$$

Therefore by (7.17), (7.18) and [Gau19, Prop. 4.6 and below], we have

$$-\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} \log \|\mathrm{jet}^\ell s(x)\|_v \leq m\widehat{h}_L(x) + \ell\left((0.6g+1)(h_{\mathrm{Fal}}(A) + \log h^0(A,L)) + 2g^2\log 10g\right). \tag{7.20}$$

7.5. **Proof of Theorem 6.1.** Let $K = \mathbb{C}(B)$. Then every place of K is non-archimedean, and hence Lemma 7.4 and (7.4) together imply (with $\epsilon = \frac{1}{8g}$)

$$\begin{aligned} -\frac{1}{[K:\mathbb{C}(\mathbb{P}^1)]}\sum_{v\in M_K}\log\|\mathrm{jet}^\ell s(x)\|_v &\geq \frac{m}{8\cdot 128e\gamma_g[K:\mathbb{C}(\mathbb{P}^1)]}\sum_{v\in\mathfrak{Bad}(A/K)}\widetilde{I}_v(A) \\ &= \frac{m}{8\cdot 128e\gamma_g[K:\mathbb{C}(\mathbb{P}^1)]}\sum_{v\in\mathfrak{Bad}(A/K)}\#(\Phi_v)^{-1/r_v}. \end{aligned} \tag{7.21}$$

Recall our choice that $x \in A(K)_{\mathrm{tor}}$; see above Proposition 7.2. So $\widehat{h}_L(x) = 0$. Applying (7.21) and the upper bound on ℓ (Proposition 7.2) to the inequality (7.19), we get

$$\frac{m}{8\cdot 128e\gamma_g[K:\mathbb{C}(\mathbb{P}^1)]}\sum_{v\in\mathfrak{Bad}(A/K)}\#(\Phi_v)^{-1/r_v} \leq \frac{g(g+2)}{2}\left(\frac{gm}{\mathfrak{N}_j}+1\right)h_{\mathrm{Fal}}(A).$$

Dividing both sides by m and letting $m \to \infty$, we get

$$\frac{1}{8\cdot 128e\gamma_g[K:\mathbb{C}(\mathbb{P}^1)]}\sum_{v\in\mathfrak{Bad}(A/K)}\#(\Phi_v)^{-1/r_v} \leq \frac{g(g+2)}{2}\frac{g}{\mathfrak{N}_j}h_{\mathrm{Fal}}(A).$$

This immediately implies the desired bound (6.2). So we are done for Theorem 6.1. □

7.6. **Proof of Theorem 6.1′.** Let K be a number field. By Lemma 7.4, Proposition 7.6, and Lemma 7.5, we have

$$-\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K}\log\|\mathrm{jet}^\ell s(x)\|_v \geq \frac{m\epsilon}{2[K:\mathbb{Q}]}\sum_{v\in M_K'}\widetilde{J}_v(A)\log N_v - \frac{g^2m}{\mathfrak{N}_j}\left(1+\log^+\frac{\pi\mathfrak{N}_j}{2g^2}\right) - \frac{10g^4m}{\mathfrak{N}_j} - o(m). \tag{7.22}$$

Now $\widehat{h}_L(x) = 0$ since x is torsion. So (7.22), (7.20) and Proposition 7.2 together imply (to ease notation, we temporarily set $\Theta := (0.6g+1)(h_{\mathrm{Fal}}(A) + \log h^0(A,L)) + 2g^2\log 10g$)

$$\frac{m\epsilon}{2[K:\mathbb{Q}]}\sum_{v\in M_K'}\widetilde{J}_v(A)\log N_v \leq g\Theta + \frac{g^2m}{\mathfrak{N}_j}\left(\Theta+1+\log^+\frac{\pi\mathfrak{N}_j}{2g^2}+10g^2\right)+o(m).$$

Multiply both sides by $\mathfrak{N}_j$, divide both sides by m and let $m \to \infty$. Recall $\epsilon = \frac{1}{8g}$. So

$$\mathfrak{N}_j\cdot\frac{1}{16g[K:\mathbb{Q}]}\sum_{v\in M_K'}\widetilde{J}_v(A)\log N_v \leq g^2\left(\Theta+1+\log^+\frac{\pi\mathfrak{N}_j}{2g^2}+10g^2\right). \tag{7.23}$$

A further coarse estimate (with $\Theta = (0.6g+1)(h_{\mathrm{Fal}}(A)+\log h^0(A,L))+2g^2\log 10g$) yields

$$\mathfrak{N}_j \leq \frac{16g^3\log^+\mathfrak{N}_j}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K'}\widetilde{J}_v(A)\log N_v} + 128g^5\log(10g)\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,L)}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K'}\widetilde{J}_v(A)\log N_v}. \tag{7.24}$$

To ease notation, denote by

$$S = \frac{16\cdot 16e\cdot g^3(g+1)^3(g+2)^3(100g+274)\gamma_g}{\pi}\frac{1}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K'}\widetilde{I}_v(A)\log N_v} > 0$$

and

$$T = \frac{128\cdot 16e\cdot g^5(g+1)^3(g+2)^3(100g+274)\log(10g)\gamma_g}{\pi}\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,L)}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K'}\widetilde{I}_v(A)\log N_v} > 0.$$

Applying the comparison of $\widetilde{I}_v(A)$ and $\widetilde{J}_v(A)$ from (7.3) to the bound (7.24), we get

$$\mathfrak{N}_j \leq S\log^+\mathfrak{N}_j + T. \tag{7.25}$$

To conclude it suffices to use following elementary lemma from [BP05, Lemma 15].

Lemma 7.7. *If* $\mathfrak{N}, S, T > 0$ *with* $\mathfrak{N} \leq S \log^+ \mathfrak{N} + T$*, then* $\mathfrak{N} \leq \max\{1, \frac{e}{e-1}(S \log S + T)\}$.

Notice that $\sum_{v \in M'_K} \widetilde{I}_v(A) \log N_v \geq \widetilde{I}_\sigma(A) \geq \frac{\sqrt{3}}{2}$; see (6.7). So

$$\log S \leq \log[K:\mathbb{Q}] + \log \frac{2 \cdot 16 \cdot 16e \cdot g^3(g+1)^3(g+2)^3(100g+274)\gamma_g}{\sqrt{3}\pi}.$$

Hence our desired bound (6.10) follows by applying Lemma 7.7 to (7.25), with

$$C_1(g) = \frac{e}{e-1} C_2(g)\,(1 + \log C_2(g))\,, \qquad C_2(g) = \frac{2048e}{\pi} g^5(g+1)^3(g+2)^3(100g+274)\log(10g)\gamma_g. \tag{7.26}$$

This concludes for Theorem 6.1′. □

7.7. **Proof of Theorem 6.2 and Theorem 6.2′.** To modify the proofs above to handle rational points of small height, we again fix $\epsilon = 1/(8g)$. For each $m > \max\{m_0, \mathfrak{N}'_j/g\}$ (with m_0 from Proposition 7.1), take a non-zero small global section $s \in H^0(A, m\overline{L}(\epsilon\mathbf{f}))$ from Proposition 7.1 and then an auxiliary point $x \in \bigcup_{i=0}^g \Gamma'_j[i]$ from Remark 7.3. Then all arguments and results in §7.3 hold true verbally, with $\mathfrak{N}_j$ replaced by $\mathfrak{N}'_j$. Thus we still have the estimate of the height of the jet as in (7.21) and (7.22), *i.e.*

- If $K = \mathbb{C}(B)$ is a function field, then
$$-\frac{1}{[K:\mathbb{C}(\mathbb{P}^1)]} \sum_{v \in M_K} \log \|\mathrm{jet}^\ell s(x)\|_v \geq \frac{m}{1024e\gamma_g[K:\mathbb{C}(\mathbb{P}^1)]} \sum_{v \in \mathfrak{Bad}(A/K)} \#(\Phi_v)^{-1/r_v}.$$
- If K is a number field, then
$$-\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} \log \|\mathrm{jet}^\ell s(x)\|_v \geq \frac{m\epsilon}{2[K:\mathbb{Q}]} \sum_{v \in M'_K} \widetilde{J}_v(A) \log N_v - \frac{g^2 m}{\mathfrak{N}_j}\left(1 + \log^+ \frac{\pi \mathfrak{N}'_j}{2g^2}\right) - \frac{10g^4 m}{\mathfrak{N}_j} - o(m).$$

We still have the inequality (7.19). But now by definition of Γ'_j (which is (6.3) when $K = \mathbb{C}(B)$ and is (6.11) when K is a number field), our bound for $\widehat{h}_L(x)$ becomes

$$\widehat{h}_L(x) \leq \begin{cases} g^2 \frac{1}{2048e \cdot g^2 \gamma_g [K:\mathbb{C}(\mathbb{P}^1)]} \sum_{v \in \mathfrak{Bad}(A/K)} \#(\Phi_v)^{-1/r_v} & \text{if } K = \mathbb{C}(B) \\ g^2 \frac{\pi}{32 \cdot 16e \cdot g^3(g+1)^3(g+2)^3(100g+274)\gamma_g[K:\mathbb{Q}]} \sum_{v \in M'_K} \widetilde{I}_v(A) \log N_v & \text{if } K \text{ is a number field.} \end{cases}$$

The vanishing order ℓ of s at x is bounded by Proposition 7.2, with $\mathfrak{N}_j$ replaced by $\mathfrak{N}'_j$. Hence we have an estimate for each term in (7.19).

When $K = \mathbb{C}(B)$ is a function field, applying these estimates to (7.19) we get

$$\frac{m}{2048e\gamma_g[K:\mathbb{C}(\mathbb{P}^1)]} \sum_{v \in \mathfrak{Bad}(A/K)} \#(\Phi_v)^{-1/r_v} \leq \frac{g(g+2)}{2}\left(\frac{gm}{\mathfrak{N}'_j} + 1\right) h_{\mathrm{Fal}}(A).$$

Dividing both sides by m and letting $m \to \infty$, we obtain the desired bound (6.4). This concludes for Theorem 6.2.

When K is a number field, we apply these estimates (as well as the comparison of $\widetilde{I}_v(A)$ and $\widetilde{J}_v(A)$ from (7.3)) to (7.19), divide both sides by m and let $m \to \infty$. Then we get, similarly to (7.23),

$$\mathfrak{N}'_j \cdot \frac{1}{32g[K:\mathbb{Q}]} \sum_{v \in M'_K} \widetilde{J}_v(A) \log N_v \leq g^2 \left(\Theta + 1 + \log^+ \frac{\pi \mathfrak{N}_j}{2g^2} + 10g^2\right). \tag{7.27}$$

This yields the desired bound (6.13) by Lemma 7.7; the proof is similar to the end of §7.6. We are done for Theorem 6.2′. From the proof we see that we can take $C'_1(g) = 3C_1(g)$. □

8. Boundary height and Faltings height

Assume A/K has semi-stable reduction. The definition of *boundary height* (1.7)-(1.9) in the number field case can be extended to the function field case, and is simpler since there are no archimedean places. For $K = \mathbb{C}(B)$, define

$$h_\partial(A) = h_{\partial,\mathrm{fin}}(A) := \frac{1}{[K:K_0]}\left(\sum_{v\in\mathfrak{Bad}(A/K)} (\#\Phi_v)^{1/r_v}\log N_v\right); \tag{8.1}$$

then $h_\partial(A_{K'}) = h_\partial(A)$ for any finite extension K'/K because A has semi-stable reduction.

Proposition 8.1. *We have*

$$h_{\partial,\mathrm{fin}}(A) \le \begin{cases} 96h_{\mathrm{Fal}}(A) & \text{when } K = \mathbb{C}(B) \\ 96h_{\mathrm{Fal}}(A) + 8g\log(\pi\sqrt{2}) & \text{when } K \text{ is a number field.}\end{cases}$$

If A carries a principal polarization, then both bounds can be further divided by 8, and

$$h_\partial(A) \le c(g)\max\{h_{\mathrm{Fal}}(A), 1\}.$$

Without the explicit constants this is Hindry–Pacheko [HP16, Thm. 1.21]. We hereby give a proof as an application of de Jong–Shokrieh's decomposition of the Faltings height [dJS22].

Proof. Denote by $K_0 = \mathbb{C}(\mathbb{P}^1)$ when $K = \mathbb{C}(B)$ and $K_0 = \mathbb{Q}$ when K is a number field. We first treat the case where A carries a principal polarization L. By [dJS22, Thm. A] (in the number field case) and [dJS22, (1,12)] (in the function field case), we have

$$h_{\mathrm{Fal}}(A) \ge \frac{2}{[K:K_0]}\sum_{v\in M_K} I_v(A,L)\log N_v - g\log(\pi\sqrt{2}) \tag{8.2}$$

where $I_v(A,L)\ge 0$ is a local invariant at the place v, and the term $-g\log(\pi\sqrt{2})$ can be removed when $K = \mathbb{C}(B)$. We will not need the precise definition of $I_v(A,L)$.

For $v \in M_{K,\infty}$, we have $A_v(\mathbb{C}) = \mathbb{C}^g/(\mathbb{Z}^g + \tau_v\mathbb{Z}^g)$ for a Siegel reduced matrix τ_v and $\alpha_v(A,L) = \mathrm{Tr}(\mathrm{Im}\tau_v)$. Now $I_v(A,L) \ge c_1(g)\alpha_v(A,L) - c_2(g)$ by [Aut06, Prop. 4.1].

Let $v \in \mathfrak{Bad}(A/K)$. Let us prove that $I_v(A,L) \ge \frac{1}{24}(\#\Phi_v)^{1/r_v}$. Indeed, $I_v(A,L)$ is half of the tropical moment $I(\Sigma_v)$ by [dJS22, Thm. B], where $I(\Sigma_v)$ is defined as follows. The metrized line bundle $(L_v^{\mathrm{an}}, \|\cdot\|_v)$ induces a metric on the skeleton Σ_v of A_v^{an}, and hence a metric $\|\cdot\|_{L,v}$ on $X^*_{v,\mathbb{R}}$ via the uniformization $X^*_{v,\mathbb{R}} \to \Sigma_v = X^*_{v,\mathbb{R}}/Y_v$. We have a subset $\mathrm{Vor}(0) \subseteq X^*_{v,\mathbb{R}}$ from [dJS22, above Thm. B] which is a fundamental subset for $X^*_{v,\mathbb{R}} \to \Sigma_v$, and the tropical moment is defined to be $I(\Sigma_v) := \frac{1}{\mathrm{vol}_{\mathrm{d}\mu}(\mathrm{Vor}(0))}\int_{\mathrm{Vor}(0)}\|\mathbf{x}\|^2_{L,v}\mathrm{d}\mu_v$ for the standard Lebesgue measure $\mathrm{d}\mu_v$ on $X^*_{v,\mathbb{R}}$. Lemma 8.2 then gives the desired lower bound of $I_v(A,L)$.

Now the conclusion for principally polarized A follows from (8.2) and the bounds in the paragraphs above. In general, consider $Z(A) := A^4 \times A^{\vee 4}$ which carries a principal polarization by Zarhin's trick. Note that $h_{\partial,\mathrm{fin}}(A) = h_{\partial,\mathrm{fin}}(Z(A))$. Moreover, $h_{\mathrm{Fal}}(Z(A)) = 4h_{\mathrm{Fal}}(A) + 4h_{\mathrm{Fal}}(A^\vee) = 8h_{\mathrm{Fal}}(A)$. So we get the desired inequality for general A. □

Lemma 8.2. *For each $v \in \mathfrak{Bad}(A/K)$, we have*

$$I(\Sigma_v) \ge \frac{r_v}{\pi(r_v+2)}\Gamma\Big(\frac{r_v}{2}+1\Big)^{2/r_v}(\#\Phi_v)^{1/r_v} \ge \frac{1}{12}(\#\Phi_v)^{1/r_v}.$$

Proof. To ease notation, we omit the subscript v in the proof, and denote by $d := [X : \lambda^*_{T,L}(Y)] \geq 1$; see (4.9) for the notation $\lambda^*_{T,L}$.

We have $\Sigma = X^*_{\mathbb{R}}/Y$. Take a $\mathbb{Z}$-basis of X^* and write $\mathbf{x} = (x_1, \ldots, x_r)$ to be the coordinates of $X^*_{\mathbb{R}}$ with respect to this basis. Now Y is a lattice in $X^*_{\mathbb{R}}$ of rank r of covolume $\#\Phi$. Let B be the matrix of q_0 under this standard basis. Then $\det(B) = d/\#\Phi$; see (4.18). Then the metric $\|\cdot\|_{L,v}$ on $X^*_{\mathbb{R}}$ is: $\|\mathbf{x}\|_{L,v} = \mathbf{x}^{\mathrm{t}}B\mathbf{x}$ for any $\mathbf{x} \in X^*_{\mathbb{R}}$.

Consider the volume form $\mathrm{d}\mathbf{x} := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_r$ on $X^*_{\mathbb{R}}$. Then $\mathrm{vol}_{\mathrm{d}\mathbf{x}}(\mathrm{Vor}(0)) = \#\Phi$, so

$$I(\Sigma) = \frac{1}{\#\Phi}\int_{\mathrm{Vor}(0)} (\mathbf{x}^{\mathrm{t}}B\mathbf{x})\mathrm{d}\mathbf{x}.$$

Since B is symmetric and positive definite, there exists a real symmetric positive definite matrix S such that $B = S^2$. Apply the change of coordinates $\mathbf{y} := S\mathbf{x}$. Then $\mathbf{x}^{\mathrm{t}}B\mathbf{x} = |\mathbf{y}|^2$ for the standard metric on $X^*_{\mathbb{R}}$, *i.e.* $|(y_1, \ldots, y_r)|^2 = \sum_{j=1}^r y_j^2$. We have $\mathrm{d}\mathbf{y} = \det(S)\mathrm{d}\mathbf{x} = d^{1/2}(\#\Phi)^{-1/2}\mathrm{d}\mathbf{x}$. From now on, we will consider the volume form $\mathrm{d}\mathbf{y}$ on $X^*_{\mathbb{R}}$.

Now $\Omega := S \cdot \mathrm{Vor}(0)$ is a Euclidean region such that

$$\mathrm{vol}_{\mathrm{d}\mathbf{y}}(\Omega) = \det(S)\,\mathrm{vol}_{\mathrm{d}\mathbf{x}}(\mathrm{Vor}(0)) = \sqrt{d\#\Phi},$$

and

$$I(\Sigma) = \frac{1}{\sqrt{d\#\Phi}}\int_{\Omega} |\mathbf{y}|^2\mathrm{d}\mathbf{y}. \tag{8.3}$$

Take the ball $B(0,R)$ centered at 0 of radius $R := \Gamma(\frac{r}{2}+1)^{1/r}\pi^{-1/2}(d\#\Phi)^{1/2r}$; then its Euclidean volume $\mathrm{vol}_{\mathrm{d}\mathbf{y}}(B(0,R))$ equals $\sqrt{d\#\Phi} = \mathrm{vol}_{\mathrm{d}\mathbf{y}}(\Omega)$. Now $|\mathbf{y}|^2 \leq R^2$ for all $\mathbf{y} \in B(0,R) \setminus \Omega$ and $|\mathbf{y}|^2 \geq R^2$ for all $\mathbf{y} \in \Omega \setminus B(0,R)$. So

$$\int_{\Omega\setminus B(0,R)} |\mathbf{y}|^2\mathrm{d}\mathbf{y} \geq R^2\,\mathrm{vol}_{\mathrm{d}\mathbf{y}}(B(0,R)\setminus\Omega) = R^2\,\mathrm{vol}_{\mathrm{d}\mathbf{y}}(\Omega\setminus B(0,R)) \geq \int_{B(0,R)\setminus\Omega} |\mathbf{y}|^2\mathrm{d}\mathbf{y}.$$

It follows that

$$\int_{\Omega} |\mathbf{y}|^2\mathrm{d}\mathbf{y} \geq \int_{B(0,R)} |\mathbf{y}|^2\mathrm{d}\mathbf{y} = \int_0^R t^2 t^{r-1} 2\pi^{r/2}\Gamma(r/2)^{-1}\mathrm{d}t = 2\pi^{r/2}\Gamma(r/2)^{-1}(r+2)^{-1}R^{r+2}.$$

This yields the conclusion by (8.3) and our choice of R and because $d \geq 1$. ☐

9. Proofs of the main results over function fields: semi-stable case

The goal of this section is to prove our main results over function fields when A/K has semi-stable reduction. More precisely, we prove Theorem 1.3 as well as Theorem 1.5 (if L defines a principal polarization) assuming A/K has semi-stable reduction. In the whole section, denote by $K_0 := \mathbb{C}(\mathbb{P}^1)$. Then $[K : K_0] = \mathrm{gon}(B)$. Denote by $g(B)$ the genus of B. We prove:

Theorem 9.1. *If A/K has semi-stable reduction and no abelian subvariety of A defined over K has good reduction everywhere, then we have*

$$\#A(K)_{\mathrm{tor}} \leq h^0(A,L)\cdot\left(24\cdot 512e\cdot g^4(g+2)\gamma_g\right)^g(2g(B)-1)^{2g} \tag{9.1}$$

for any polarization L on A.

Before moving on, let us point out that Theorem 9.1 immediately implies Theorem 1.3 when A/K has semi-stable reduction by Zarhin's trick and the (short) argument in §11.1.2.

9.1. **Local and global Faltings height.** In this subsection, assume that A has semi-stable reduction. Assume also that A does *not* have good reduction everywhere, *i.e.* $\#\mathfrak{Bad}(A/K) \geq 1$. To ease notation, denote by

$$\mathfrak{b} := \#\mathfrak{Bad}(A/K).$$

We prove that the stable Faltings height $h_{\mathrm{Fal}}(A)$ is comparable to the sum of the local invariants $\sum_{v\in\mathfrak{Bad}}(\#\Phi_v)^{-1/r_v}$. The key is to use is the following *Arakelov Inequality* proved by Deligne [Del87], which is the function field version of Szpiro's conjecture.

Theorem 9.2 (Deligne)**.** *Assume A/K has semi-stable reduction. Then*

$$[K:K_0]h_{\mathrm{Fal}}(A) \leq \frac{g}{2}\big(2g(B)-2+\mathfrak{b}\big)$$

whenever the right hand side is ≥ 0.

We use Jensen's Inequality and Theorem 9.2 to prove the following proposition:

Proposition 9.3. *Assume $\mathfrak{b}\geq 1$ (and $\mathfrak{b}\geq 3$ if $g(B)=0$). Then*

$$\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}} \leq 24g^2\left(2g(B)-1\right)^2. \tag{9.2}$$

and

$$\frac{[K:K_0]\max\{h_{\mathrm{Fal}}(A),1\}}{\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}} \leq 48g^2[K:K_0](2g(B)-1)^2. \tag{9.3}$$

Proof. By Jensen's Inequality applied to the function $t\mapsto t^{-1}$, we have

$$\left(\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{1/r_v}\right)\left(\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}\right) \geq \big(\#\mathfrak{Bad}(A/K)\big)^2 = \mathfrak{b}^2.$$

Moreover $\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{1/r_v} \leq 96[K:K_0]h_{\mathrm{Fal}}(A)$ by Proposition 8.1. So the inequality above further implies

$$\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v} \geq \frac{1}{96}\frac{\mathfrak{b}^2}{[K:K_0]h_{\mathrm{Fal}}(A)}. \tag{9.4}$$

Thus

$$\begin{aligned}\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}} &\leq 96\left(\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\mathfrak{b}}\right)^2\\ &\leq 96\left(\frac{g\left(2g(B)-2+\mathfrak{b}\right)}{2\mathfrak{b}}\right)^2 \qquad \text{by Theorem 9.2}\\ &= 24g^2\left(\frac{2g(B)-2}{\mathfrak{b}}+1\right)^2 \leq 24g^2\max\{(2g(B)-1)^2,1\}\end{aligned}$$

The last inequality holds true since $\mathfrak{b}\geq 1$ (and $\mathfrak{b}\geq 3$ if $g(B)=0$). This proves (9.2).

On the other hand, applying Theorem 9.2 to (9.4), we get

$$\frac{1}{\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}} \leq \frac{96g(2g(B)-2+\mathfrak{b})}{2\mathfrak{b}^2} = 48g\left(\frac{2g(B)-2}{\mathfrak{b}^2}+\frac{1}{\mathfrak{b}}\right) \leq 48g\max\{2g(B)-1,1\}, \tag{9.5}$$

where the last inequality follows from $\mathfrak{b}\geq 1$ (and $\mathfrak{b}\geq 3$ if $g(B)=0$). Then we immediately get (9.3) by comparing (9.2) and $[K:K_0]\cdot$(9.5). $\square$

9.2. **Proof of Theorem 9.1.** By the Poincaré Irreducibility Theorem, A is isogenous to a product of K-simple abelian varieties. Each such K-simple abelian varieties (counted with multiplicity) is called a *factor* of A. By our assumption, we know that no factor of A has good reduction everywhere.

We will proceed by induction on the number of factors of A, which we denote by n.

$\boxed{\text{Base step: } n=1}$ In this case A is simple. Then our assumption becomes: A does not have good reduction everywhere, *i.e.* $\mathfrak{b} = \#\mathfrak{Bad}(A/K) \geq 1$ (and $\mathfrak{b} \geq 3$ if $g(B) = 0$). Hence we can apply Theorem 6.1 and get

$$\#A(K)_{\mathrm{tor}} \leq h^0(A,L)\left(512e \cdot g^2(g+2)\gamma_g \frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}}\right)^g. \tag{9.6}$$

Hence the conclusion follows immediately from (9.2).

$\boxed{\text{Induction step}}$ For arbitrary $n \geq 2$. By Theorem 6.1, there exists a proper abelian subvariety A' of A defined over K such that

$$\#(A/A')(K)_{\mathrm{tor}} \leq \frac{h^0(A,L)}{h^0(A',L)}\left(512e \cdot g^2(g+2)\gamma_g \frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}}\right)^{g-\dim A'}.$$

Hence by (9.2), we have

$$\#(A/A')(K)_{\mathrm{tor}} \leq \frac{h^0(A,L)}{h^0(A',L)}\left(24\cdot 512e \cdot g^4(g+2)\gamma_g\right)^{g-\dim A'}(2g(B)-1)^{2(g-\dim A')}.$$

Since $A' \neq A$, the number of factors of A' is $\leq n-1$. So we can apply the induction hypothesis to A' and get

$$\#A'(K)_{\mathrm{tor}} \leq h^0(A',L)\left(24\cdot 512e \cdot g^4(g+2)\gamma_g\right)^{\dim A'}(2g(B)-1)^{2\dim A'}.$$

Thus the desired (9.1) follows from the product of the two bounds above. We are done. □

9.3. **Proof of a weak version of Theorem 1.5 in the semi-stable case.** To present the idea how our bounds allow to treat small rational points, we hereby present a simple proof of Theorem 1.5 when A/K has semi-stable reduction, *provided that* L is a principal polarization on A. For general L, we need to use *Zarhin's box* as in §11.2.

Let A, L be as in Theorem 1.5, and A/K has semi-stable reduction. Let us prove prove:

$$\widehat{h}_L(P) > \frac{c_0'''(g)}{h^0(A,L)^2}\frac{\max\{h_{\mathrm{Fal}}(A),1\}}{\mathrm{gon}(B)\cdot(2g(B)-1)^{4g+2}}. \tag{9.7}$$

for any $P \in A(K)$ such that $\overline{\mathbb{Z}\cdot P}^{\mathrm{Zar}} = A$, with

$$c_0'''(g) = \left(48\cdot 2048(24\cdot 1024)^{2g}e^{2g+1}g^{8g+4}(g+2)^{2g}\gamma_g^{2g+1}\right)^{-1}.$$

Denote by $K_0 = \mathbb{C}(\mathbb{P}^1)$. Then $\mathrm{gon}(B) = [K:K_0]$. By (9.3), we have

$$\frac{1}{2048g^2\gamma_g[K:K_0]}\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v} \geq \frac{1}{48\cdot 2048g^4\gamma_g}\cdot\frac{\max\{h_{\mathrm{Fal}}(A),1\}}{[K:K_0](2g(B)-1)^2}.$$

So the set Γ' defined in (6.3) contains the following susbset

$$\Gamma'' := \left\{x \in A(K) : \widehat{h}_L(x) \leq \frac{1}{48\cdot 2048e\cdot g^4\gamma_g}\cdot\frac{\max\{h_{\mathrm{Fal}}(A),1\}}{[K:K_0](2g(B)-1)^2}\right\}.$$

We wish to invoke Theorem 6.2. Set

$$m := h^0(A,L)\cdot\left(24\cdot 1024e\cdot g^4(g+2)\gamma_g\right)^g(2g(B)-1)^{2g}.$$

Then the right hand side of (9.7) is $\frac{1}{m^2 48\cdot 2048e\cdot g^4\gamma_g}\frac{\max\{h_{\mathrm{Fal}}(A),1\}}{[K:K_0](2g(B)-1)^2}$.

Assume (9.7) is false. Then the rational points

$$\pm P,\ [\pm 2]P,\ \ldots,\ [\pm m]P$$

are in Γ'', and hence in Γ'.

For *any* proper abelian subvariety A' of A defined over K, our assumption $\overline{\mathbb{Z}\cdot P}^{\mathrm{Zar}} = A$ implies that

$$q(\pm P), q([\pm 2]P), \ldots, q([\pm m]P)$$

are 2-by-2 distinct for the quotient $q\colon A\to A/A'$. In particular, $\#\left(\frac{\Gamma''+A'}{A'}\right)\geq 2m > m$. So

$$\left(\#\left(\frac{\Gamma'+A'}{A'}\right)\cdot\frac{h^0(A',L)}{h^0(A,L)}\right)^{\frac{1}{g-\dim A'}} > \left(m\cdot\frac{h^0(A',L)}{h^0(A,L)}\right)^{1/g}\geq\left(\frac{m}{h^0(A,L)}\right)^{1/g},$$

and hence, by our choice of m and (9.2),

$$\left(\#\left(\frac{\Gamma'+A'}{A'}\right)\cdot\frac{h^0(A',L)}{h^0(A,L)}\right)^{\frac{1}{g-\dim A'}} > 1024e\cdot g^2(g+2)\gamma_g\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}}.$$

This violates Theorem 6.2. So (9.7) holds true. We are done. □

10. From semi-stable to general case

For simplicity let $K=\mathbb{C}(B)$ in this section. The treatment also works for number fields K *if* there exists an extension K'/K such that (i) $A_{K'}/K'$ has semi-stable reduction, (ii) K'/K is *tamely* ramified at all $v\in M_K^0$ where A has bad but potentially good reduction.[8]

Throughout the whole section, denote by $K_0 := \mathbb{C}(\mathbb{P}^1)$. Then $[K:K_0]=\mathrm{gon}(B)$.

Theorem 10.1. *Assume A/K does not have good reduction everywhere. Then there exists a proper abelian subvariety A' of A defined over K such that*

$$\left(\#\left(\frac{A(K)_{\mathrm{tor}}+A'}{A'}\right)\cdot\frac{h^0(A',L)}{h^0(A,L)}\right)^{\frac{1}{g-\dim A'}}\leq 512\cdot 3^{12g^2+2}g^4(g+2)\gamma_g\,(g(B)+1)^2. \tag{10.1}$$

In previous sections, we proved (10.1) when A/K furthermore has semi-stable reduction; this is the combination of Theorem 6.1 and Proposition 9.3. Theorem 10.1 is the generalization to arbitrary A/K not having good reduction everywhere. Its proof is an adaption of the semi-stable case by adding local contributions at places of bad but potentially good reductions.

Again, define $\mathfrak{N}$ to be the minimum of the left hand side of (10.1) when A' runs over all proper abelian subvarieties of A defined over K (or defined over $\overline{K}$ – this does not change $\mathfrak{N}$ by Gaudron–Rémond [GR25, Lem. 6.1]). Then Theorem 10.1 is equivalent to:

$$\mathfrak{N}\leq 512\cdot 3^{12g^2+2}g^4(g+2)\gamma_g\,(g(B)+1)^2. \tag{10.2}$$

We can also prove a similar bound for small rational points. Define

$$\Gamma := \left\{P\in A(K) : \widehat{h}_L(P)\leq\frac{\max\{h_{\mathrm{Fal}}(A),1\}}{2048\cdot 3^{12g^2+2}g^4\gamma_g[K:K_0]\,(g(B)+1)^2}\right\}. \tag{10.3}$$

[8]So the semi-stable assumption in Theorem 1.8 and Theorem 1.9 can be relaxed to this (with different constants), if one combines the discussion in this section and all other treatments over number fields.

Theorem 10.2. *Assume A/K does not have good reduction everywhere. Then there exists a proper abelian subvariety A' of A defined over K such that*

$$\left(\#\left(\frac{\Gamma + A'}{A'}\right) \cdot \frac{h^0(A', L)}{h^0(A, L)}\right)^{\frac{1}{g-\dim A'}} \leq 1024 \cdot 3^{12g^2+2} g^4 (g+2) \gamma_g \left(g(B)+1\right)^2. \tag{10.4}$$

10.1. **Setting up.** Let $K' := K(A[3])$. Then $A_{K'}$ has semi-stable reduction over K'. We have that K'/K is Galois, and $[K' : K] \leq \#\mathrm{GL}_{2g}(\mathbb{Z}/3\mathbb{Z}) \leq 3^{4g^2}$. Moreover. $K' = \mathbb{C}(B')$ for a smooth projective irreducible curve B' defined over $\mathbb{C}$, whose genus $g(B') \geq 1$.

Most computation in this section is done over K' instead of K.

10.2. **Local discussion.** Let $v \in M_K$ be a place over which A has bad but potentially good reduction. Denote by $F := K_v$.

Let $v' \in M_{K'}$ be a place lying over v; then v' can be seen as a point $B'(\mathbb{C})$. Let $e(v'|v)$ be the ramification index of v' over v. Then $e(v'|v) \leq [K' : K] \leq 3^{4g^2}$.

Let $F' \subseteq K_{v'}$ be the totally ramified extension of F such that $A_{F'}$ has good reduction. Then $[F' : F] = e(v'|v)$. Let ϖ' be the uniformizer of F', and take $\log|\varpi'| = -1$; this is compatible with the convention in §2.5 when we work with K'. Since K is a function field of characteristic 0, the extension F'/F is tamely ramified, and hence we can apply results of Edixhoven [Edi92] – this is the only place we need to work over $K = \mathbb{C}(B)$.

Let $\mathcal{N}$ be the Néron model of A over $\mathcal{O}_F$, and let $\mathcal{A}'$ be the Néron model of $A_{F'}$ over $\mathcal{O}_{F'}$. The universal property of Néron models yields a natural morphism

$$\varphi \colon \mathcal{N} \otimes_{\mathcal{O}_F} \mathcal{O}_{F'} \longrightarrow \mathcal{A}',$$

which induces $\varphi_{v'} \colon \mathcal{N}_{v'} \to \mathcal{A}'_{v'}$. Denote by

$$q := g - \dim \varphi_{v'}(\mathcal{N}_{v'}) = \mathrm{codim}_{\mathcal{A}'_{v'}}(\varphi_{v'}(\mathcal{N}_{v'})) \geq 1. \tag{10.5}$$

In suitable formal coordinates, the differential map $\mathrm{d}\varphi$ is given by

$$\mathrm{d}\varphi \colon \mathrm{Lie}(\mathcal{N}) \otimes_{\mathcal{O}_F} \mathcal{O}_{F'} \longrightarrow \mathrm{Lie}(\mathcal{A}'), \quad (u_1, \ldots, u_{g-q}, y_1, \ldots, y_q) \mapsto (u_1, \ldots, u_{g-q}, \varpi'^{b_1} y_1, \ldots, \varpi'^{b_q} y_q)$$

for some positive integers (called the *positive elementary divisors* of $\mathrm{d}\varphi$)

$$b_1, \ldots, b_q \in \mathbb{Z}_{\geq 1}, \tag{10.6}$$

where $u_1, \ldots, u_{g-q}$ is a local coordinate system of $\varphi_{v'}(\mathcal{N}_{v'})$ at the origin and $y_1, \ldots, y_q$ is a local coordinate system of the complementary abelian variety of $\varphi_{v'}(\mathcal{N}_{v'})^\circ$ at the origin.

Lemma 10.3. $(g-q)! \cdot h^0(\varphi_{v'}(\mathcal{N}_{v'}), \mathcal{L}) \leq 3^q e(v'|v)^{q+1} g! \cdot h^0(A, L)$.

Proof. This algebro-gemetric result is proved by Looper–Yap [LY26, Cor. 7.4], and the key ingredient is a result of Edixhoven (reinterpreted as [HN11, Lem. 3.2.(2)]). For the sake of completeness, we include a sketch of the proof here.

The Galois group $\mathrm{Gal}(F'/F) \simeq \mu_{e(v'|v)}$, and by [HN11, Lem. 3.2.(2)] we have $\varphi_{v'}(\mathcal{N}_{v'}) = (\mathcal{A}'_{v'})^{\mu_{e(v'|v)}}$. Let $\mathcal{M} := \bigotimes_{a \in \mu_{e(v'|v)}} a^*\mathcal{L}$. Then $\mathcal{M}$ carries a natural $\mu_{e(v'|v)}$-linearization since the action of $\mu_{e(v'|v)}$ permutes the tensor factors. Since $a^*\mathcal{L}$ is numerically equivalent to $\mathcal{L}$ for every $a \in \mu_{e(v'|v)}$, we have $\mathcal{M} \equiv \mathcal{L}^{\otimes e(v'|v)}$. The line bundle $\mathcal{M}^{\otimes 3}$ is very ample and gives a $\mu_{e(v'|v)}$-equivariant projective embedding. Since the fixed locus in the ambient projective space is a union of at most $e(v'|v)$ linear subspaces, Bézout gives

$$\deg_{\mathcal{M}^{\otimes 3}}(\varphi_{v'}(\mathcal{N}_{v'})) \leq e(v'|v) \deg_{\mathcal{M}^{\otimes 3}}(\mathcal{A}'_{v'}).$$

As $\dim \varphi_{v'}(\mathcal{N}_{v'}) = g - q$, this becomes

$$(3e(v'|v))^{g-q}(g-q)! h^0(\varphi_{v'}(\mathcal{N}_{v'}), \mathcal{L}) = (3e(v'|v))^{g-q} \deg_{\mathcal{L}}(\varphi_{v'}(\mathcal{N}_{v'})) \leq e(v'|v)(3e(v'|v))^g \deg_{\mathcal{L}}(\mathcal{A}'_{v'}).$$

We are done since $\deg_{\mathcal{L}}(\mathcal{A}'_{v'}) = \deg_L(A) = g!h^0(A, L)$. $\square$

To proceed, we need to study formal neighborhoods of $\varphi_{v'}(\mathcal{N}_{v'})$ in $\mathcal{A}'$ as follows. Denote by $\mathfrak{m}'$ the maximal ideal of $\mathcal{O}_{F'}$. For any $n \geq 0$, let $\varphi_n \colon \mathcal{N} \otimes_{\mathcal{O}_F} \mathcal{O}_{F'}/\mathfrak{m}'^{n+1} \to \mathcal{A}' \otimes_{\mathcal{O}_{F'}} \mathcal{O}_{F'}/\mathfrak{m}'^{n+1}$ be induced by φ and set

$$N_n := \text{the schematic image of } \mathcal{N} \otimes_{\mathcal{O}_F} \mathcal{O}_{F'}/\mathfrak{m}'^{n+1}.$$

Then $N_0 = \varphi_{v'}(\mathcal{N}_{v'})$ and, for any $0 \leq i \leq n$, we have

$$\widehat{\mathcal{O}}_{N_i} \simeq \widehat{\mathcal{O}}_{N_0} \widehat{\otimes}_{\mathbb{C}} \mathbb{C}[\varpi, y_1, \ldots, y_q]/(\varpi'^j y_1^{\nu_1} \cdots y_q^{\nu_q} : j + b_1\nu_1 + \ldots + b_q\nu_q \geq i+1).$$

So the rank of the $\mathcal{O}_{N_0}$-module $\ker(\mathcal{O}_{N_i} \to \mathcal{O}_{N_{i-1}})$ is

$$l_i := \#\left\{(j, \nu_1, \ldots, \nu_q) \in \mathbb{Z}_{\geq 0}^{q+1} : j + b_1\nu_1 + \cdots + b_q\nu_q = i\right\}.$$

Along the generic point of each component of N_0, the generic length of N_n is

$$\tau_n := \sum\nolimits_{i=0}^n l_i = \#\left\{(j, \nu_1, \ldots, \nu_q) \in \mathbb{Z}_{\geq 0}^{q+1} : j + b_1\nu_1 + \cdots + b_q\nu_q \leq n\right\}.$$

Lemma 10.4. (i) $\tau_n = \dfrac{n^{q+1}}{(q+1)!b_1 \cdots b_q} + O(n^q)$, *as* $n \to \infty$.

(ii) $h^0(N_n, m\mathcal{L}) \leq \tau_n \cdot h^0(N_0, m\mathcal{L})$.

Proof. The generating series of the numbers l_i is

$$\sum_{i\geq 0} l_i T^i = \sum_{j,\nu_1,\ldots,\nu_q \geq 0} T^{j+b_1\nu_1+\cdots+b_q\nu_q} = \Big(\sum_{j\geq 0} T^j\Big) \prod_{k=1}^q \Big(\sum_{\nu_i \geq 0} T^{b_k\nu_k}\Big) = \frac{1}{1-T} \prod_{k=1}^q \frac{1}{1-T^{b_k}}.$$

Hence the generating series of τ_n is

$$\sum_{n\geq 0} \tau_n T^n = \sum_{n\geq 0}\sum_{i=0}^n l_i T^n = \sum_{i\geq 0} l_i \sum_{n \geq i} T^n = \frac{1}{1-T}\sum_{i\geq 0} l_i T^i = \frac{1}{(1-T)^2 \prod_{k=1}^q (1-T^{b_k})}.$$

Near $T = 1$, we have $1 - T^{b_k} = b_k(1-T) + O((1-T)^2)$, and hence

$$\frac{1}{(1-T)^2 \prod_{k=1}^q (1-T^{b_k})} = \frac{1}{b_1 \cdots b_q} \frac{1}{(1-T)^{q+2}} + O\left(\frac{1}{(1-T)^{q+1}}\right).$$

Taking the coefficient of T^{r-1}, we get

$$\tau_n = \frac{1}{b_1 \cdots b_q} \binom{n+q+1}{q+1} + O(n^q), \quad \text{as } n \to \infty.$$

This establishes (i).

For (ii), consider the (regular) embedding $N_0 \hookrightarrow \mathcal{A}'_{v'}$ of smooth algebraic groups defined over $\mathbb{C}$. The normal bundle is $\simeq (\operatorname{Lie}\mathcal{A}'_{v'}/\operatorname{Lie} N_0) \otimes_{\mathbb{C}} \mathcal{O}_{N_0} \simeq \mathcal{O}_{N_0}^{\oplus q}$. Taking the dual yields

$$\mathcal{I}_{N_0}/\mathcal{I}_{N_0}^2 \simeq \mathcal{O}_{N_0}^{\oplus q}.$$

Thus all symmetric powers of $\mathcal{I}_{N_0}/\mathcal{I}_{N_0}^2$ are also trivial, and therefore

$$\ker(\mathcal{O}_{N_i} \to \mathcal{O}_{N_{i-1}}) \simeq \bigoplus_{j,\nu_1,\ldots,\nu_q \geq 0,\ j+b_1\nu_1+\ldots+b_q\nu_q = i} \mathcal{O}_{N_0} = \mathcal{O}_{N_0}^{\oplus l_i}.$$

This gives a short exact sequence, for all $i \geq 1$,

$$0 \to \mathcal{O}_{N_0}^{\oplus l_i} \to \mathcal{O}_{N_i} \to \mathcal{O}_{N_{i-1}} \to 0.$$

Tensoring with $\mathcal{L}^{\otimes m}$ and taking global sections, we get

$$h^0(N_i, \mathcal{L}^{\otimes m}) \leq h^0(N_{i-1}, \mathcal{L}^{\otimes m}) + l_i h^0(N_0, \mathcal{L}^{\otimes m})$$

We are done for (ii) by taking the sum of the inequality above over $1 \leq i \leq n$. □

Set

$$\lambda := \frac{1}{3e(v'|v)}\left(\binom{g}{q}^{-1}\frac{b_1\cdots b_q}{e(v'|v)}\right)^{1/q} \geq \frac{1}{3g}e(v'|v)^{-1-1/q}. \tag{10.7}$$

Corollary 10.5. *Let $n = \lfloor \lambda m \rfloor$. Then we have*

$$h^0(A, mL)\log|\varpi'|^{-n} - h^0(N_n, m\mathcal{L}) \geq h^0(A, L)m^{g+1}\frac{\lambda}{2} - O(m^g).$$

Proof. We have

$$\begin{aligned}
h^0(N_n, m\mathcal{L}) &\leq \left(\frac{n^{q+1}}{(q+1)!b_1\cdots b_q} + O(n^q)\right)m^{g-q}h^0(N_n,\mathcal{L}) \qquad \text{by Lemma 10.4}\\
&\leq \left(\frac{n^{q+1}}{(q+1)!b_1\cdots b_q} + O(n^q)\right)m^{g-q}\frac{3^q e(v'|v)^{q+1}g!h^0(A,L)}{(g-q)!} \qquad \text{by Lemma 10.3}\\
&= 3^q h^0(A,L)\frac{1}{g+1}\binom{g+1}{q+1}\frac{\lambda^{q+1}e(v'|v)^{q+1}m^{g+1}}{b_1\cdots b_q} + O(m^g) \text{ since } n = \lfloor \lambda m \rfloor = \lambda m + O(1)\\
&= \frac{\lambda}{q+1}h^0(A,L)m^{g+1} + O(m^g) \qquad \text{by choice of } \lambda \text{ from (10.7).}
\end{aligned}$$

But $h^0(A, mL) = m^g h^0(A, L)$ and $\log|\varpi'| = -1$ and $n = \lambda m + O(1)$, so

$$h^0(A, mL)\log|\varpi'|^{-n} - h^0(N_n, m\mathcal{L}) \geq h^0(A, L)m^{g+1}\frac{q}{q+1}\lambda - O(m^g). \tag{10.8}$$

But $1 \leq q \leq g$, so $\frac{q}{q+1} \geq \frac{1}{2}$. We are done. □

10.3. **Global computation and Proof of Theorem 10.1.** Let us proceed to prove Theorem 10.1 in several steps. The proof is an adaption of §7. Since most computation is done over K', by abuse of notation we let $\overline{L}$ be the canonical adelic extension of $L_{K'} = L \otimes_K K'$. Then we adopt the notation $\epsilon = 1/(8g)$ and the construction of the perturbed adelic line bundle $\overline{L}(\epsilon\mathbf{f})$ from §7.1 (which is over $A_{K'}$ now).

10.3.1. *Set up of the local invariants.* Set

$$\mathfrak{B}^0_{A/K'} := \{v' \in M_{K'} : A_{K'} \text{ has good reduction at } v', \text{ but } A \text{ has bad reduction at } v'|_K\}.$$

Let $v' \in \mathfrak{B}^0_{A/K'}$, and let v be its restriction to K. Write $e(v'|v)$ for the ramification index. In particular, $e(v'|v) \leq [K' : K] \leq 3^{4g^2}$. Let $q_{v'} \geq 1$ be the integer appearing in (10.5), and let $\lambda_{v'}$ be the number defined in (10.7), with q replaced by $q_{v'}$ and with the positive integers $b_{v',1}, \ldots, b_{v',q_{v'}}$ chosen as in (10.6). Then

$$\lambda_{v'} \geq \frac{1}{3g}e(v'|v)^{-2} \geq \frac{1}{3^{8g^2+1}g}. \tag{10.9}$$

10.3.2. *Modified norm and a small section over K'.* For $m \geq 1$, set

$$E_m := H^0(A_{K'}, mL).$$

Then $h^0(A_{K'}, mL) = m^g h^0(A, L)$. The adelic line bundle $m\overline{L}(\epsilon\mathbf{f})$ defines an adelic vector space over K'

$$\overline{E}_m := \left(E_m, \{\|\cdot\|_{m\overline{L}(\epsilon\mathbf{f}),v',\sup}\}_{v'\in M_{K'}}\right).$$

Moreover, for the Néron model $\pi' \colon \mathcal{A}' \to B'$ of $A_{K'}/K'$ and the canonical extension $\mathcal{L}$ of $L_{K'}$, we know that $\|\cdot\|_{m\overline{L}(\epsilon\mathbf{f}),v',\sup}$ is induced by the vector bundle $\mathcal{E}_m := \pi'_*\mathcal{L}^{\otimes m}|_{U_0}$ on U_0, for all $v' \in U_0 := B' \setminus \mathfrak{Bad}(A/K)$, *i.e.* $\mathcal{E}_m|_{\operatorname{Spec}\mathcal{O}_{v'}}$ is the unit ball/lattice for $\|\cdot\|_{m\overline{L}(\epsilon\mathbf{f}),v',\sup} = \|\cdot\|_{m\overline{L},v',\sup}$.[9]

Now we modify the metrics at each $v' \in \mathfrak{B}^0_{A/K'}$. For the infinitesimal neighborhood $N_{v'} := N_{v',n_{v'}} \subseteq \mathcal{A}'_{v'}$ introduced in the previous subsection, restriction to $N_{v'}$ gives a homomorphism $\mathrm{red}_{m,v'} \colon \mathcal{E}_m|_{\operatorname{Spec}\mathcal{O}_{v'}} \longrightarrow H^0\left(N_{v'}, \mathcal{L}_{v'}^{\otimes m}|_{N_{v'}}\right)$. Set

$$\mathcal{S}_{m,v'} := \ker(\mathrm{red}_{m,v'}).$$

Then $\mathcal{S}_{m,v'}$ is an $\mathcal{O}_{v'}$-lattice in E_m, and

$$\mathrm{length}_{\mathcal{O}_{v'}}\left(\mathrm{im}(\mathrm{red}_{m,v'})\right) \le h^0(N_{v'}, m\mathcal{L}).$$

Denote by $\|\cdot\|_{\mathcal{S}_{m,v'}}$ the lattice norm induced by $\mathcal{S}_{m,v'}$. We now define a new adelic structure $\overline{E}_m^{\mathrm{new}} := (E_m, \{\|\cdot\|_{\mathrm{new},v'}\}_{v'\in M_{K'}})$ on E_m by

$$\|s\|_{\mathrm{new},v'} := \begin{cases} \|s\|_{m\overline{L}(\epsilon\mathbf{f}),v',\sup}, & v' \notin \mathfrak{B}^0_{A/K'} \\ |\varpi_{v'}|_{v'}^{n_{v'} - \left\lfloor \frac{\lambda_{v'}}{4} m \right\rfloor} \|s\|_{\mathcal{S}_{m,v'}}, & v' \in \mathfrak{B}^0_{A/K'}. \end{cases}$$

At each $v' \in \mathfrak{B}^0_{A/K'}$, replacing $\mathcal{E}_m|_{\operatorname{Spec}\mathcal{O}_{v'}}$ by $\mathcal{S}_{m,v'}$ decreases $\widehat{\deg}$ by $\mathrm{length}\left(\mathrm{im}(\mathrm{red}_{m,v'})\right)$, and multiplying the metric by $|\varpi_{v'}|_{v'}^{n_{v'} - \lfloor \lambda_{v'} m/4 \rfloor}$ decreases $\widehat{\deg}$ by $h^0(A, mL)\log|\varpi_{v'}|_{v'}^{n_{v'} - \lfloor \lambda_{v'} m/4 \rfloor}$. Therefore we have (recall that $\log|\varpi_{v'}|_{v'} = -1$)

$$\begin{aligned} \widehat{\deg}(\overline{E}_m^{\mathrm{new}}) &= \widehat{\deg}(\overline{E}_m) + \sum\nolimits_{v'\in\mathfrak{B}^0_{A/K'}} \left(h^0(A, mL)\log|\varpi_{v'}|_{v'}^{-n_{v'} + \left\lfloor \frac{\lambda_{v'}}{4} m \right\rfloor} - \mathrm{length}\left(\mathrm{im}(\mathrm{red}_{m,v'})\right) \right) \\ &\ge \widehat{\deg}(\overline{E}_m) + \sum\nolimits_{v'\in\mathfrak{B}^0_{A/K'}} \left(h^0(A, mL)\log|\varpi_{v'}|_{v'}^{-n_{v'} + \left\lfloor \frac{\lambda_{v'}}{4} m \right\rfloor} - h^0(N_{v'}, m\mathcal{L}) \right) \qquad (10.10) \\ &\ge \widehat{\deg}(\overline{E}_m) + m^{g+1}h^0(A, L)\sum\nolimits_{v'\in\mathfrak{B}^0_{A/K'}} \frac{\lambda_{v'}}{4} - O(m^g) \text{ by Corollary 10.5.} \end{aligned}$$

We wish to apply Theorem 2.3 to $\overline{E}_m^{\mathrm{new}}$. Set

$$U := U_0 \setminus \mathfrak{B}^0_{A/K'} = B' \setminus \left(\mathfrak{Bad}(A_{K'}/K') \sqcup \mathfrak{B}^0_{A/K'}\right).$$

Then for each m and each $v' \in U$, the metric $\|\cdot\|_{\mathrm{new},v'}$ is induced by the vector bundle $\mathcal{E}_m|_U$ on U. Now

$$\begin{aligned} &\frac{\widehat{\deg}(\overline{E}_m^{\mathrm{new}}) - h^0(A, mL)\left(g(B) - 1 + \#(B\setminus U)\right)}{m^{g+1}h^0(A, L)} \\ \ge& \frac{\overline{L}(\epsilon\mathbf{f})^{g+1}}{(g+1)!h^0(A, L)} + \sum\nolimits_{v'\in\mathfrak{B}^0_{A/K'}} \frac{\lambda_{v'}}{4} + o(1) \qquad \text{by (10.10) and (2.6)} \\ \ge& \epsilon\left(2(1-\epsilon)^g - 2(1+\epsilon)^g + 1\right) \sum_{v'\in\mathfrak{Bad}(A_{K'}/K')} \widetilde{J}_{v'}(A_{K'}) + \sum\nolimits_{v'\in\mathfrak{B}^0_{A/K'}} \frac{\lambda_{v'}}{4} + o(1) \quad \text{by (7.8)} \end{aligned}$$

as $m \to \infty$. The right hand side > 0 when $m \gg 1$, because $\mathfrak{Bad}(A_{K'}/K') \bigcup \mathfrak{B}^0_{A/K'} \ne \emptyset$. Thus by Theroem 2.3, there exists a non-zero global section $s' \in H^0(A_{K'}, mL)$ such that

$$\|s'\|_{\mathrm{new},v'} \le 1 \quad \text{for all } v' \in M_{K'}.$$

[9]Here we use the notation from §2.5 and $\mathcal{O}_{v'} = \widehat{\mathcal{O}_{B',v'}}$ is the discrete valuation ring of $K'_{v'}$.

10.3.3. *Small section over K and auxiliary point.* Since we aim to find an abelian subvariety of A defined over K, we need to construct a section s defined over K and an auxiliary point $x \in A(K)$. We proceed as follows.

Recall that the extension K'/K is Galois. Set $G := \mathrm{Gal}(K'/K)$. The local data $q_{v'}$, $d_{v'}$, $d_{0,v}$ and the multiset of elementary divisors $\{b_{v',i}\}$ are constant on G-orbits, so $\lambda_{\sigma v'} = \lambda_{v'}$ and $n_{\sigma v'} = n_{v'}$ for all $\sigma \in G$. Thus the modified adelic norms are G-equivariant. Set

$$s := N_{K'/K}(s') = \prod\nolimits_{\sigma\in G} \sigma(s') \in H^0(A, [K' : K]mL) \setminus \{0\}.$$

Similar to Proposition 7.2, we apply Nakamaye's [Nak07] to get a point $x \in A(K)_{\mathrm{tor}}$ such that the vanishing order of s at x satisfies

$$\mathrm{ord}_x(s) \le g\left(\frac{g[K' : K]m}{\mathfrak{N}} + 1\right). \tag{10.11}$$

10.3.4. *Jet estimates for s'.* From now on we go back to work over K' for our computation.

Let ℓ be the vanishing order of s' at $x \in A(K)$. Then ℓ is also the vanishing order of $\sigma(s')$ at x for each $\sigma \in G$. The Leibniz rule then implies

$$\mathrm{jet}^{[K':K]\ell} s(x) = \bigotimes_{\sigma\in G} \mathrm{jet}^{\ell}(\sigma(s'))(x) \ne 0.$$

Therefore $[K' : K]\ell \le \mathrm{ord}_x(s)$, and hence by (10.11) we have

$$\ell \le g\left(\frac{gm}{\mathfrak{N}} + 1\right). \tag{10.12}$$

At $v' \in \mathfrak{Bad}(A_{K'}/K')$, we still have $\log \|\mathrm{jet}^{\ell} s'(x)\|_{v'} \le -m\epsilon \widetilde{J}_{v'}(A_{K'})$ by Lemma 7.4. Hence by (7.4) (and $\epsilon = \frac{1}{8g}$), we have

$$\log \|\mathrm{jet}^{\ell} s'(x)\|_{v'} \le -\frac{m}{8 \cdot 128e \cdot \gamma_g} (\#\Phi_{v'})^{-1/r_{v'}}. \tag{10.13}$$

At $v' \in \mathfrak{B}^0(A/K')$, again we have $\|\mathrm{jet}^{\ell} s'(x)\|_{v'} \le \|s'(x)\|_{\overline{L},v'} = \|s'(x)\|_{\mathcal{L}_{v'}}$ as in the proof of Lemma 7.4. Next we shall estimate $\|s'(x)\|_{\mathcal{L}_{v'}}$. Indeed, we have

Lemma 10.6. *For every $x \in A(K)$ and $0 \ne s' \in E_m$, we have*

$$\|s'(x)\|_{\mathcal{L}_{v'}} \le \|s'\|_{\mathrm{new},v'} |\varpi'|_{v'}^{\lfloor \frac{\lambda_{v'}}{4} m \rfloor}.$$

An upshot of this lemma and (10.9) is then (recall $\log |\varpi'|_{v'} = -1$)

$$\log \|\mathrm{jet}^{\ell} s'(x)\|_{v'} \le -\lfloor \frac{\lambda_{v'}}{4} m \rfloor = -\frac{m}{4 \cdot 3^{8g^2+1} g} + o(m), \text{ for any } v \in \mathfrak{B}^0_{A/K'}. \tag{10.14}$$

Proof. Let $\xi \in K'_{v'}$ with minimal valuation, such that $\xi^{-1}s' \in \mathcal{S}_{m,v'}$. Then we have, by definition, $|\xi|_{v'} = \|s'\|_{\mathcal{S}_{m,v'}}$.

Note that the Zariski closure $\overline{x}$ of x in $\mathcal{A}'_{v'}$ is contained in the schematic image of $\varphi \colon \mathcal{N} \otimes_{\mathcal{O}_{K_v}} \mathcal{O}_{K'_{v'}} \to \mathcal{A}'_{v'}$, and $\xi^{-1}s' \in \mathcal{S}_{m,v'}$ vanishes on the $n_{v'}$-th infinitesimal neighborhood $N_{v'}$. Hence the restriction of $\xi^{-1}s'$ on $\overline{x}$ also vanishes on the n_v-th infinitesimal neighborhood of the special fiber of $\overline{x}$. This implies $\|\xi^{-1}s'(x)\|_{\mathcal{L}_{v'}} \le |\varpi'|_{v'}^{n_v}$. So

$$\|s'(x)\|_{\mathcal{L}_{v'}} \le |\xi|_{v'} |\varpi'|_{v'}^{n_v} = \|s'\|_{\mathcal{S}_{m,v'}} |\varpi'|_{v'}^{n_v} = \|s'\|_{\mathrm{new},v'} |\varpi'|_{v'}^{\lfloor \frac{\lambda_{v'}}{4} m \rfloor}$$

as desired. $\square$

The inequality (7.19) is still true with K replaced by K' and s replaced by s'. We have $\widehat{h}_L(x)=0$ since $x\in A(K)_{\mathrm{tor}}$, and so the inequalities (10.12), (10.13), and (10.14) imply

$$\frac{1}{[K':K_0]}\left(\frac{m}{8\cdot 128e\gamma_g}\sum_{v'\in\mathfrak{Bad}(A_{K'}/K')}(\#\Phi_{v'})^{-1/r_{v'}}+\frac{m\cdot\#\mathfrak{B}^0_{A/K'}}{4\cdot 3^{8g^2+1}g}+o(m)\right)\le\frac{g(g+2)}{2}\left(\frac{gm}{\mathfrak{N}}+1\right)h_{\mathrm{Fal}}(A). \tag{10.15}$$

Divide both sides by m and let $m\to\infty$. Together with $[K':K_0]=[K':K][K:K_0]\le 3^{4g^2}[K:K_0]$ and $e<3$, we get

$$\mathfrak{N}\le 4\cdot 128\cdot g^2(g+2)3^{4g^2+1}\gamma_g\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\sum_{v'\in\mathfrak{Bad}(A_{K'}/K')}(\#\Phi_{v'})^{-1/r_{v'}}+\frac{256e\gamma_g\#\mathfrak{B}^0_{A/K'}}{3^{8g^2+1}g}}. \tag{10.16}$$

10.3.5. *Final conclusion.* To ease notation, denote by

$$\mathfrak{b}:=\#\mathfrak{Bad}(A_{K'}/K'),\quad \mathfrak{b}_0:=\#\mathfrak{B}^0_{A/K'}.$$

Then the number of non-semi-stable reduction places of A/K is $\le\mathfrak{b}+\mathfrak{b}_0$, and hence

$$g(B')\le[K':K](g(B)+\mathfrak{b}+\mathfrak{b}_0). \tag{10.17}$$

Now that $A_{K'}/K'$ has semi-stable reduction, we can apply the Arakelov Inequality (Theorem 9.2) to $A_{K'}/K'$. Combining with (10.17), we get the following crude bound

$$[K:K_0]h_{\mathrm{Fal}}(A)\le\frac{3g}{2}\left(g(B)+\mathfrak{b}+\mathfrak{b}_0\right). \tag{10.18}$$

To ease notation, denote by (recall our assumption that $\mathfrak{b}+\mathfrak{b}_0\ge 1$)

$$\Delta:=\sum_{v'\in\mathfrak{Bad}(A_{K'}/K')}(\#\Phi_{v'})^{-1/r_{v'}}+\frac{256e\gamma_g\mathfrak{b}_0}{3^{8g^2+1}g}>0.$$

Notice that $A_{K'}/K'$ has semi-stable reduction, and hence we can apply (9.4) and get

$$\Delta\ge\sum_{v'\in\mathfrak{Bad}(A_{K'}/K')}(\#\Phi_{v'})^{-1/r_{v'}}\ge\frac{\mathfrak{b}^2}{96\cdot 3^{4g^2}[K:K_0]h_{\mathrm{Fal}}(A)}. \tag{10.19}$$

We claim:

$$\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\Delta}\le 3^{8g^2+1}g^2\left(g(B)+1\right)^2 \tag{10.20}$$

and

$$\frac{[K:K_0]\max\{h_{\mathrm{Fal}}(A),1\}}{\Delta}\le 3^{8g^2+1}g^2[K:\mathbb{Q}]\left(g(B)+1\right)^2. \tag{10.21}$$

$\boxed{\text{Case } \mathfrak{b}_0=0}$ In this case we have $\mathfrak{b}\ge 1$. So taking the inverse of (10.19) and multiplying by $[K:K_0]h_{\mathrm{Fal}}(A)$, we get in this case

$$\begin{aligned}\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\Delta}&\le 96\cdot 3^{4g^2}\left(\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\mathfrak{b}}\right)^2\\&\le 96\cdot 3^{4g^2}\frac{9g^2}{4}\left(\frac{g(B)}{\mathfrak{b}}+1\right)^2\le 24\cdot 3^{4g^2+2}g^2\left(g(B)+1\right)^2\quad\text{by (10.18).}\end{aligned}$$

This establishes (10.20) in this case. For (10.21), it suffices to notice

$$\frac{1}{\Delta}\le\frac{96\cdot 3^{4g^2}}{\mathfrak{b}}\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\mathfrak{b}}\le 96\cdot 3^{4g^2}\cdot\frac{3g}{2}(g(B)+1)=48\cdot 3^{4g^2+1}g\left(g(B)+1\right).$$

$\boxed{\text{Case } \mathfrak{b}_0 \geq 1}$ Now (10.18) implies

$$\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\Delta} \leq \frac{3g}{2}\left(\frac{g(B)}{\Delta}+\frac{\mathfrak{b}_0}{\Delta}+\frac{\mathfrak{b}}{\Delta}\right) \leq \frac{3g}{2}\left(\frac{3^{8g^2+1}g\cdot g(B)}{256e\gamma_g}+\frac{3^{8g^2+1}g}{256e\gamma_g}+\frac{\mathfrak{b}}{\Delta}\right). \tag{10.22}$$

By (10.19), we get

$$\frac{3g}{2}\frac{\mathfrak{b}}{\Delta} \leq \frac{3g}{2}\sqrt{\frac{96\cdot 3^{4g^2}[K:K_0]h_{\mathrm{Fal}}(A)}{\Delta}} \leq \frac{1}{2}\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\Delta}+\frac{1}{2}\left(\frac{3g}{2}\sqrt{96\cdot 3^{4g^2}}\right)^2$$

Putting this estimate in (10.22), we get

$$\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\Delta} \leq \frac{3^{8g^2+2}g^2}{256e\gamma_g}\left(g(B)+1\right)+24g^2 3^{4g^2+2} \leq 3^{8g^2+1}g^2(g(B)+1).$$

This establishes (10.20) in this case. For (10.21), it suffices to notice $\Delta \geq \frac{256e\gamma_g}{3^{8g^2+1}g}$.

Our desired bound (10.2) now follows immediately from (10.16) and (10.20). □

10.4. **Proof of Theorem 10.2.** The proof of Theorem 10.1 executed in §10.3 can be modified to prove Theorem 10.2. Let us explain the modifications.

Define $\mathfrak{N}'$ to be the minimum of the left hand side of (10.4) when A' runs over all proper abelian subvarieties of A_K (or of $A_{\overline{K}}$ – they are the same by Gaudron–Rémond [GR25, Lem. 6.1]). Then Theorem 10.2 is equivalent to:

$$\mathfrak{N}' < 8\cdot 128\cdot 3^{12g^2+2}g^4(g+2)\gamma_g\left(g(B)+1\right)^2. \tag{10.23}$$

Run the arguments in §10.3. The first modification is the choice of the auxiliary point x above (10.11). In the current situation, similar to Remark 7.3 our auxiliary point x is chosen to be in $\Gamma[g] := \{x_1+\cdots+x_g : x_i\in\Gamma\}$. Hence by definition (10.3) of Γ we have

$$\widehat{h}_L(x) \leq g^2\frac{\max\{h_{\mathrm{Fal}}(A),1\}}{2048\cdot 3^{12g^2+2}g^4\gamma_g[K:K_0]\left(g(B)+1\right)^2} \leq \frac{1}{2048\gamma_g}\frac{\Delta}{3^{4g^2+1}[K:K_0]} \leq \frac{1}{16\cdot 128e\gamma_g}\frac{\Delta}{[K':K_0]},$$

where the second inequality follows from (10.21) and the third inequality follows from $[K':K] \leq 3^{4g^2}$ and $2048 = 16\cdot 128$ and $e<3$.

Then we can proceed before applying the bounds (10.12), (10.13), and (10.14) to the inequality (7.19). In the current situation, the upper bound of $\widehat{h}_L(x)$ is given above, and hence instead of (10.15) we get

$$\frac{1}{[K':K_0]}\left(\frac{\Delta\cdot m}{16\cdot 128e\gamma_g}+o(m)\right) \leq \frac{g(g+2)}{2}\left(\frac{gm}{\mathfrak{N}'}+1\right)h_{\mathrm{Fal}}(A). \tag{10.24}$$

Dividing both sides by m and letting $m\to\infty$, we get (noticing that $[K':K_0] = [K':K][K:K_0] \leq 3^{4g^2}[K:K_0]$ and $e<3$)

$$\mathfrak{N}' \leq 8\cdot 128\cdot g^2(g+2)3^{4g^2+1}\gamma_g\frac{[K:K_0]h_{\mathrm{Fal}}(A)}{\Delta}.$$

Therefore, the desired bound (10.23) follows immediately from (10.20). We are done. □

11. Proof of the main results over function fields: general case

11.1. **Proof of the bound on $\#A(K)_{\mathrm{tor}}$ (Theorem 1.3).** By the Poincaré Irreducibility Theorem, A is isogenous to a product of K-simple abelian varieties. Each such K-simple abelian varieties (counted with multiplicity) is called a *factor* of A.

11.1.1. *Case: no factor of A has good reduction everywhere.* Assume no factor of A has good reduction everywhere. Let us prove:

$$\#A(K)_{\mathrm{tor}} \leq h^0(A,L)\left(512\cdot 3^{12g^2+2}g^4(g+2)\gamma_g\right)^g (g(B)+1)^{2g}. \tag{11.1}$$

Notice that (1.2) follows immediately from (11.1) by Zarhin's trick (which asserts that $(A\times A^\vee)^4$ carries a principal polarization).

We will proceed by induction on the number of factors of A, which we denote by n. Denote by $K_0 = \mathbb{C}(\mathbb{P}^1)$; then $[K : K_0] = \mathrm{gon}(B)$.

$\boxed{\text{Base step: } n = 1}$ In this case A is simple. Our assumption becomes: A does not have good reduction everywhere. The conclusion then follows immediately from Theorem 10.1.

$\boxed{\text{Induction step}}$ For arbitrary $n \geq 2$. By Theorem 10.1, there exists a proper abelian subvariety A' of A defined over K such that

$$\#(A/A')(K)_{\mathrm{tor}} \leq \frac{h^0(A,L)}{h^0(A',L)}\left(512\cdot 3^{12g^2+2}g^4(g+2)\gamma_g\right)^{g-\dim A'}(g(B)+1)^{2(g-\dim A')}.$$

Since $A'\neq A$, the number of factors of A' is $\leq n-1$. So we can apply the induction hypothesis to A' and get

$$\#A'(K)_{\mathrm{tor}} \leq h^0(A',L)\left(512\cdot 3^{12g^2+2}g^4(g+2)\gamma_g\right)^{\dim A'}(g(B)+1)^{2\dim A'}.$$

Thus the desired bound (11.1) follows from the product of the two bounds above.

11.1.2. *Case: A has good reduction everywhere and is traceless.* Next let us prove: If A has good reduction everywhere, is traceless *i.e.* $(\mathrm{Tr}_{K/\mathbb{C}}A)(\mathbb{C}) = 0$, and carries a principle polarization given by a symmetric ample line bundle L defined over K, then we have

$$\#A(K)_{\mathrm{tor}} \leq 2^{2^{2N'^2+4N'\log_2 N'}}, \qquad \text{where } N' = 4^g(g(B)+3)-1. \tag{11.2}$$

Since A/K has good reduction everywhere, it extends to an abelian scheme $\mathcal{A}\to B$. The dual abelian scheme $\pi\colon \mathcal{A}^\vee \to B$ has generic fiber $A^\vee$, and the total space is a projective variety defined over $\mathbb{C}$. Write $\iota\colon A^\vee \to \mathcal{A}^\vee$ for the natural inclusion. We have the following exact sequence of groups:

$$0\to \mathrm{Pic}(B)\xrightarrow{\pi^*}\mathrm{Pic}(\mathcal{A}^\vee)\xrightarrow{\iota^*}\mathrm{Pic}(A^\vee)\to 0. \tag{11.3}$$

Indeed, π^* is clearly injective, ι^* is surjective as $\mathcal{A}^\vee$ is regular (so Weil divisors and Cartier divisors are the same), and the exactness at $\mathrm{Pic}(\mathcal{A}^\vee)$ follows from [Kle13, Thm. 2.5].

We have $\iota^*\big(\mathrm{Pic}^0(\mathcal{A}^\vee)\big)\subseteq \mathrm{Pic}^0(A^\vee) = A(K)$. Moreover $\mathrm{Pic}^0(\mathcal{A}^\vee)$ is an abelian variety defined over $\mathbb{C}$, and hence $\iota^*\big(\mathrm{Pic}^0(\mathcal{A}^\vee)\big)\subseteq (\mathrm{Tr}_{K/\mathbb{C}}A)(\mathbb{C}) = 0$ by the universal property of the trace. Therefore the exact sequence (11.3) induces an exact sequence

$$0\longrightarrow \mathrm{Pic}(B)/\mathrm{Pic}^0(B)\longrightarrow \mathrm{NS}(\mathcal{A}^\vee) := \mathrm{Pic}(\mathcal{A}^\vee)/\mathrm{Pic}^0(\mathcal{A}^\vee)\longrightarrow \mathrm{Pic}(A^\vee)\longrightarrow 0. \tag{11.4}$$

But B is a smooth projective irreducible curve defined over $\mathbb{C}$, and hence $\mathrm{Pic}(B)/\mathrm{Pic}^0(B) = \mathrm{NS}(B)\simeq \mathbb{Z}$. Therefore (11.4) induces a natural isomorphism $\mathrm{NS}(\mathcal{A}^\vee)_{\mathrm{tor}} = \mathrm{Pic}(A^\vee)_{\mathrm{tor}}$.

On the other hand, we have that $\mathrm{Pic}(A^\vee)/\mathrm{Pic}^0(A^\vee)\subseteq \mathrm{NS}(A^\vee_{0,\overline{K}})$ is torsion-free. So the inclusion $\mathrm{Pic}^0(A^\vee)\subseteq \mathrm{Pic}(A^\vee)$ induces $\mathrm{Pic}(A^\vee)_{\mathrm{tor}} = \mathrm{Pic}^0(A^\vee)_{\mathrm{tor}} = A(K)_{\mathrm{tor}}$.

The upshot of the previous two paragraphs is that we have a natural isomorphism

$$A(K)_{\mathrm{tor}} = \mathrm{NS}(\mathcal{A}^\vee)_{\mathrm{tor}}.$$

Hence to bound $\#A(K)_{\mathrm{tor}}$, it suffices to bound $\#\mathrm{NS}(\mathcal{A}^\vee)_{\mathrm{tor}}$. We shall invoke [Kwe21], for which we need to embed $\mathcal{A}^\vee$ smoothly into a projective space $\mathbb{P}^N$. First the principle polarization on A defined by L gives rise to a symmetric ample line bundle L' on $A^\vee$ such that $h^0(A^\vee, L') = 1$. Then there exists a symmetric relatively ample line bundle $\mathcal{L}'$ on $\mathcal{A}^\vee$, whose restriction to the zero section of $\mathcal{A}^\vee/B$ is trivial, such that $\mathcal{L}'|_{A^\vee} = 4L_0'$; see [Ray70, Thm. XI.1.4]. Moreover the complete linear system of $4L$ embeds $A^\vee$ into $\mathbb{P}^{4^g-1}$, whose homogeneous ideal is generated by degree 2 polynomials. Next classical algebraic geometry asserts: there exists a very ample line bundle $\mathcal{M}$ on B such that B is embedded into $\mathbb{P}^{g(B)+2}$ as a smooth curve via the global sections of $\mathcal{M}$, and the homogeneous ideal is generated by degree 2 polynomials. Then the global sections of $\mathcal{L}'+\pi^*\mathcal{M}$ embeds $\mathcal{A}^\vee$ into $\mathbb{P}^{4^g-1}\times\mathbb{P}^{g(B)+2}$ as a smooth subvariety. Composing furthermore with the Veronese embedding $\mathbb{P}^{4^g-1}\times\mathbb{P}^{g(B)+2} \subseteq \mathbb{P}^{N'}$ with $N' = 4^g(g(B)+3)-1$, we have embedded $\mathcal{A}^\vee$ into $\mathbb{P}^{N'}$ as the complete intersection of degree 4 polynomials. Hence $\mathrm{NS}(\mathcal{A}^\vee)_{\mathrm{tor}} \leq 2^{4^{N'^2+2N'\log_2 N'}}$ by [Kwe21, Thm. 4.12]. Thus we get the desired bound (11.2).

11.1.3. *General case.* Now for an arbitrary A, let A_0 be the maximal abelian subvariety defined over K which has good reduction everywhere. Then no K-factor of A/A_0 has good reduction everywhere. The same holds true for $(A/A_0)^4 \times (A/A_0)^{\vee 4}$, which by Zarhin's trick carries a principle polarization. Hence the conclusion of the case §11.1.1 implies that

$$\#(A/A_0)(K)_{\mathrm{tor}} \leq \#\left((A/A_0)^4 \times (A/A_0)^{\vee 4}\right)(K)_{\mathrm{tor}} \leq c_0(g)\,(g(B)+1)^{16g}$$

for the explicit number $c_0(g) = \left(512\cdot 3^{12(8g)^2+2}(8g)^4(8g+2)\gamma_{8g}\right)^{8g} > 0$.

Next let us turn to A_0, which is traceless because $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$ by our assumption. By Zarhin's trick, $(A_0\times A_0^\vee)^4$ carries a principle polarization and hence satisfies all assumptions of the case §11.1.2 (and has dimension $8g$). Hence

$$\#A_0(K)_{\mathrm{tor}} \leq \#(A_0\times A_0^\vee)^4(K)_{\mathrm{tor}} \leq 2^{2^{2N^2+4N\log_2 N}}, \qquad \text{where } N = 4^{8g}(g(B)+3)-1.$$

Hence we can conclude by taking the product of the two inequalities above (notice that $2N^2+4N\log_2 N \leq 4N^2 \leq 4^{16g+1}(g(B)+3)^2$). We are done. ☐

11.2. **Proof of** (1.4) **in Theorem 1.5.** The argument in §9.3 allows to prove a bound in flavor of (1.4) but with an extra dependence on $h^0(A,L)$. We now upgrade this argument to remove this extra dependence. The key observation is to use a fancier version of Zarhin's trick and use a certain *Zarhin's box* to replace the multiple of a fixed point.

Let A, L be as in Theorem 1.5. Let $P \in A(K)$ be such that $\overline{\mathbb{Z}\cdot P}^{\mathrm{Zar}} = A$.

11.2.1. *Case: A/K has good reduction everywhere and $A \neq \mathrm{Tr}_{K/k}(A)\otimes_k K$.* In this case the Néron model of A/K is an abelian scheme $\mathcal{A}\to B$, and L extends to a symmetric ample line bundle $\mathcal{L}$ on B. The Zariski closure $\overline{P}$ of P in $\mathcal{A}$ is a section of $\mathcal{A}\to B$, and hence $\overline{P}^*\mathcal{L}$ is a line bundle on the projective curve B. In this situation, we have

$$\widehat{h}_L(P) = \frac{1}{[K:\mathbb{C}(\mathbb{P}^1)]}\deg(\overline{P}^*\mathcal{L}),$$

where $\deg(\overline{P}^*\mathcal{L})$ means the usual degree of a line bundle and hence is in $\mathbb{Z}$. But $\widehat{h}_L(P) > 0$ by our assumption $A \neq \mathrm{Tr}_{K/k}(A)\otimes_k K$ and $\overline{\mathbb{Z}\cdot P}^{\mathrm{Zar}} = A$. So $\deg(\overline{P}^*\mathcal{L})$ is a positive integer, and thus $\widehat{h}_L(P) \geq \frac{1}{[K:\mathbb{C}(\mathbb{P}^1)]}$. Combined with Deligne's Arakelov Inequality (Theorem 9.2) and $\mathrm{gon}(B) = [K:\mathbb{C}(\mathbb{P}^1)]$, we get

$$\widehat{h}_L(P) \geq \frac{\max\{h_{\mathrm{Fal}}(A),1\}}{\max\{\mathrm{gon}(B), g(g(B)-1)\}}. \tag{11.5}$$

This establishes (1.4) in this case.

11.2.2. *Case: A/K does not have good reduction everywhere.* Set

$$Z(A) := A^4 \times (A^\vee)^4.$$

By the line bundle version of Zarhin's trick (Lemma B.1), $Z(A)$ carries a principal polarization induced by a symmetric ample line bundle M on $Z(A)$ defined over K such that for a suitable K-isogeny

$$\pi\colon A^8 \longrightarrow Z(A),$$

π^*M induces the same polarization on A^8 as $L^{\boxtimes 8}$. Hence

$$\widehat{h}_M\big(\pi(x_1,\ldots,x_8)\big) = \sum\nolimits_{j=1}^{8} \widehat{h}_L(x_j). \tag{11.6}$$

For each $j \in \{1,\ldots,8\}$, define

$$Q_j := \pi(0,\ldots,0,P,0,\ldots,0),$$

where P occurs in the i-th coordinate. Then

$$\overline{\mathbb{Z}Q_1 + \ldots + \mathbb{Z}Q_8}^{\mathrm{Zar}} = Z(A). \tag{11.7}$$

Denote by $K_0 = \mathbb{C}(\mathbb{P}^1)$; then $[K : K_0] = \mathrm{gon}(B)$. We wish to apply Theorem 10.2 to the polarized abelian variety $(Z(A), M)$. Set

$$\Gamma_{Z(A)} := \left\{ x \in Z(A)(K) : \widehat{h}_M(x) \leq \frac{\max\{h_{\mathrm{Fal}}(Z(A)), 1\}}{2048 \cdot 3^{12(8g)^2+2}(8g)^4\gamma_{8g}[K : K_0]\,(g(B)+1)^2} \right\}.$$

Then $\Gamma_{Z(A)}$ is contained in the subsets Γ defined in (10.3), with A replaced by $Z(A)$ and L replaced by M. Since $Z(A)$ does not have good reduction everywhere, Theorem 10.2 (joint with $h^0(Z(A), M) = 1$) implies: there exists a proper abelian subvariety A' of A defined over K such that

$$\left(\#\left(\frac{\Gamma_{Z(A)} + A'}{A'}\right)\right)^{\frac{1}{8g-\dim A'}} \leq 1024 \cdot 3^{12(8g)^2+2}(8g)^4(8g+2)\gamma_{8g}\,(g(B)+1)^2. \tag{11.8}$$

To ease notation denote by $c_2(g) := 1024 \cdot 3^{12(8g)^2+2}(8g)^4(8g+2)\gamma_{8g}$. We claim:

$$\widehat{h}_L(P) > \frac{\max\{h_{\mathrm{Fal}}(A), 1\}}{2c_2(g)^{2g+1}[K : K_0]\,(g(B)+1)^{4g+2}}. \tag{11.9}$$

Assume (11.9) is false. Set

$$m := c_2(g)^g\,(g(B)+1)^{2g}.$$

and consider

$$\mathcal{B}_m := \left\{ \sum_{j=1}^{8} [a_j]Q_j : 0 \leq a_j \leq m \right\}.$$

As (11.9) is false, (11.6) then implies

$$\widehat{h}_M(x) \leq 8m^2\widehat{h}_L(P) \leq \frac{8\max\{h_{\mathrm{Fal}}(A), 1\}}{2c_2(g)[K : K_0]\,(g(B)+1)^2} \qquad \text{for any } x \in \mathcal{B}_m,$$

and hence $\mathcal{B}_m$ is contained in the subset $\Gamma_{Z(A)}$ defined above, because $h_{\mathrm{Fal}}(Z(A)) = 8h_{\mathrm{Fal}}(A)$. Hence (11.8) implies

$$\#\left(\frac{\mathcal{B}_m + A'}{A'}\right)^{\frac{1}{8g-\dim A'}} \leq c_2(g)\,(g(B)+1)^2 = m^{1/g}. \tag{11.10}$$

On the other hand, we claim:

$$\#\left(\frac{\mathcal{B}_m + A'}{A'}\right) \geq (m+1)^{\frac{8g-\dim A'}{g}} > m^{\frac{8g-\dim A'}{g}}. \tag{11.11}$$

Indeed, let $q\colon Z(A) \to Z(A)/A'$ be the quotient map. Let r be the rank of the subgroup $\Xi := \mathbb{Z}q(Q_1) + \ldots + \mathbb{Z}q(Q_8)$ of $(Z(A)/A')(K)$. Then $\#\left(\frac{\mathcal{B}_m+A'}{A'}\right) \geq (m+1)^r$ because there are r independent $q(Q_j)$'s with $j \in \{1,\ldots,8\}$. On the other hand by definition of Q_j, each generator of Ξ is of the form $\varphi(P)$ for some morphism of abelian varieties $\varphi\colon A \to Z(A)/A'$. So $\dim Z(A)/A' \leq gr$ because $\overline{\Xi}^{\mathrm{Zar}} = Z(A)/A'$ by (11.7), and thus $gr \geq 8g - \dim A'$. This establishes (11.11).

Now (11.10) and (11.11) contradict each other. So (11.9) must hold. This establishes (1.4) since $[K : K_0] = \mathrm{gon}(B)$. $\square$

11.3. **Proof of** (1.5) **in Theorem 1.5.** Denote by $K_0 = \mathbb{C}(\mathbb{P}^1)$. Then $\mathrm{gon}(B) = [K : K_0]$.

Let A, L be as in Theorem 1.5. Let $P \in A(K)$ be such that $\overline{\mathbb{Z}\cdot P}^{\mathrm{Zar}} = A$. Since the Faltings height of $\mathrm{Tr}_{K/\mathbb{C}}(A) \otimes K$ is 0, it suffices to prove the result after taking quotient by $\mathrm{Tr}_{K/\mathbb{C}}(A) \otimes K$. So we may assume $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$.

Consider the Weil restriction $A_1 := \mathrm{Res}_{K/K_0}(A)$ and the norm line bundle L_1 on A_1. Then $\mathrm{Tr}_{K_0/\mathbb{C}}(A_1) = 0$ because $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$. Let $P_1 \in A_1(K_0)$ be the image of P under the canonical identification $A(K) = A_1(K_0)$. Then $\widehat{h}_L(P) = \widehat{h}_{L_1}(P_1)$.

Consider the abelian subvariety $A_1' := \left(\overline{\mathbb{Z}P_1}^{\mathrm{Zar}}\right)^0$ of A_1. The image of P_1 under $A_1 \to A_1/A_1'$ is a K_0-torsion point, whose order we denote by m'. Applying (1.3) to A_1/A_1' which is traceless and has dimension $\leq g\cdot\mathrm{gon}(B)$, we get

$$m' \leq c_0(g\cdot\mathrm{gon}(B)^2)\cdot 2^{2^{2^{32g\cdot\mathrm{gon}(B)^2+6}}} \tag{11.12}$$

Moreover, $[m']P_1 \in A_1'(K_0)$ and $\mathbb{Z}\cdot[m']P_1$ is Zariski dense in A_1'. Now apply (1.4) to $(A_1', L_1|_{A_1'})$ and the point $[m']P_1$. Then we get

$$\begin{aligned} m'^2\widehat{h}_L(P) = m'^2\widehat{h}_{L_1}(P_1) = \widehat{h}_{L_1}([m']P_1) &\geq c_0'(\dim A_1')\max\{h_{\mathrm{Fal}}(A_1'),1\} \\ &\geq c_0'(g\cdot\mathrm{gon}(B))\max\{h_{\mathrm{Fal}}(A_1'),1\}. \end{aligned} \tag{11.13}$$

Next, over $\overline{K_0}$ we have

$$A_1 \otimes_{K_0} \overline{K_0} = \prod\nolimits_{\sigma'\colon K\hookrightarrow\overline{K_0}} A^{\sigma'}.$$

The projection $A_1' \to A^{\sigma'}$ is surjective for each σ', because the image of contains the Zariski-dense subset $\mathbb{Z}\cdot[m']P^{\sigma'}$. Hence

$$h_{\mathrm{Fal}}(A_1') \geq \frac{1}{[K:K_0]}h_{\mathrm{Fal}}(A). \tag{11.14}$$

Now (1.5) follows from (11.12), (11.13), and (11.14). We are done. $\square$

12. Proofs of the main results over number fields

12.1. **Generalized Szpiro Conjecture.** The following conjecture is proposed by Hindry. See (2.1) for the definition of $\mathfrak{f}_{A/K}$.

Conjecture 12.1 (Generalized Szpiro Conjecture, strong version, [Hin07, Conj. 3.4])**.** *There exist positive constants $\kappa_2' = \kappa_2'(g,K)$ and $\kappa_3 = \kappa_3(g,K)$ such that*

$$h_{\mathrm{Fal}}(A) \leq \kappa_2'\log\mathfrak{f}_{A/K} + \kappa_3. \tag{12.1}$$

We have proposed in §1.3.1 another version of the generalized Szpiro Conjecture, which is closer to the original Szpiro's Conjecture. Recall the non-archimedean boundary height (1.7), which in view of the generalized Szpiro ratio $\mathfrak{s}_{A/K} \geq 1/g$ from (1.6) is

$$h_{\partial,\mathrm{fin}}(A) = \frac{1}{[K:\mathbb{Q}]}\mathfrak{s}_{A/K}\log \mathfrak{f}_{A/K} \geq \frac{\log \mathfrak{f}_{A/K}}{g[K:\mathbb{Q}]}. \tag{12.2}$$

Lemma 12.2. *[Hin07, Conj. 3.4] implies Conjecture 1.7.*

Proof. Assume Conjecture 12.1 holds true. Then $h_{\mathrm{Fal}}(A) \leq \kappa_2' g[K:\mathbb{Q}]h_{\partial,\mathrm{fin}}(A)+\kappa_3$. But $h_\partial(A) \geq h_{\partial,\mathrm{fin}}(A)$ and $h_\partial(A) \geq \sqrt{3}g/2$. So $h_{\mathrm{Fal}}(A) \leq \kappa_2 h_\partial(A)$ with $\kappa_2 = \kappa_2' g[K:\mathbb{Q}]+\frac{2\kappa_3}{\sqrt{3}g}$. This establishes (1.11). It remains to prove (1.10), *i.e.* $\mathfrak{s}_{A/K} \leq \kappa_1$ for some $\kappa_1 = \kappa_1(g,K)$.

If $\mathfrak{Bad}(A/K) = \emptyset$, then $\mathfrak{s}_{A/K} = 1/g$. So we can take $\kappa_1 = 1/g$ in this case. Assume $\mathfrak{Bad}(A/K) \neq \emptyset$. Then $\log \mathfrak{f}_{A/K} \geq \log 2$ by its definition (2.1). By (12.2) and Proposition 8.1, we have $\mathfrak{s}_{A/K}\log\mathfrak{f}_{A/K} \leq 96[K:\mathbb{Q}]h_{\mathrm{Fal}}(A)+8g\log(\pi\sqrt{2})$. Bounding $h_{\mathrm{Fal}}(A)$ from above by (12.1), we furthermore get

$$\mathfrak{s}_{A/K}\cdot\log\mathfrak{f}_{A/K} \leq 96[K:\mathbb{Q}]\left(\kappa_2'\log\mathfrak{f}_{A/K}+\kappa_3+8g\log\pi+8g\log\sqrt{2}\right).$$

So in this case, we can take $\kappa_1 = 96[K:\mathbb{Q}]\left(\kappa_2'+\frac{\kappa_3+8g\log\pi}{\log 2}+4g\right)$ since $\log\mathfrak{f}_{A/K} \geq \log 2$. To sum up, we are done by letting

$$\kappa_1 := 96[K:\mathbb{Q}]\left(\kappa_2'+\frac{\kappa_3}{\log 2}+8g\log_2\pi+4g\right). \qquad \square$$

12.2. **Bounds in terms of Szpiro ratio.** Let L be a polarization on A. For each $v \in M_{K,\infty}$, consider $\alpha_v(A,L)$ from (1.8). Define

$$h_\partial(A,L) := h_{\partial,\mathrm{fin}}(A)+h_{\partial,\infty}(A,L), \text{ and } h_{\partial,\infty}(A,L) := \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_{K,\infty}}\alpha_v(A,L)\log N_v. \tag{12.3}$$

Then $h_{\partial,\infty}(A,L) \geq \sqrt{3}/2$ since $\alpha_v(A,L) \geq \sqrt{3}/2$ for each $v \in M_{K,\infty}$.

Proposition 12.3. *We have*

$$\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,L)+\log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K'}\widetilde{I}_v(A)\log N_v} \leq \frac{2}{\sqrt{3}}g^2\mathfrak{s}_{A/K}^2[K:\mathbb{Q}]\log[K:\mathbb{Q}]+\frac{\log h^0(A,L)}{h_\partial(A,L)}$$
$$+g^2\mathfrak{s}_{A/K}^2\cdot\max\left\{\frac{h_{\mathrm{Fal}}(A)}{h_\partial(A,L)},\frac{2}{\sqrt{3}}\right\}\cdot[K:\mathbb{Q}].$$

Proof. Use the notation and the choice of $\sigma\colon K\hookrightarrow\mathbb{C}$ from §6.2-6.3. In particular, $\widetilde{I}_\sigma(A) \geq h_{\partial,\infty}(A,L)$.

By Jensen's Inequality applied to the function $t\mapsto t^{-1}$, we have

$$\left(\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{1/r_v}\log N_v\right)\left(\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}\log N_v\right) \geq \left(\sum_{v\in\mathfrak{Bad}(A/K)}\log N_v\right)^2.$$

For the left hand side, we have $\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{1/r_v}\log N_v = \mathfrak{s}_{A/K}\log\mathfrak{f}_{A/K}$ by definition of $\mathfrak{s}_{A/K}$. For the right hand side, we have $\sum_{v\in\mathfrak{Bad}(A/K)}\log N_v \geq \sum_{v\in\mathfrak{Bad}(A/K)}(r_v/g)\log N_v = (\log\mathfrak{f}_{A/K})/g$. So the inequality above implies

$$\mathfrak{s}_{A/K}\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}\log N_v \geq \frac{\log\mathfrak{f}_{A/K}}{g^2},$$

and therefore

$$\sum_{v\in M'_K}\widetilde{I}_v(A)\log N_v=\sum_{v\in\mathfrak{Bad}(A/K)}(\#\Phi_v)^{-1/r_v}\log N_v+\widetilde{I}_\sigma(A)\geq\frac{\frac{1}{g^2}\log\mathfrak{f}_{A/K}+\mathfrak{s}_{A/K}\widetilde{I}_\sigma(A)}{\mathfrak{s}_{A/K}}.$$

Combing this with

$$h_\partial(A,L)=h_{\partial,\mathrm{fin}}(A)+h_{\partial,\infty}(A,L)=\frac{1}{[K:\mathbb{Q}]}\mathfrak{s}_{A/K}\log\mathfrak{f}_{A/K}+h_{\partial,\infty}(A,L)\leq\mathfrak{s}_{A/K}\log\mathfrak{f}_{A/K}+\widetilde{I}_\sigma(A)$$

and (12.2), we get

$$\frac{h_\partial(A,L)}{\sum_{v\in M'_K}\widetilde{I}_v(A)\log N_v}\leq\frac{\mathfrak{s}^2_{A/K}\log\mathfrak{f}_{A/K}+\mathfrak{s}_{A/K}\widetilde{I}_\sigma(A)}{\frac{1}{g^2}\log\mathfrak{f}_{A/K}+\mathfrak{s}_{A/K}\widetilde{I}_\sigma(A)}\leq g^2\mathfrak{s}^2_{A/K},\tag{12.4}$$

where the last inequality follows from $\mathfrak{s}^2_{A/K}\geq 1/g^2$. It follows, combined with $h_\partial(A,L)\geq\sqrt{3}/2$, that

$$\begin{aligned}&\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,L)+\log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M'_K}\widetilde{I}_v(A)\log N_v}\\
=&\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,L)+\log[K:\mathbb{Q}]}{h_\partial(A,L)}g^2\mathfrak{s}^2_{A/K}[K:\mathbb{Q}]\\
\leq&\left(\max\left\{\frac{h_{\mathrm{Fal}}(A)}{h_\partial(A,L)},\frac{2}{\sqrt{3}}\right\}+\frac{\log h^0(A,L)}{h_\partial(A,L)}+\frac{2}{\sqrt{3}}\log[K:\mathbb{Q}]\right)g^2\mathfrak{s}^2_{A/K}[K:\mathbb{Q}].\qquad\square\end{aligned}$$

Corollary 12.4. *Assume Conjecture 1.7. Then there exists a polarization M_A on A of minimal degree such that*

$$\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,M_A)+\log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M'_K}\widetilde{I}_v(A)\log N_v}\leq\kappa_3[K:\mathbb{Q}]\max\{\log[K:\mathbb{Q}],1\}.$$

for $\kappa_3:=g^2\kappa_1^2\max\{\kappa_2+\frac{2}{\sqrt{3}},\frac{4}{\sqrt{3}}\}+2^{17}g^5(\kappa_2+2)\log(14g)$.

Proof. Let M_A be a polarization on A of minimal degree. Then $h_\partial(A,M_A)\geq h_\partial(A)$ by definition. So $h_{\mathrm{Fal}}(A)\leq\kappa_2 h_\partial(A,M_A)$ by Conjecture 1.7. Thus Proposition 12.3 implies

$$\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,L)+\log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M'_K}\widetilde{I}_v(A)\log N_v}\leq\kappa_4[K:\mathbb{Q}]\max\{1,\log[K:\mathbb{Q}]\}+\frac{\log h^0(A,L)}{h_\partial(A,M_A)}$$

for $\kappa_4:=g^2\kappa_1^2\max\{\kappa_2+\frac{2}{\sqrt{3}},\frac{4}{\sqrt{3}}\}$. Next by Gaudrons–Rémond's [GR14a], we have

$$\log h^0(A,M_A)\leq 2^{10}g^3\left(64g^2\log(14g)+\log[K:\mathbb{Q}]+2\log\max\{h_{\mathrm{Fal}}(A),1\}\right).$$

Since $h_{\mathrm{Fal}}(A)\leq\kappa_2 h_\partial(A,M_A)$ and $h_\partial(A,M_A)\geq\sqrt{3}/2$, we are done by the crude bound

$$\frac{\log h^0(A,M_A)}{h_\partial(A)}\leq 2^{17}g^5\log(14g)\cdot(\kappa_2+1+\log[K:\mathbb{Q}]).\qquad\square$$

12.3. **Proof of Theorem 1.8.** Let (A,L) be a polarized abelian variety defined over K, such that A is K-simple and L is a principle polarization. Then $h_\partial(A)\leq h_\partial(A,L)$ by their definitions (1.9) and (12.3).

By Theorem 6.1′, there exists an explicit positive constant $C_1=C_1(g)>0$ such that

$$\#A(K)_{\mathrm{tor}}\leq(2g)^g(g+2)^g\cdot h^0(A,L)\cdot\left(C_1\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,L)+\log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M'_K}\widetilde{I}_v(A)\log N_v}\right)^g.$$

Now $h^0(A,L)=1$. Hence Proposition 12.3 simplifies to

$$\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,L)+\log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M'_K}\widetilde{I}_v(A)\log N_v} \leq \frac{2}{\sqrt{3}}\mathfrak{s}_{A/K}^2[K:\mathbb{Q}]\left(\log[K:\mathbb{Q}]+\max\left\{\frac{h_{\mathrm{Fal}}(A)}{h_\partial(A,L)},1\right\}\right). \tag{12.5}$$

Now (1.12) follows immediately from the two bounds above, with (also use $\frac{4}{\sqrt{3}}\leq 2.4$)

$$c_1(g)=(2.4g)^g(g+2)^gC_1(g)^g, \qquad \text{with } C_1(g) \text{ from (7.26)}. \tag{12.6}$$

Let us turn to non-torsion rational points. To ease notation, denote by

$$C_6(g):=\frac{32\cdot 16eg^3(g+1)^3(g+2)^3(100g+274)\gamma_g}{\pi}. \tag{12.7}$$

By (12.4), we know that

$$\Gamma'':=\left\{x\in A(K):\widehat{h}_L(x)\leq\frac{1}{C_6(g)}\frac{h_\partial(A,L)}{g^2\mathfrak{s}_{A/K}^2[K:\mathbb{Q}]}\right\}$$

is contained in $\bigcup_{j=1}^{N_g}\Gamma'_j$ defined in (6.11), with $N_g=(2g)^g(g+2)^g$.

For the number $C'_1(g)=3C_1(g)$ from Theorem 6.2′, set

$$m:=\left(\frac{2}{\sqrt{3}}C'_1(g)\mathfrak{s}_{A/K}^2[K:\mathbb{Q}]\Big(\log[K:\mathbb{Q}]+\max\big\{\frac{h_{\mathrm{Fal}}(A,L)}{h_\partial(A)},1\big\}\Big)\right)^g.$$

Let $P\in A(K)$ be such that $\overline{\mathbb{Z}\cdot P}^{\mathrm{Zar}}=A$. We claim:

$$\widehat{h}_L(P)>\frac{1}{(N_gm)^2}\frac{1}{C_6(g)}\frac{h_\partial(A,L)}{g^2\mathfrak{s}_{A/K}^2[K:\mathbb{Q}]}. \tag{12.8}$$

Assume not, then there are $2N_gm+1$ distinct points $0,\pm P,\ldots,[\pm N_gm]P$ contained in Γ''. The Pigeonhole Principle then implies that Γ'_j contains $>2m$ of these points, for some $j\in\{1,\ldots,N_g\}$. But $h^0(A,L)=1$, so Theorem 6.2′ gives an upper bound $\#\Gamma'_j\leq m$ when combined with (12.5). We get a contradiction. So (12.8) must hold true.

We can simplify (12.8). Using the trivial bound $h_\partial(A,L)\geq\frac{\sqrt{3}}{2}$ (see below (12.3)), we get $\frac{\max\{h_{\mathrm{Fal}}(A),1\}}{h_\partial(A,L)}\leq\frac{2}{\sqrt{3}}\max\left\{\frac{h_{\mathrm{Fal}}(A)}{h_\partial(A,L)},1\right\}$. Together with $\frac{2}{\sqrt{3}}\leq 1.2$, we can easily obtain (1.13) from (12.8) and the choices of N_g and m, with the constant

$$c'_1(g)=1.2(7.2g)^{2g}(g+2)^{2g}C_1(g)^{2g}C_6(g)g^2, \tag{12.9}$$

for $C_1(g)$ from (7.26) and $C_6(g)$ as in (12.7). □

12.4. **Proof of Theorem 1.9.(i): rational torsion points.** Let A be an abelian variety of dimension $g\geq 1$ defined over the number field K.

Assume Conjecture 1.7 holds true for K up to dimension g. Let us prove the following bound by induction on the number of K-simple factors of A (call this number n): There exists a polarization M_A on A of minimal degree satisfying

$$\#A(K)_{\mathrm{tor}}\leq h^0(A,M_A)\cdot C_4(g)\kappa_3^g\cdot([K:\mathbb{Q}]\max\{\log[K:\mathbb{Q}],1\})^g \tag{12.10}$$

for an explicit constant $C_4(g)>0$ depending only on g. Then Theorem 1.9.(i) follows immediately by applying Zarhin's trick to A.

$\boxed{\text{Base step: } n=1}$ In this case A is K-simple. Apply Theorem 6.1′ to get

$$\#A(K)_{\mathrm{tor}}\leq(2g)^g(g+2)^g\cdot h^0(A,M_A)\left(C_1\frac{\max\{h_{\mathrm{Fal}}(A),1\}+\log h^0(A,M_A)+\log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M'_K}\widetilde{I}_v(A)\log N_v}\right)^g$$

for a constant $C_1 = C_1(g) > 0$. Hence we can conclude by Corollary 12.4.

$\boxed{\text{Induction step}}$ For arbitrary $n \geq 2$. We still apply Theorem 6.1′, but in a finer way. We have divided $A(K)_{\mathrm{tor}}$ into the union of $N_g := (2g)^g(g+2)^g$ subsets $\Gamma_1, \ldots, \Gamma_{N_g}$. For each $j \in \{1, \ldots, N_g\}$, there exists a proper abelian subvariety A'_j of A defined over K such that

$$\#\left(\frac{\Gamma_j + A'_j}{A'_j}\right) \leq \frac{h^0(A, M_A)}{h^0(A'_j, M_A)} \left(C_1 \frac{\max\{h_{\mathrm{Fal}}(A), 1\} + \log h^0(A, M_A) + \log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v \in M'_K} \widetilde{I}_v(A) \log N_v}\right)^{g - \dim A'_j}$$

for an explicit constant $C_1 = C_1(g) > 0$. So Corollary 12.4 implies

$$\#\left(\frac{\Gamma_j + A'_j}{A'_j}\right) \leq \frac{h^0(A, M_A)}{h^0(A'_j, M_A)} \left(C_1 \kappa_3 [K:\mathbb{Q}] \max\{\log[K:\mathbb{Q}], 1\}\right)^{g - \dim A'_j}.$$

Since $A'_j \subsetneq A$, the number of factors of A'_j is $\leq n-1$. So we can apply the induction hypothesis to A'_j and get

$$\#A'_j(K)_{\mathrm{tor}} \leq h^0(A'_j, M_{A'_j}) \left(C_4(g-1) \cdot \kappa_3 [K:\mathbb{Q}] \max\{\log[K:\mathbb{Q}], 1\}\right)^{\dim A'_j}.$$

Taking the product of the two inequalities above, we get

$$\#\Gamma_j \leq h^0(A, M_A) \frac{h^0(A'_j, M_{A'_j})}{h^0(A'_j, M_A)} C_3 \left(\kappa_3 [K:\mathbb{Q}] \max\{\log[K:\mathbb{Q}], 1\}\right)^g$$

with $C_3 = C_3(g) > 0$ explicit. But $h^0(A'_j, M_{A'_j}) \leq h^0(A'_j, M_A|_{A'_j})$ by definition of $M_{A'_j}$, so

$$\#\Gamma_j \leq h^0(A, M_A) C_3 \left(\kappa_3 [K:\mathbb{Q}] \max\{\log[K:\mathbb{Q}], 1\}\right)^g.$$

Therefore we are done by

$$\#A(K)_{\mathrm{tor}} \leq \sum_{j=1}^{(2g)^g(g+2)^g} \#\Gamma_j \leq h^0(A, M_A) \cdot (2g)^g (g+2)^g C_3 \kappa_3^g \cdot \left([K:\mathbb{Q}] \max\{\log[K:\mathbb{Q}], 1\}\right)^g. \quad \square$$

12.5. **Proof of Theorem 1.9.(ii): small rational points.** Let A, L be as in Theorem 1.9.(ii). Let $P \in A(K)$ be such that $\overline{\mathbb{Z} \cdot P}^{\mathrm{Zar}} = A$. If L defines a principal polarization, we can apply the arguments as in §12.3 and hence relax the requirement (1.11) to $h_{\mathrm{Fal}}(A) \leq \kappa_2 h_\partial(A, L)$ (see (12.3) for definition; we trivially have $h_\partial(A, L) \geq h_\partial(A)$).

For general L, set

$$Z(A) := A^4 \times (A^\vee)^4.$$

By the line bundle version of Zarhin's trick (Lemma B.1), $Z(A)$ carries a principal polarization induced by a symmetric ample line bundle M on $Z(A)$ defined over K, such that for a suitable K-isogeny

$$\pi \colon A^8 \longrightarrow Z(A),$$

$\pi^* M$ and $L^{\boxtimes 8}$ induce the same polarization on A^8. Hence

$$\widehat{h}_M\big(\pi(x_1, \ldots, x_8)\big) = \sum\nolimits_{j=1}^{8} \widehat{h}_L(x_j). \tag{12.11}$$

For each $j \in \{1, \ldots, 8\}$, define

$$Q_j := \pi(0, \ldots, 0, P, 0, \ldots, 0),$$

where P occurs in the i-th coordinate. Then

$$\overline{\mathbb{Z}Q_1 + \ldots + \mathbb{Z}Q_8}^{\mathrm{Zar}} = Z(A). \tag{12.12}$$

By Proposition 12.3 applied to $Z(A)$ and M and Conjecture 1.7, we have

$$\frac{\max\{h_{\mathrm{Fal}}(Z(A)),1\}+\log h^0(Z(A),M)+\log[K:\mathbb{Q}]}{\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M'_K}\widetilde{I}_v(Z(A))\log N_v}\leq \kappa_5[K:\mathbb{Q}]\max\{\log[K:\mathbb{Q}],1\} \tag{12.13}$$

where $\kappa_5 := (8g)^2\kappa_1(8g,K)^2\max\{\kappa_2(8g,K)+\frac{2}{\sqrt{3}},\frac{4}{\sqrt{3}}\}$. To ease notation, denote by $C_5(g) := 32\cdot 16e(8g)^3(8g+1)^3(8g+2)^3(800g+274)\gamma_{8g}/\pi$. So the set

$$\Gamma'_{Z(A)} := \left\{x\in Z(A)(K) : \widehat{h}_M(x)\leq \frac{1}{C_5(g)}\frac{\max\{h_{\mathrm{Fal}}(Z(A)),1\}}{\kappa_5[K:\mathbb{Q}]\max\{\log[K:\mathbb{Q}],1\}}\right\}.$$

is contained in the set $\bigcup_{j=1}^{N_{8g}}\Gamma'_j$ defined in (6.11) but with (A,L) replaced by $(Z(A),M)$. Here $N_{8g}=(2\cdot 8g)^{8g}(8g+2)^{8g}$.

For the number $C'_1>0$ from Theorem 6.2′, set

$$m := \left(2C'_1(8g)\kappa_5\right)^g\left([K:\mathbb{Q}]\max\{\log[K:\mathbb{Q}],1\}\right)^g.$$

To conclude for Theorem 1.9.(ii), it suffices to prove:

$$\widehat{h}_L(P) > \frac{\max\{h_{\mathrm{Fal}}(A),1\}}{(mN_{8g}^g)^2C_5(g)\kappa_5[K:\mathbb{Q}]\max\{\log[K:\mathbb{Q}],1\}}. \tag{12.14}$$

Assume (12.14) is false. Consider

$$\mathcal{B}_m := \left\{\sum_{j=1}^{8}[a_j]Q_j : 0\leq a_j\leq mN_{8g}^g\right\}.$$

As (12.14) is false, (12.11) then implies

$$\widehat{h}_M(x)\leq 8(mN_{8g}^g)^2\widehat{h}_L(P)\leq \frac{1}{C_5(g)}\frac{8\max\{h_{\mathrm{Fal}}(A),1\}}{\kappa_5[K:\mathbb{Q}]\max\{\log[K:\mathbb{Q}],1\}} \qquad \text{for any } x\in\mathcal{B}_m,$$

and hence $\mathcal{B}_m$ is contained in the subset $\Gamma'_{Z(A)}$ defined above, because $h_{\mathrm{Fal}}(Z(A)) = 8h_{\mathrm{Fal}}(A)$. Thus $\mathcal{B}_m\subseteq\bigcup_{j=1}^{N_{8g}}\Gamma'_{Z(A),j}$.

Similar to (11.11), we can show, for any proper abelian subvariety A' of $Z(A)$ defined over K

$$\#\left(\frac{\mathcal{B}_m+A'}{A'}\right)\geq (mN_{8g}^g+1)^{\frac{8g-\dim A'}{g}} > m^{\frac{8g-\dim A'}{g}}N_{8g};$$

the last inequality holds true since $8g-\dim A'\geq 1$. The Pigeonhole Principle then implies that there exists $j\in\{1,\ldots,N_{8g}\}$ with

$$\#\left(\frac{\Gamma'_{Z(A),j}+A'}{A'}\right) > m^{\frac{8g-\dim A'}{g}}. \tag{12.15}$$

By our choice of m and since $h^0(Z(A),M)=1$, we get

$$\left(\#\left(\frac{\Gamma'_{Z(A),j}+A'}{A'}\right)\frac{h^0(A',M)}{h^0(Z(A),M)}\right)^{\frac{1}{8g-\dim A'}} > \frac{m^{1/g}}{2} = C'_1(8g)\kappa_5[K:\mathbb{Q}]\max\{\log[K:\mathbb{Q}],1\}.$$

This contradicts Theorem 6.2′ in view of (12.13). So (12.14) must hold. We are done. ☐

13. Improvements in the case of elliptic curves

In the end, let us explain how to improve our results for elliptic curves, *i.e.* $g = 1$. There are two improvements: getting better constants, and proving uniformity for function fields of arbitrary characteristic.

Let $k = \overline{k}$ be an algebraically closed field. Let B be a smooth projective geometrically connected curve defined over k. Let $K := k(B)$ be the function field of B.

Let E/K be a traceless semi-stable elliptic curve, equipped with the canonical principal polarization $L = O(0)$. In this case, E/K does not have everywhere good reduction because the modular map $j\colon B \to \mathbb{P}^1_k$ is surjective.

Let p^ν be the inseparable degree of the modular map j. When $\mathrm{char}(k) = 0$, set $p^\nu = 1$.

13.1. **Bump functions.** We start by constructing a bump function at each non-archimedean place $v \in \mathfrak{Bad}(E/K)$, which is better than the one from §5.1′.

Continue to use the notations from §4. In this case, E_v is a Tate curve $F^\times/\mathfrak{q}^{\mathbb{Z}}$. The horizontal line of the Raynaud cross (4.1) becomes $\mathbb{G}_\mathrm{m} = \mathbb{G}_\mathrm{m} \to 0$, and the map $\mathfrak{v}\colon Y = \mathbb{Z} \to E(F) = F^\times$ maps a generator 1 of Y to some $\mathfrak{q} \in F^\times$ such that $|\mathfrak{q}|_v < 1$. Then $n_v := \mathrm{ord}_v\,\mathfrak{q}$ is the number of Néron components of A at v. The skeleton of A_v^{an} is $\Sigma_v := X^*_{\mathbb{R}}/Y$, while $X^*_{\mathbb{R}} = \mathbb{R}$, $Y = \mathbb{Z}$ and X^* is the lattice $\frac{1}{n_v}\mathbb{Z}$. See the end of §4.

Let x be the standard coordinate on $X^*_{\mathbb{R}} = \mathbb{R}$. Define the following function on $X^*_{\mathbb{R}}$

$$F_v(x) := \frac{1}{2n_v}\{n_v x\}(1 - \{n_v x\}) \qquad \text{where } \{\cdot\} \text{ denotes the fraction part.}$$

Then F_v descends to a function $f_v\colon \Sigma_v = X^*_{\mathbb{R}}/Y \overset{x}{\simeq} \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ with the following properties:

(i) $f_v|_{\Phi_v} = 0$, where $\Phi_v = X^*/Y$;
(ii) f_v is $\frac{1}{n_v}\mathbb{Z}/\mathbb{Z}$-periodic;
(iii) $\overline{L}(f_v)$ is semipositive;
(iv) $\int_{\Sigma_v} f_v \mathrm{d}\mu_v = (12n_v)^{-1}$;
(v) $\overline{L}(2f_v)$ coincides with the model metric of the admissible $\mathbb{Q}$-extension of L on the the minimal regular (proper) model $\mathcal{C}/O_F$.

See [Fal84] for the theory of admissible $\mathbb{Q}$-extensions. Indeed, (i) and (ii) follow immediately from the construction of f_v, and (iv) is a direct computation. For (iii), it suffices to see that $c_1(\overline{L}(f_v)) = \frac{1}{2n_v}\sum_{u\in\Phi_v}\delta_u + \frac{1}{2}c_1(\overline{L}_v)$, which is semipositive. For (v), the chern form is $c_1(\overline{L}(2f_v)) = \frac{1}{n_v}\sum_{u\in\Phi_v}\delta_u$. This coincides with the chern form of the admissible $\mathbb{Q}$-extension of L on the the minimal regular (proper) model $\mathcal{C}/O_F$. Therefore they are equal by for example the non-archimedean Calabi Theorem [YZ17, Thm. 1.5].

13.2. **Application of Arithmetic Hilbert Samuel.** Let $\mathbf{f} = (f_v)_{v\in\mathfrak{Bad}(E/K)}$ be the collection of bump functions.

Proposition 13.1. $\mathrm{vol}(\overline{L}(\mathbf{f})) = \frac{1}{12}\sum_{v\in\mathfrak{Bad}(E/K)} n_v^{-1}$.

Proof. By (iii), $\overline{L}(\mathbf{f})$ is nef. So Arithmetic Hilbert Samuel implies $\mathrm{vol}(\overline{L}(\mathbf{f})) = \overline{L}(\mathbf{f})^2$.

Let $\mathcal{C}/B$ be the minimal regular (proper) model of E/K. By property (v), $\overline{L}(2\mathbf{f})$ carries the model metric of the Faltings–Hiljac admissible $\mathbb{Q}$-extension $\mathcal{L}$ of L on $\mathcal{C}/B$. By their theory of admissible $\mathbb{Q}$-extension, we have $\overline{L}(2\mathbf{f})^2 = \mathcal{L}^2 = 0$. Thus

$$\overline{L}(\mathbf{f})^2 = \frac{1}{2}(\overline{L}+\overline{L}(2\mathbf{f}))^2 = \frac{1}{4}\overline{L}^2+\frac{1}{2}\overline{L}\cdot\overline{L}(2\mathbf{f})+\frac{1}{4}\overline{L}(2\mathbf{f})^2 = \frac{1}{2}\overline{L}\cdot\overline{L}(2\mathbf{f}) = \frac{1}{2}\overline{L}\cdot(\overline{L}+\overline{\mathcal{O}}(2\mathbf{f})) = \frac{1}{2}\overline{L}\cdot\overline{\mathcal{O}}(2\mathbf{f}).$$

Therefore

$$\overline{L}(\mathbf{f})^2 = \frac{1}{2} \sum_{v \in \mathfrak{Bad}(E/K)} \int_{A_v^{\mathrm{an}}} 2 f_v c_1(\overline{L}_v) = \sum_{v \in \mathfrak{Bad}(E/K)} (12 n_v)^{-1}. \qquad \square$$

13.3. **A bound for torsion points and small points.** We use slightly different notations from previous sections. For $\varepsilon \in [0, 1/24)$, define

$$\Gamma_\varepsilon := \left\{ x \in E(K) : \widehat{h}_L(x) \le \frac{\varepsilon}{[K : k(\mathbb{P}^1)]} \sum_{v \in \mathfrak{Bad}(E/K)} {n_v}^{-1} \right\}$$

to be the set of small points. Notice that $\Gamma_0 = E(K)_{\mathrm{tor}}$. In the following we shall prove

Theorem 13.2. *For the inseparable degree p^ν of the modular map $j \colon B \to \mathbb{P}^1_k$, we have*

$$\#\Gamma_\varepsilon \le \frac{108}{1 - 24\varepsilon} (p^\nu)^2 \max\{1, 2g(B) - 1\}^2.$$

The proof follows the guideline in §7.

13.3.1. *Existence of small section.* By Zhang's fundamental inequality (see the proof of the Theorem of Successive Minima, [Zha95a, Theorem 5.2]) and the calculation of arithmetic volume Proposition 13.1, the following holds true: For $m \gg 1$, there is a non-zero section $s \in H^0(A, mL)$ such that

$$\frac{1}{m} h_{m\overline{L}(\mathbf{f})}(s) \ge \frac{\mathrm{vol}(\overline{L}(\mathbf{f}))}{[K : k(\mathbb{P}^1)] 2 \deg L} + o(1) = \frac{1}{[K : k(\mathbb{P}^1)]} \sum_{v \in \mathfrak{Bad}(E/K)} \frac{1}{24 n_v} + o(1), \tag{13.1}$$

where

$$h_{m\overline{L}(\mathbf{f})}(s) := \frac{1}{[K : k(\mathbb{P}^1)]} \sum_{v \in M_B} - \log \|s\|_{m\overline{L}(\mathbf{f}), v, \sup}.$$

13.3.2. *Zero estimate.* In this case, we only need to apply the Pigeonhole Principle to get: there is a point $x \in \Gamma_\varepsilon$ with vanishing order $\ell = \mathrm{ord}_x s \le \frac{m}{\#\Gamma_\varepsilon}$.

13.3.3. *Liouville inequality.* We still have the inequality (7.19), which in the current case becomes

$$\frac{1}{m} h_J(\mathrm{jet}^\ell s(x)) \le \widehat{h}_L(x) + \frac{3}{2m} \ell h_{\mathrm{Fal}}(E) \le \widehat{h}_L(x) + \frac{3}{2 \#\Gamma_\varepsilon} h_{\mathrm{Fal}}(E). \tag{13.2}$$

The left hand side $h_J(\mathrm{jet}^\ell s(x)) = -\frac{1}{[K:k(\mathbb{P}^1)]} \sum_{v \in M_K} \log \|\mathrm{jet}^\ell s(x)\|_v$ can be estimated as in Lemma 7.4: We have

$$\|\mathrm{jet}^\ell s(x)\|_v \le \|s(x)\|_v.$$

But $f_v(x) = 0$ because $f_v|_{\Phi_v} = 0$ by our choice of $\mathbf{f} = (f_v)$, for any $v \in \mathfrak{Bad}(E/K)$. So

$$\|s(x)\|_v = \|s(x)\|_{m\overline{L}(\mathbf{f}), v} \le \|s\|_{m\overline{L}(\mathbf{f}), v, \sup} \quad \text{for all } v \in M_K.$$

Taking $-\log$ and adding all places up, we get

$$h_J(\mathrm{jet}^\ell s(x)) \ge h_{\overline{L}(\mathbf{f})}(s).$$

Therefore by (13.1) and (13.2) and letting $m \to \infty$, we get

$$\#\Gamma_\varepsilon \le \frac{36}{1 - 24\varepsilon} \cdot \frac{h_{\mathrm{Fal}}(E)}{\frac{1}{[K:k(\mathbb{P}^1)]} \sum_{v \in \mathfrak{Bad}(E/K)} {n_v}^{-1}}. \tag{13.3}$$

Notice that $\mathfrak{Bad}(E/K) \neq \emptyset$, so the denominator is > 0

13.3.4. *Applying Arakelov inequality.* Since E/K is a traceless elliptic curve, we have $\#\mathfrak{Bad}(E/K) \geq 1$ (and $\#\mathfrak{Bad}(E/K) \geq 3$ if $g(B) = 0$).

Recall the Arakelov inequality for family of elliptic curves from [GS95, Thm. 3].

Theorem 13.3. *For the inseparable degree p^ν of the modular map $j\colon B \to \mathbb{P}^1_k$, we have*

(i) *$p^\nu = 1$ if and only if E/K is not a Frobenius pullback.*

(ii) *The following Arakelov inequality holds*

$$[K : k(\mathbb{P}^1)]h_{\mathrm{Fal}}(E) \leq \frac{p^\nu}{2}(2g(B) - 2 + \#\mathfrak{Bad}(E/K)). \tag{13.4}$$

Proof of Theorem 13.2. Cauchy inequality implies

$$\left(\sum_{v\in\mathfrak{Bad}(E/K)} {n_v}^{-1}\right)\left(\sum_{v\in\mathfrak{Bad}(E/K)} n_v\right) \geq (\#\mathfrak{Bad}(E/K))^2.$$

But $\sum_{v\in\mathfrak{Bad}(E/K)} n_v = 12[K : k(\mathbb{P}^1)]h_{\mathrm{Fal}}(E)$ by the Faltings–Silverman Formula. So

$$\frac{1}{\sum_{v\in\mathfrak{Bad}(E/K)} {n_v}^{-1}} \leq 12\frac{[K : k(\mathbb{P}^1)]h_{\mathrm{Fal}}(E)}{(\#\mathfrak{Bad}(E/K))^2}. \tag{13.5}$$

$$\begin{aligned} \frac{[K : k(\mathbb{P}^1)]h_{\mathrm{Fal}}(E)}{\sum_{v\in\mathfrak{Bad}(E/K)} {n_v}^{-1}} &\leq 12\frac{([K : k(\mathbb{P}^1)]h_{\mathrm{Fal}}(E))^2}{(\#\mathfrak{Bad}(E/K))^2} \\ &\leq 12\left(\frac{p^\nu(g(B)-1)}{\#\mathfrak{Bad}(E/K)} + \frac{p^\nu}{2}\right)^2 \qquad \text{by (13.4)} \\ &\leq 3(p^\nu)^2\left(1 + \frac{2g(B)-2}{\#\mathfrak{Bad}(E/K)}\right)^2 \leq 3(p^\nu)^2\max\{1, 2g(B)-1\}^2. \end{aligned} \tag{13.6}$$

We are done by (13.3). □

Moreover as an immediate consequence of (13.5) with (13.4), we have

$$\frac{[K : k(\mathbb{P}^1)]}{\sum_{v\in\mathfrak{Bad}(E/K)} n_v^{-1}} \leq 6p^\nu[K : k(\mathbb{P}^1)]\max\{1, 2g(B)-1\}. \tag{13.7}$$

13.4. **Handling inseparable points.** The traceless semi-stable elliptic curve $E/K = k(B)$ induces the modular map $j\colon B \to \mathbb{P}^1_k$, whose inseparable degree is denoted by p^ν. Hence there exists an elliptic curve E_0/K such that $E = E_0^{(p^\nu)}$. Then E_0 is also traceless and semi-stable because $F^e\colon E_0 \to E_0^{(p^\nu)} = E$ is a K-isogeny. There is a natural identification $E(K) = E_0(K^{\frac{1}{p^\nu}})$.

Let $K^{\mathrm{insep}} := \bigcup_{n\geq 1} K^{\frac{1}{p^n}}$. The following proposition is essentially due to Rössler [Rös15]; see also Yuan [Yua21, Prop. 3.6],.

Proposition 13.4. *We have $[K(P) : K] \leq 12\max\{1, 2g(B)-1\}$ for all $P \in E_0(K^{\mathrm{insep}})$.*

Proof. Let $\pi\colon \mathcal{E}_0 \to B$ be the Néron model of E_0/K. Let $\mathcal{P}_0$ be the schematic closure of P in $\mathcal{E}_0$, and $\mathrm{n}\colon \mathcal{P} \to \mathcal{P}_0$ be its normalization. Then the structural morphism $\psi\colon \mathcal{P} \to B$ is a radical quasi-finite morphism (it is not necessarily finite since $\mathcal{E}_0 \to B$ need not be proper). It suffices to bound the degree of the morphism ψ.

As $\mathcal{P}_0$ is purely inseparable over B, we have a canonical isomorphism

$$\Omega^1_{\mathcal{P}_0/k} = \Omega^1_{\mathcal{P}_0/B}.$$

So the canonical surjection $(\Omega^1_{\mathcal{E}_0/B})|_{\mathcal{P}_0} \to \Omega^1_{\mathcal{P}_0/B}$ can be written as

$$\tau_0 \colon (\Omega^1_{\mathcal{E}_0/B})|_{\mathcal{P}_0} \to \Omega^1_{\mathcal{P}_0/k}.$$

Next let us pull back τ_0 along the normalization $\mathrm{n}\colon \mathcal{P} \to \mathcal{P}_0$. We have $\mathrm{n}^*(\Omega^1_{\mathcal{E}_0/B})|_{\mathcal{P}_0} \simeq \psi^*\pi_*\Omega_{\mathcal{E}_0}$ since $\mathcal{E}_0/B$ is a group scheme. Hence from τ_0, we obtain a non-zero morphism between line bundles on $\mathcal{P}$

$$\tau \colon \psi^*\pi_*\Omega_{\mathcal{E}_0} \to \Omega^1_{\mathcal{P}/k}.$$

Let $\mathcal{P}^{\mathrm{c}}$ be the unique smooth compactification of $\mathcal{P}$. Then by properness the morphism ψ extends to $\psi^{\mathrm{c}} \colon \mathcal{P}^{\mathrm{c}} \to B$. Let E_0 be the reduced closed subscheme of B on which $\mathcal{E}_0$ has bad reduction, and let $E := (\psi^{\mathrm{c}})^{-1}(E_0)$ endowed with the reduced closed scheme structure. Then [Yua21, Prop. 3.6] asserts that τ extends to a morphism $\tau^{\mathrm{c}} \colon (\psi^{\mathrm{c}})^*\pi_*\Omega_{\mathcal{E}_0} \to \Omega^1_{\mathcal{P}^{\mathrm{c}}/k}(E)$. Taking the degrees of both sides, we get

$$\deg(\psi^{\mathrm{c}}) \cdot \deg(\pi_*\Omega_{\mathcal{E}_0}) \leq 2g(B) - 2 + \deg(E_0) = 2g(B) - 2 + \#\mathfrak{Bad}(E/K).$$

Let $n_{v,0}$ be the size of the Néron component group of E_0 at the place v. The Faltings–Silverman Formula says that

$$12\deg(\pi_*\Omega_{\mathcal{E}_0}) = \sum\nolimits_{v\in\mathfrak{Bad}(E/K)} n_{v,0} \geq \#\mathfrak{Bad}(E/K).$$

Therefore

$$[K(P):K] = \deg(\psi^{\mathrm{c}}) \leq 12\frac{2g(B) - 2 + \#\mathfrak{Bad}(E/K)}{\#\mathfrak{Bad}(E/K)} \leq 12\max\{1, 2g(B)-1\}. \qquad \square$$

13.5. **Proof of Theorem 1.6, part (i) and (ii): rational torsion points.** If $\mathrm{char}(k) = 0$, then Theorem 13.2 suffices to conclude that $\#E(K)_{\mathrm{tor}} \leq 108(g(B)-1)^2$.

Assume $\mathrm{char}(k) = p > 0$. Let E_0/K be the elliptic curve constructed at the beginning of §13.4. Then $E(K) \subseteq E_0(K^{\mathrm{insep}})$.

Fix p^{ν_0} to be the largest power of p satisfying $p^{\nu_0} \leq 12\max\{1, 2g(B)-1\}$. Then Proposition 13.4 implies $E_0(K^{\mathrm{insep}}) = E_0^{(p^{\nu_0})}(K)$.

Apply Theorem 13.2 to $E_0^{(p^{\nu_0})}$. Then we get

$$\#E_0^{(p^{\nu_0})}(K)_{\mathrm{tor}} \leq 108p^{2e_0}\max\{1, 2g(B)-1\}^2 \leq 108\cdot 144\max\{1, 2g(B)-1\}^4.$$

We are done since $E(K)_{\mathrm{tor}} \subseteq E_0^{(p^{\nu_0})}(K)_{\mathrm{tor}}$. $\square$

13.6. **Proof of Theorem 1.6.(iii): Lang-Silverman Conjecture.** Take $\varepsilon = \frac{1}{72}$. Let $P \in E(K)$ be a non-torsion point. Let N be the smallest positive integer with $2N \geq \frac{108}{1-24\varepsilon}p^{2e}\max\{1, 2g(B)-1\}^2$. Then $[N]P \notin \Gamma_\varepsilon$; otherwise there are $2N+1$ distinct points, namely $0, \pm P, \cdots, \pm[N]P$, contained in Γ_ε, contradicting Theorem 13.2. So

$$N^2\widehat{h}_L(P) = \widehat{h}_L([N]P) \geq \frac{\varepsilon}{[K:k(\mathbb{P}^1)]}\sum_{v\in M'_K} n_v^{-1} \geq \frac{\varepsilon\max\{h_{\mathrm{Fal}}(E), 1\}}{6(p^\nu)^2[K:k(\mathbb{P}^1)]\max\{1, 2g(B)-1\}^2}$$

where the last inequality follows from (13.6) and (13.7). Substituting $\varepsilon = \frac{1}{72}$, we obtain

$$\widehat{h}_L(P) \geq \frac{\max\{h_{\mathrm{Fal}}(E), 1\}}{2^4\cdot 3^{11}(p^\nu)^6[K:k(\mathbb{P}^1)]\,(2g(B)-1)^6}. \tag{13.8}$$

When $\mathrm{char}\, k = 0$, we are done since $\mathrm{gon}(B) = [K : k(\mathbb{P}^1)]$. From now on, assume $\mathrm{char} k = p > 0$. Let E_0/K be the elliptic curve constructed at the beginning of §13.4. Then $E(K) \subseteq E_0(K^{\mathrm{insep}})$. Fix p^{ν_0} to be the largest power of p satisfying $p^{\nu_0} \leq 12\max\{1, 2g(B)-1\}$. Then Proposition 13.4 implies $E_0(K^{\mathrm{insep}}) = E_0^{(p^{\nu_0})}(K)$. We therefore regard $P \in E(K)$

as a non-torsion point in $E_0^{(p^{\nu_0})}(K)$. Denote by $\widehat{h}_0(\cdot)$ the canonical height associated to the principal polarization on $E_0^{(p^{\nu_0})}$. Applying (13.8) to $P \in E_0^{(p^{\nu_0})}(K)$, we get

$$\begin{aligned}
\widehat{h}_L(P) = p^{\nu_0-\nu}\widehat{h}_0(P) &\geq p^{\nu_0-\nu}\frac{\max\{h_{\mathrm{Fal}}(E_0^{(p^{\nu_0})}),1\}}{16\cdot 3^{11}p^{6\nu_0}[K:k(\mathbb{P}^1)]\,(2g(B)-1)^6} \\
&= (p^{\nu})^{-1}\frac{\max\{p^{\nu_0-\nu}h_{\mathrm{Fal}}(E),1\}}{2^4\cdot 3^{11}p^{5\nu_0}[K:k(\mathbb{P}^1)]\,(2g(B)-1)^6} \\
&= \frac{(p^{\nu})^{-2}}{2^4\cdot 3^{11}[K:k(\mathbb{P}^1)]\,(2g(B)-1)^6}\max\left\{\frac{h_{\mathrm{Fal}}(E)}{p^{4\nu_0}},\frac{p^{\nu}}{p^{5\nu_0}}\right\} \\
&\geq \frac{(p^{\nu})^{-2}}{2^4\cdot 3^{11}[K:k(\mathbb{P}^1)]\,(2g(B)-1)^6}\frac{\max\{h_{\mathrm{Fal}}(E),1\}}{p^{5\nu_0}} \\
&\geq \frac{\max\{h_{\mathrm{Fal}}(E),1\}}{2^{14}\cdot 3^{16}(p^{\nu})^2[K:k(\mathbb{P}^1)]|2g(B)-1|^{11}}. \qquad \square
\end{aligned}$$

Appendix A. Maximal Slope of (Co)Tangent Space

In this appendix, we give an upper bound on the maximal slope of $\overline{\mathrm{Lie}\,A}^{\vee}$, where A/K is an abelian variety over a function field $K = \mathbb{C}(B)$. Our proof is an adaptation of the proof of [Gau19] in the case of number fields, and also applies to arbitrary $k = \overline{k}$.

Theorem A.1. $\widehat{\mu}_{\max}(\overline{\mathrm{Lie}\,A}^{\vee}) \leq \left(\frac{g}{2}+1\right)h_{\mathrm{Fal}}(A)$.

The same argument also shows $\widehat{\mu}_{\max}(\overline{\mathrm{Lie}\,A}) \leq h_{\mathrm{Fal}}(A)$. See the remark at the end.

A.1. **The Harder-Narasimhan Polygon.** For every vector bundle $\mathcal{E}$ on B, there is a canonical filtration (called the Harder-Narasimhan filtration, or HN filtration for short)

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_m = \mathcal{E},$$

with every subquotient $\mathcal{E}_i/\mathcal{E}_{i+1}$ semistable, and $\mu(\mathcal{E}_1/\mathcal{E}_0) \geq \mu(\mathcal{E}_2/\mathcal{E}_1) \geq \cdots \geq \mu(\mathcal{E}_m/\mathcal{E}_{m-1})$.

The *Harder-Narasimhan polygon (HN polygon)* is the piecewise linear function

$$P_{\mathcal{E}}\colon [0,\operatorname{rank}\mathcal{E}] \to \mathbb{R}$$

joining the points given by the Harder-Narasimhan filtration: $\{(\operatorname{rank}\mathcal{E}_i, \deg\mathcal{E}_i)\}$; here we set $\deg 0 = 0$. In particular, $\mu_{\max}(\mathcal{E}) = P_{\mathcal{E}}(1)$.

We list some basic properties of HN polygon.

Lemma A.2. *The followings hold true:*

(i) $P_{\mathcal{E}^{\vee}}(x) = P_{\mathcal{E}}(\operatorname{rank}\mathcal{E} - x) - \deg\mathcal{E}$.
(ii) $P_{\mathcal{E}\otimes\mathcal{L}}(x) = P_{\mathcal{E}}(x) + x\deg\mathcal{L}$ *for any line bundle* $\mathcal{L}$ *on* B.
(iii) *Let* $\varphi\colon \mathcal{E}_K \to \mathcal{E}'_K$ *be an injective* K*-linear map between generic fibers of vector bundles* $\mathcal{E}$ *and* $\mathcal{E}'$. *Assume that* φ *extends to a morphism* $\mathcal{E} \to \mathcal{E}'$. *Then*

$$P_{\mathcal{E}}(x) \leq P_{\mathcal{E}'}(x) \quad \textit{for all } x \in [0,\operatorname{rank}\mathcal{E}].$$

(iv) *Let* $\varphi\colon \mathcal{E}_K \to \mathcal{E}'_K$ *be a surjective* K*-linear map between generic fibers of vector bundles* $\mathcal{E}$ *and* $\mathcal{E}'$. *Assume that* φ *extends to a* B*-morphism* $\mathcal{E} \to \mathcal{E}'$. *Then*

$$P_{\mathcal{E}'}(y) \leq P_{\mathcal{E}}(y + \dim\ker\varphi) - \deg\mathcal{E} + \deg\mathcal{E}', \quad \textit{for all } y \in [0,\operatorname{rank}\mathcal{E}'].$$

In (iii), without assuming the extension of φ to a B-morphism $\mathcal{E} \to \mathcal{E}'$, we have $P_{\mathcal{E}}(x) \leq P_{\mathcal{E}'}(x) + xh(\varphi)$. Then this recovers the classical slope inequality $\mu_{\max}(\mathcal{E}) \leq \mu_{\max}(\mathcal{E}') + h(\varphi)$ by taking $x = 1$. Similarly in (iv), without assuming the extension of φ, we have $P_{\mathcal{E}'}(y) \leq P_{\mathcal{E}}(y + \dim\ker\varphi) + (\operatorname{rank}\mathcal{E}' - y)h(\varphi) - \deg\mathcal{E} + \deg\mathcal{E}'$.

A.2. **Evaluation maps.** Fix a torsion point $a \in A(K)$. Define:

$$\delta_0 \colon H^0(A, L^{\otimes 2}) \longrightarrow L_a^{\otimes 2}, \quad s \longmapsto s(a),$$
$$\delta_1 \colon \ker\delta_0 \longrightarrow \Omega_A \otimes L_a^{\otimes 2}, \quad s \longmapsto \operatorname{jet}^1 s(a).$$

The map δ_0 is surjective since $L^{\otimes 2}$ is globally generated. The map δ_1 is surjective if and only if the complete linear system $H^0(A, L^{\otimes 2})$ separates tangent vectors at a. This holds for generic $a \in A(\overline{K})$, by generic smoothness of $A \to \mathbb{P}(H^0(A, L^{\otimes 2}))$. Moreover, we can take a to be a torsion point, by possibly enlarging K (the integral structure on $\operatorname{Lie} A$ is that on the tensor product, so this do not affect later arguments).

By [MB85, §II, 1.2.2], there exists a *Moret-Bailly model* $(\mathcal{A}, \mathcal{L}_2, \varepsilon_a)$ of $(A, L^{\otimes 2}, a)$ defined over a finite field extension K' of K, *i.e.*

- $\mathcal{A}$ is a semi-abelian scheme over B', whose generic fiber is $A_{K'}$;
- $\mathcal{L}$ is a cubical line bundle on $\mathcal{A}$, whose generic fiber is $L^{\otimes 2} \otimes_K K'$;
- ε_a is a section of $\pi' \colon \mathcal{A} \to B'$ whose geometric generic fiber is a.

such that

$$\mathcal{K}(\mathcal{L}_2) := \{a \in \mathcal{A}(B') : T_a^*\mathcal{L}_2 \simeq \mathcal{L}_2\}$$

is a finite group scheme over B'. For a quick survey on this subject, see also [Bos96a, §4.3].

Endow the sources and targets of δ_0, δ_1 with the integral structures given by $(\mathcal{A}, \mathcal{L}_2, \varepsilon_a)$. Then $\pi'_*\mathcal{L}_2$ is a vector bundle on B'; see [MB85, Intro., (7.3)]. Both δ_0 and δ_1 extend to

$$\boldsymbol{\delta}_0 \colon \pi'_*\mathcal{L}_2 \longrightarrow \varepsilon_a^*\mathcal{L}_2,$$
$$\boldsymbol{\delta}_1 \colon \ker\delta_0 \longrightarrow \operatorname{Lie}(\mathcal{A}/B')^\vee \otimes \varepsilon_a^*\mathcal{L}_2.$$

A.3. **Proof of Theorem A.1.** Since ε_a is a torsion section, we have $\deg(\varepsilon_a^*\mathcal{L}_2) = 0$. So

$$-\deg(\pi'_*\mathcal{L}_2)^\vee + \deg(\ker\boldsymbol{\delta}_0)^\vee = \deg(\pi'_*\mathcal{L}_2) - \deg(\ker\boldsymbol{\delta}_0) = \deg(\varepsilon_a^*\mathcal{L}_2) = 0.$$

Apply Lemma A.2.(iv) for the dual of $\ker(\boldsymbol{\delta}_0) \hookrightarrow \pi'_*\mathcal{L}_2$, we then get

$$P_{(\ker\boldsymbol{\delta}_0)^\vee}(y) \leq P_{(\pi'_*\mathcal{L}_2)^\vee}(y+1) - \deg(\pi'_*\mathcal{L}_2)^\vee + \deg(\ker\boldsymbol{\delta}_0)^\vee = P_{(\pi'_*\mathcal{L}_2)^\vee}(y+1). \tag{A.1}$$

Now we compute

$$\begin{aligned} P_{\operatorname{Lie}(\mathcal{A}/B')}(g-1) &= P_{\operatorname{Lie}(\mathcal{A}/B') \otimes \varepsilon_a^*\mathcal{L}_2^\vee}(g-1) && \text{by Lemma A.2.(ii)} \\ &\leq P_{(\ker\boldsymbol{\delta}_0)^\vee}(g-1) && \text{by Lemma A.2.(iii) applied to } \boldsymbol{\delta}_1^\vee \\ &\leq P_{(\pi'_*\mathcal{L}_2)^\vee}(g) && \text{by (A.1) with } y = g-1. \end{aligned} \tag{A.2}$$

Notice that $\mathcal{K}(\mathcal{L}_2)$ is a group of isometric automorphisms of $\pi'_*\mathcal{L}_2$ which acts irreducibly on $(\pi'_*\mathcal{L}_2)_{K'} = H^0(A_{K'}, L^{\otimes 2})$. So $\pi'_*\mathcal{L}_2$ is semi-stable.[10] So $(\pi'_*\mathcal{L}_2)^\vee$ is also semistable. So

$$P_{(\pi'_*\mathcal{L}_2)^\vee}(g) = g \cdot \mu((\pi'_*\mathcal{L}_2)^\vee), \tag{A.3}$$

and Moret–Bailly's formula [MB85, Chap. VIII, Thm. 1.3, $FCC(\mathbf{Spec}\,\mathbb{Z}[\frac{1}{d}], g, d)$] gives

$$\mu((\pi'_*\mathcal{L}_2)^\vee) = \frac{1}{2}[K' : \mathbb{C}(\mathbb{P}^1)] h_{\mathrm{Fal}}(A). \tag{A.4}$$

[10]For number fields, this is [Bos96b, Prop. A.3]. The proof works verbally for function fields.

Therefore we are done by

$$\begin{aligned}\mu_{\max}(\mathrm{Lie}(\mathcal{A}/B')^\vee) &= P_{\mathrm{Lie}(\mathcal{A}/B')^\vee}(1)\\ &= P_{\mathrm{Lie}(\mathcal{A}/B')}(g-1) - \deg \mathrm{Lie}(\mathcal{A}/B') &&\text{by Lemma A.2.(i)}\\ &\leq \frac{g}{2}[K':\mathbb{C}(\mathbb{P}^1)]h_{\mathrm{Fal}}(A) + [K':\mathbb{C}(\mathbb{P}^1)]h_{\mathrm{Fal}}(A) &&\text{by (A.2), (A.3), (A.4).}\end{aligned}$$

□

Remark A.3. *We can also prove $\widehat{\mu}_{\max}(\overline{\mathrm{Lie}\, A}) \leq h_{\mathrm{Fal}}(A)$ as follows:*

$$\begin{aligned}\mu_{\max}(\mathrm{Lie}(\mathcal{A}/B')) &= P_{\mathrm{Lie}(\mathcal{A}/B')}(1)\\ &= P_{\mathrm{Lie}(\mathcal{A}/B')\otimes\varepsilon_a^*\mathcal{L}_2^\vee}(1) &&\textit{by Lemma A.2.(ii)}\\ &\leq P_{(\ker \boldsymbol{\delta}_0)^\vee}(1) &&\textit{by Lemma A.2.(iii) applied to } \boldsymbol{\delta}_1^\vee\\ &\leq P_{(\pi'_*\mathcal{L}_2)^\vee}(2) &&\textit{by (A.1) applied to } y=1\\ &= 2\mu((\pi'_*\mathcal{L}_2)^\vee) &&\textit{by the semi-stability of } \pi'_*\mathcal{L}_2\\ &= [K:\mathbb{C}(\mathbb{P}^1)]h_{\mathrm{Fal}}(A) &&\textit{by (A.4).}\end{aligned}$$

Appendix B. Line bundle version of Zarhin's trick

Lemma B.1. *Let A/K be an abelian variety and let L be a symmetric ample line bundle on A defined over K. Set $Z(A) := A^4 \times (A^\vee)^4$. Then there exist a symmetric ample line bundle M on $Z(A)$, defined over K, such that $h^0(Z(A), M) = 1$, and a K-isogeny*

$$\pi\colon A^8 \longrightarrow Z(A)$$

such that π^*M and $L^{\boxtimes 8}$ define the same polarization on A^8.

Proof. For any line bundle N on an abelian variety, denote by ϕ_N the map $t_x^*N \otimes N^{\otimes -1}$.

Denote by $D := h^0(A, L)$. Then $\deg \phi_L = D^2$. Let P_A be the rigidified Poincaré bundle on $A\times A^\vee$, and let q_1, q_2 be the projections of $A\times A^\vee$ to both factors. By [BL03, Thm. 4.1],

$$L^\dagger := (\det q_{2*}(q_1^*L \otimes P_A))^{\otimes -1}$$

is an ample line bundle on $A^\vee$, defined over K, and $\phi_{L^\dagger} \circ \phi_L = [D]_A$. Moreover, $L^\dagger$ is symmetric since (using $([-1]_A \times 1_A)^*P_A \simeq (1_A \times [-1]_{A^\vee})^*P_A \simeq P_A^{\otimes -1}$)

$$[-1]_{A^\vee}^*(q_1^*L\otimes P_A) \simeq q_1^*L \otimes P_A^{\otimes -1} \simeq ([-1]_A \times 1_{A^\vee})^*(q_1^*L\otimes P_A) \simeq ([-1]_A^* q_1^*L)\otimes P_A \simeq q_1^*L\otimes P_A.$$

The four-square theorem gives $a, b, c, d \in \mathbb{Z}$ such that $a^2 + b^2 + c^2 + d^2 = D - 1$. Then

$$\mathcal{I} := \begin{bmatrix} a & -b & -c & -d\\ b & a & d & -c\\ c & -d & a & b\\ d & c & -b & a\end{bmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{Z}) \subseteq \mathrm{End}(A^4)$$

satisfies $\mathcal{I}\circ\mathcal{I}^{\mathrm{t}} = \mathcal{I}^{\mathrm{t}}\circ\mathcal{I} = [D-1]$; here $\mathcal{I}^{\mathrm{t}}$ is also seen as an element in $\mathrm{End}(A^4)$.

Since $\phi_{L^{\boxtimes 4}} = (\phi_L, \phi_L, \phi_L, \phi_L)$ is diagonal, we have

$$\mathcal{I}\circ\phi_{L^{\boxtimes 4}} = \phi_{L^{\boxtimes 4}}\circ\mathcal{I}, \qquad \mathcal{I}^{\mathrm{t}}\circ\phi_{L^{\boxtimes 4}} = \phi_{L^{\boxtimes 4}}\circ\mathcal{I}^{\mathrm{t}}.$$

To ease notation, denote by $X := A^4$. Then $Z(A) = X\times X^\vee$. Let p_1, p_2 be the projections, and let P_X be the normalized Poincaré bundle on $X \times X^\vee$. Define an endomorphism

$$\rho = \begin{bmatrix} -\mathcal{I}^{\mathrm{t}} & 0\\ 0 & 1\end{bmatrix} : X\times X^\vee \longrightarrow X\times X^\vee, \qquad (x,y)\mapsto(-\mathcal{I}^{\mathrm{t}}x, y).$$

We claim that the line bundle

$$M := p_1^*L^{\boxtimes 4}\otimes p_2^*L^{\dagger\boxtimes 4}\otimes\rho^*P_X$$

is our desired line bundle on $Z(A) = X \times X^\vee$, with

$$\pi = \begin{bmatrix} \mathcal{I} & -1 \\ \phi_{L^{\boxtimes 4}} & 0 \end{bmatrix} : A^8 = X \times X \longrightarrow Z(A) = X \times X^\vee, \qquad (x,y) \mapsto (\mathcal{I}x - y, \phi_{L^{\boxtimes 4}}(x)).$$

First M and π are clearly defined over K. Moreover, M is symmetric: the first two factors are symmetric, while $([-1]_X \times [-1]_{X^\vee})^* P_X \simeq P_X$ and ρ commutes with $[-1]$.

Next, it is easy to compute that

$$\ker \pi = \{(x, \mathcal{I}x) : x \in \ker \phi_{L^{\boxtimes 4}}\}, \quad \phi_M = \begin{bmatrix} \phi_{L^{\boxtimes 4}} & -\mathcal{I} \\ -\mathcal{I}^{\mathrm{t}} & \phi_{L^{\dagger \boxtimes 4}} \end{bmatrix}, \quad \pi^\vee = \begin{bmatrix} \mathcal{I}^{\mathrm{t}} & \phi_{L^{\boxtimes 4}} \\ -1 & 0 \end{bmatrix}.$$

So π is an isogeny and $\deg \pi = \deg \phi_{L^{\boxtimes 4}} = D^8$ (similarly $\deg \pi^\vee = D^8$), and

$$\pi^\vee \circ \phi_M \circ \pi = \begin{bmatrix} \phi_{L^{\boxtimes 4}} & 0 \\ 0 & \phi_{L^{\boxtimes 4}} \end{bmatrix} = \phi_{L^{\boxtimes 8}} \quad \text{using } \phi_{L^\dagger} \circ \phi_L = [D]_A. \tag{B.1}$$

But $\phi_{\pi^* M} = \pi^\vee \circ \phi_M \circ \pi$. So $\pi^* M$ and $L^{\boxtimes 8}$ are algebraically equivalent up to an element of $\mathrm{Pic}^0(A^8)$. In particular, $\pi^* M$ is ample. Since π is finite and surjective, M is ample.

Finally to prove $h^0(Z(A), M) = 1$, it suffices to show $\deg \phi_M = 1$. Now (B.1) implies

$$(\deg \pi^\vee)(\deg \phi_M)(\deg \pi) = \deg \phi_{L^{\boxtimes 8}} = D^{16}.$$

So $\deg \phi_M = 1$ since $\deg \pi = \deg \phi_{L^{\boxtimes 4}} = D^8$. We are done. □

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