The mixed Ax-Lindemann theorem and its applications to the Zilber-Pink conjecture

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Le théorème d'Ax-Lindemann mixte et ses applications à la conjecture de Zilber-Pink

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Contents

Introduction (Français)

Le but de cette thèse est d'étudier la géométrie diophantienne des variétés de Shimura mixtes. L'un des résultats principaux est le théorème d'Ax-Lindemann. Nous en déduirons ensuite un théorème de répartition et nous utiliserons ces deux résultats pour étudier la conjecture de Zilber-Pink. Dans cette thèse deux aspects de cette conjecture seront étudiés : la conjecture d'André-Oort et la conjecture d'André-Pink-Zannier.

Toute sous-variété algébrique d'une variété algébrique est supposée fermée sauf indication contraire.

La famille universelle des variétés abéliennes

Considérons le couple $(\mathrm{GSp}_{2g}, \mathbb{H}_g^+)$, où

- GSp_{2g} est le $\mathbb{Q}$-groupe
$$\mathrm{GSp}_{2g} := \left\{ h \in \mathrm{GL}_{2g} \mid h \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} h^t = \nu(h) \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \text{ avec } \nu(h) \in \mathbb{G}_m \right\}.$$
- $\mathbb{H}_g^+ := \{Z = X + iY \in M_g(\mathbb{C}) |\ Z = Z^t,\ Y > 0\}$.

Un fait élémentaire sur ce couple est que $\mathrm{GSp}_{2g}(\mathbb{R})^+$, la composante connexe de $\mathrm{GSp}_{2g}(\mathbb{R})$ dans la topologie archimédienne contenant 1, agit transitivement sur $\mathbb{H}_g^+$ par
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$
De plus, l'inclusion $\mathbb{H}_g^+ \subset M_g(\mathbb{C}) \simeq \mathbb{C}^{g^2}$ induit une structure complexe sur $\mathbb{H}_g^+$. Dans la théorie classique, ce couple correspond à l'espace de modules des variétés abéliennes principalement polarisées.

Pour avoir un autre couple correspondant à la famille universelle, il faut élargir $(\mathrm{GSp}_{2g}, \mathbb{H}_g^+)$. Définissons maintenant un deuxième couple $(P_{2g,\mathrm{a}}, \mathcal{X}_{2g,\mathrm{a}}^+)$[1] de la manière suivante :

- $P_{2g,\mathrm{a}}$ est le $\mathbb{Q}$-groupe $V_{2g} \rtimes \mathrm{GSp}_{2g}$, où V_{2g} est le $\mathbb{Q}$-groupe vectoriel de dimension $2g$ et GSp_{2g} agit sur V_{2g} par la représentation naturelle;
- $\mathcal{X}_{2g,\mathrm{a}}^+$ est $\mathbb{R}^{2g} \times \mathbb{H}_g^+$ comme ensembles, muni de l'action de $P_{2g,\mathrm{a}}(\mathbb{R})^+$ sur $\mathcal{X}_{2g,\mathrm{a}}^+$ définie par
$$(v, h) \cdot (v', x) := (v + hv', hx)$$
pour $(v, h) \in P_{2g,\mathrm{a}}(\mathbb{R})^+$ et $(v', x) \in \mathcal{X}_{2g,\mathrm{a}}^+$. On peut vérifier que cette action est aussi transitive. De plus, cette action est algébrique.

[1] La lettre « a » en indice est l'initiale du mot « abélien » pour désigner que ce couple correspond à la famille universelle des variétés abéliennes. On n'utilise pas $(P_{2g}, \mathcal{X}_{2g}^+)$ parce que cette notation plus simple est utilisée pour un autre couple correspondant au $\mathbb{G}_m$-torseur ample canonique sur la famille universelle.

Il est plus délicat de définir la structure complexe sur $\mathcal{X}^+_{2g,\mathrm{a}}$: tout d'abord par la transitivité de l'action de $P_{2g,\mathrm{a}}(\mathbb{R})^+$ sur $\mathcal{X}^+_{2g,\mathrm{a}}$, on a (pour un point $x_0 \in \mathcal{X}^+_{2g,\mathrm{a}}$)

$$\mathcal{X}^+_{2g,\mathrm{a}} = P_{2g,\mathrm{a}}(\mathbb{R})^+ \cdot x_0.$$

Par ailleurs on rappelle que le $P_{2g,\mathrm{a}}(\mathbb{R})^+$-ensemble $\mathcal{X}^+_{2g,\mathrm{a}}$ se plonge de manière équivariante dans un $P_{2g,\mathrm{a}}(\mathbb{C})$-ensemble[2]. On a donc

$$\mathcal{X}^+_{2g,\mathrm{a}} = P_{2g,\mathrm{a}}(\mathbb{R})^+ \cdot x_0 \hookrightarrow P_{2g,\mathrm{a}}(\mathbb{C}) \cdot x_0 = P_{2g,\mathrm{a}}(\mathbb{C})/\mathrm{Stab}_{P_{2g,\mathrm{a}}(\mathbb{C})}(x_0) =: \mathcal{X}^\vee.$$

Alors $\mathcal{X}^\vee$ est par une variété complexe algébrique. L'inclusion ci-dessus réalise $\mathcal{X}^+_{2g,\mathrm{a}}$ comme un ensemble ouvert (dans la topologie archimédienne) semi-algébrique de $\mathcal{X}^\vee$, et ainsi induit une structure complexe sur $\mathcal{X}^+_{2g,\mathrm{a}}$.

Remarque. *Une façon plus concrète de voir cette structure complexe sur $\mathcal{X}^+_{2g,\mathrm{a}}$ est (essentiellement) la suivante (prenons le cas $g = 1$) : sur chaque point $\tau \in \mathbb{H}^+$, la fibre de la projection $\mathcal{X}^+_{2,\mathrm{a}} \to \mathbb{H}^+$ est*

$$(\mathcal{X}^+_{2,\mathrm{a}})_\tau = \begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\sim} & \mathbb{C} \\ (a,b) & \mapsto & a + b\tau \end{array}.$$

L'analogue de cette identification pour les dimensions supérieures est aussi correcte. Voir Remark 1.3.4.

Maintenant prenons un groupe de congruence net $\Gamma := \mathbb{Z}^{2g} \rtimes \Gamma_G < P_{2g}(\mathbb{Z})$, on a alors

$$\mathfrak{A}_g := \Gamma\backslash\mathcal{X}^+_{2g} \xrightarrow{[\pi]} \mathcal{A}_g := \Gamma_G\backslash\mathbb{H}^+_g.$$

La fibre de $[\pi]$ sur un point $[x] \in \mathcal{A}_g$ est $\mathbb{Z}^{2g}\backslash\mathbb{R}^{2g}$ munie de la structure complexe de $(\mathcal{X}^+_{2g,\mathrm{a}})_x$. En dimension 1 ($g = 1$ et $x = \tau \in \mathbb{H}$) elle n'est que $\mathbb{R}^2 \simeq \mathbb{C}$, $(a,b) \mapsto a + b\tau$ comme expliqué ci-dessus.

Théorème (Kuga, Brylinski, Pink)**.** $\mathfrak{A}_g \xrightarrow{[\pi]} \mathcal{A}_g$ *est la famille universelle des variétés abéliennes principalement polarisées (munie d'une structure de niveau Γ_G) sur l'espace de modules fin $\mathcal{A}_g$. De plus $\mathfrak{A}_g$ et $\mathcal{A}_g$ sont des variétés algébriques complexes.*

Les variétés de Shimura connexes mixtes arbitraires

La famille universelle $\mathfrak{A}_g$ est un exemple de variété de Shimura connexe mixte. D'autres exemples incluent:

1. Le $\mathbb{G}_m$-torseur ample canonique sur $\mathfrak{A}_g$;

[2] Pour ceux qui connaissent bien la théorie de Hodge, ce nouvel ensemble est l'ensemble des $\mathbb{Q}$-structures de Hodge mixtes de type $\{(-1,0),(0,-1),(-1,-1)\}$ sur le $\mathbb{Q}$-espace vectoriel de dimension $2g+1$. Nous n'en parlerons pas beaucoup dans l'introduction. Voir le début de §1.3.1 pour plus de détails.

2. La biextension de Poincaré sur $\mathcal{A}_g$.

Les définitions des données de Shimura connexes mixtes et des variétés de Shimura connexes mixtes seront précisées dans §1.1.2.1. Il suffit ici de savoir qu'une donnée de Shimura connexe mixte est un couple $(P, \mathcal{X}^+)$ qui partage des propriétés élémentaires de $(P_{2g,\mathrm{a}}, \mathcal{X}^+_{2g,\mathrm{a}})$, par exemple P est un $\mathbb{Q}$-groupe et $P(\mathbb{R})^+U(\mathbb{C})$[3] agit transitivement sur $\mathcal{X}^+$ et cette action est algébrique. Une variété de Shimura connexe mixte S associée à $(P, \mathcal{X}^+)$ est le quotient $\Gamma\backslash\mathcal{X}^+$ de $\mathcal{X}^+$ par un sous-groupe de congruence Γ de $P(\mathbb{Q})$. D'après un théorème de Pink, S admet une structure canonique de variété algébrique. Ce théorème généralise un résultat de Baily-Borel pour les variétés de Shimura pures.

Historique du théorème d'Ax-Lindemann

Dans cette section, nous rappelons brièvement l'historique du théorème d'Ax-Lindemann et on voit comment il est une généralisation naturelle de l'analogue fonctionnel du théorème classique de Lindemann-Weierstrass. Commençons par le théorème classique de Lindemann-Weierstrass.

Théorème (Lindemann-Weierstrass)**.** *Soient* $\alpha_1, ..., \alpha_n \in \overline{\mathbb{Q}}$. *S'ils sont linéairement indépendants sur* $\mathbb{Q}$, *alors* $\exp(\alpha_1), ..., \exp(\alpha_n)$ *sont algébriquement indépendants sur* $\overline{\mathbb{Q}}$.

L'analogue fonctionnelle de ce théorème est la suivante :

Théorème (Analogue fonctionnel, démontré par Ax [5, 6])**.** *Soient* $\mathcal{Z}$ *une variété algébrique irréductible sur* $\mathbb{C}$ *et* $f_1, ..., f_n \in \mathbb{C}[\mathcal{Z}]$ *des fonctions régulières sur* $\mathcal{Z}$. *Si les fonctions* $f_1, ..., f_n$ *sont* $\mathbb{Q}$*-linéairement indépendantes à constantes près, c'est-à-dire qu'il n'existe pas* $a_1, ..., a_n \in \mathbb{Q}$ *(ne pas tous nuls) tels que* $a_1 f_1 + ... + a_n f_n \in \mathbb{C}$, *alors les fonctions*

$$\exp(f_1), ..., \exp(f_n) \colon \mathcal{Z} \to \mathbb{C}$$

sont algébriquement indépendantes sur $\mathbb{C}$.

Cet analogue fonctionnel peut s'écrire de la façon géométrique de la manière suivante (reformulée par Pila-Zannier). C'est cette forme-là que l'on généralisera aux variétés de Shimura connexes mixtes arbitraires.

Théorème (Ax-Lindemann pour les tores algébriques sur $\mathbb{C}$)**.** *Soient* $\mathrm{unif} = (\exp, \cdots, \exp) \colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ *et* $\mathcal{Z}$ *une sous-variété algébrique irréductible de* $\mathbb{C}^n$. *Alors* $\overline{\mathrm{unif}(\mathcal{Z})}^{\mathrm{Zar}}$ *est le translaté d'un sous-tore de* $(\mathbb{C}^*)^n$.

D'après l'énoncé de ce théorème d'Ax-Lindemann, nous sommes dans la **situation bi-algébrique** suivante : $\mathbb{C}^n$ et $(\mathbb{C}^*)^n$ sont des variétés algébriques,

[3] Ici U est un sous-groupe distingué de P. C'est un groupe vectoriel qui est uniquement déterminé par P (voir Definition 1.1.12). Pour $\mathfrak{A}_g$ il est trivial.

pourtant le morphisme $\mathrm{unif}\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ est transcendant. Donc à priori, il n'existe aucune relation entre les deux structures algébriques de $\mathbb{C}^n$ et de $(\mathbb{C}^*)^n$. Néanmoins nous avons trouvé par Ax-Lindemann une collection des sous-variétés, les $\overline{\mathrm{unif}(\mathcal{Z})}^{\mathrm{Zar}}$ avec $\mathcal{Z}$ algébrique dans $\mathbb{C}^n$, qui sont toutes bi-algébriques. Ici on dit qu'un sous-ensemble V de $\mathbb{C}^n$ est **bi-algébrique pour** $\mathbb{C}^n \xrightarrow{\mathrm{unif}} (\mathbb{C}^*)^n$ si V est fermé, algébrique, irréductible et son image sous unif est aussi algébrique. On dit qu'un sous-ensemble V' de $(\mathbb{C}^*)^n$ est **bi-algébrique pour** $\mathbb{C}^n \xrightarrow{\mathrm{unif}} (\mathbb{C}^*)^n$ s'il est l'image d'un sous-ensemble bi-algébrique de $\mathbb{C}^n$. Notons qu'Ax-Lindemann a comme conséquence directe la déscription des sous-variétés bi-algébriques : les sous-variétés bi-algébriques de $(\mathbb{C}^*)^n$ sont précisément les translatés des sous-tores de $(\mathbb{C}^*)^n$.

Il existe un résultat similaire pour les variétés abéliennes complexes :

Théorème (Ax-Lindemann pour les variétés abéliennes complexes)**.** *Soient A une variété abélienne complexe,* $\mathrm{unif}\colon \mathbb{C}^n \to A$ *et $\mathcal{Z}$ une sous-variété algébrique irréductible de $\mathbb{C}^n$. Alors* $\overline{\mathrm{unif}(\mathcal{Z})}^{\mathrm{Zar}}$ *est le translaté d'une sous-variété abélienne de A.*

Nous sommes alors dans une **situation bi-algébrique** similaire : $\mathbb{C}^n$ et A sont des variétés algébriques, pourtant le morphisme $\mathrm{unif}\colon \mathbb{C}^n \to A$ est transcendant. Donc à priori, il n'existe aucune relation entre les deux structures algébriques de $\mathbb{C}^n$ et de A. Néanmoins, nous avons trouvé par Ax-Lindemann une collection des sous-variétés, les $\overline{\mathrm{unif}(\mathcal{Z})}^{\mathrm{Zar}}$ avec $\mathcal{Z}$ algébrique dans $\mathbb{C}^n$, qui sont toutes bi-algébriques. Ici on dit qu'un sous-ensemble V de $\mathbb{C}^n$ est **bi-algébrique pour** $\mathbb{C}^n \xrightarrow{\mathrm{unif}} A$ si V est fermé, algébrique, irréductible et son image sous unif est aussi algébrique. On dit qu'un sous-ensemble V' de A est **bi-algébrique pour** $\mathbb{C}^n \xrightarrow{\mathrm{unif}} A$ s'il est l'image d'un sous-ensemble bi-algébrique de $\mathbb{C}^n$. Comme auparavant, Ax-Lindemann a comme conséquence directe la déscription des sous-variétés bi-algébriques : les sous-variétés bi-algébriques de A sont précisément les translatés des sous-variétés abéliennes de A.

Ax-Lindemann pour les tores algébriques sur $\mathbb{C}$ et Ax-Lindemann pour les variétés abéliennes ont été démontrés par Ax [5, 6]. Les démonstrations par la théorie o-minimale ont été trouvées par Pila-Zannier [51] et Peterzil-Starchenko [46]. Appelons ces deux cas **Ax-Lindemann plat**. Après ces travaux, des cas variés d'**Ax-Lindemann hyperbolique** (c'est-à-dire Ax-Lindemann pour les variétés de Shimura connexes pures)[4] ont été étudiés et démontrés par Pila [48] (pour $\mathcal{A}_1^N$), Ullmo-Yafaev [67] (pour les variétés de Shimura pures compactes) et Pila-Tsimerman [50] (pour $\mathcal{A}_g$). Le résultat de Pila, étant une découverte capitale pour ce théorème, a conduit à une démonstration inconditionnelle de la conjecture d'André-Oort pour $\mathcal{A}_1^N$, qui est la deuxième preuve inconditionnelle

[4]Au lieu de donner l'énoncé précis d'Ax-Lindemann hyperbolique ici, nous allons plutôt expliquer en détailles Ax-Lindemann mixte dans la prochaine section et signaler à quel cas Ax-Lindemann hyperbolique correspond.

des cas spécifiques de cette conjecture après le travail d'André pour $\mathcal{A}_1^2$ [2]. La version complète d'Ax-Lindemann hyperbolique a été démontré récemment par Klingler-Ullmo-Yafaev [29]. Le théorème d'Ax-Lindemann hyperbolique est aussi un énoncé bi-algébrique dans une situation bi-algébrique similaire à celle d'Ax-Lindemann plat.

Ayant tous ces résultats, on peut se poser les questions suivantes :

Question.

- *Est-ce qu'il existe un résultat contenant Ax-Lindemann plat et Ax-Lindemann hyperbolique ?*
- *De plus, est-ce qu'il existe une version en famille ?*

Les réponses à ces deux questions sont positives. Un des résultats principaux de cette thèse est la démonstration du théorème d'Ax-Lindemann mixte qui est le résultat désiré.

Avant de passer à la prochaine section, faisons la remarque suivante :

Remarque. *Dans les deux cas d'Ax-Lindemann plat, les conclusions ne changent pas si Z est seulement supposée* ***semi-algébrique et complexe analytique irréductible****. Ceci est une conséquence d'un résultat de Pila-Tsimerman [49, Lemma 4.1].*

L'énoncé du théorème d'Ax-Lindemann mixte

Dans cette partie, S est toujours une variété de Shimura connexe mixte associée à la donnée de Shimura connexe mixte $(P, \mathcal{X}^+)$ et unif : $\mathcal{X}^+ \to S$ est son uniformisation. Tout d'abord, rappelons qu'Ax-Lindemann est un théorème de bi-algébricité. Donc nous expliquerons au début la situation bi-algébrique pour ce cas. La variété S a une structure algébrique naturelle, l'espace d'uniformisation $\mathcal{X}^+$ n'est pourtant que très rarement une variété algébrique. Cependant on a :

Proposition. *Pour toute donnée de Shimura connexe mixte $(P, \mathcal{X}^+)$, il existe une variété complexe algébrique $\mathcal{X}^\vee$ définie en termes de $(P, \mathcal{X}^+)$ et une inclusion $\mathcal{X}^+ \hookrightarrow \mathcal{X}^\vee$ qui réalise $\mathcal{X}^+$ comme un ensemble ouvert (dans la topologie archimédienne) semi-algébrique de $\mathcal{X}^\vee$.*

D'après la remarque de la dernière section, il suffit de considérer la « situation bi-algébrique » suivante : considérons les sous-ensembles semi-algébriques et complexes analytiques irréductibles de $\mathcal{X}^+$ et la structure algébrique naturelle de S. Rappelons que unif : $\mathcal{X}^+ \to S$ est transcendant. Comme auparavant, on souhaite trouver les objets « bi-algébriques ».

Question. *Quels sont les objets bi-algébriques (c'est-à-dire les sous-ensembles semi-algébriques et complexes analytiques irréductibles de $\mathcal{X}^+$ dont l'image dans S est algébrique) ?*

Pour répondre à cette question, nous utilisons la notion de sous-variété faiblement spéciale introduite par Pink (voir Definition 1.2.2).

Définition. *1. Un sous-ensemble $\widetilde{F} \subset \mathcal{X}^+$ est dit* ***faiblement spécial*** *s'il existe une sous-donnée de Shimura connexe mixte $(Q, \mathcal{Y}^+)$ de $(P, \mathcal{X}^+)$, un sous-groupe distingué N de Q et un point $\widetilde{y} \in \mathcal{Y}^+$ tels que*

$$\widetilde{F} = N(\mathbb{R})^+ U_N(\mathbb{C})\widetilde{y},$$

où $U_N := N \cap U$ (rappelons que U est un sous-groupe distingué de P qui est un groupe vectoriel déterminé par P). Si $(P, \mathcal{X}^+) = (P_{2g,\mathrm{a}}, \mathcal{X}^+_{2g,\mathrm{a}})$ (c'est le cas considéré dans l'introduction), alors U est trivial.

2. Une sous-variété F de S est dite ***faiblement spéciale*** *si $F = \mathrm{unif}(\widetilde{F})$ pour un $\widetilde{F} \subset \mathcal{X}^+$ faiblement spécial.*

Pour les variétés de Shimura pures, Moonen a démontré que les sous-variétés faiblement spéciales d'une variété de Shimura pure sont précisément ses sous-variétés totalement géodésiques [39, 4.3]. Donnons ici un exemple pour les variétés de Shimura mixtes.

Exemple 1 (Voir Proposition 1.2.14)**.** *Soit $\mathfrak{A} \to B$ une famille des variétés abéliennes principalement polarisées de dimension g sur une courbe algébrique complexe B. Soit $\mathcal{C}$ sa partie isotriviale, c'est-à-dire le plus grand sous-schéma abélien isotrivial de $\mathfrak{A} \to B$. Alors quitte à prendre des revêtements finis de B, on peut supposer que $\mathcal{C}$ est une famille constante et qu'il existe un diagramme cartésien*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{i} & \mathfrak{A}_g \\ \downarrow & & \downarrow{\scriptstyle [\pi]} \\ B & \xrightarrow{i_B} & \mathcal{A}_g \end{array}$$

où i_B est soit constant soit quasi-fini, auquel cas i est aussi quasi-fini. Alors

$$\{i^{-1}(E)|\ E \text{ faiblement spécial dans } \mathfrak{A}_g\} = \{\text{translatés des sous-schémas abéliens de } \mathfrak{A} \to B \text{ par une section de torsion et puis par une section constante de } \mathcal{C} \to B\}.$$

Nous démontrons dans cette thèse (voir Remark 1.3.7, le cas pur par Ullmo-Yafaev [65]):

Théorème. *Un sous-ensemble $F \subset S$ est faiblement spécial si et seulement si $\widetilde{F}$ (une composante complexe analytique irréductible de $\mathrm{unif}^{-1}(F)$) est semi-algébrique dans $\mathcal{X}^+$ et F est algébrique irréductible dans S.*

Nous sommes désormais prêts à donner l'énoncé du théorème d'Ax-Lindemann mixte dont la démonstration sera faite dans le Chapitre 3 de cette thèse (de §3.1 à §3.4).

Théorème (Ax-Lindemann mixte)**.** *Soit $\widetilde{\mathcal{Z}}$ un sous-ensemble semi-algébrique et complexe analytique irréductible de $\mathcal{X}^+$. Alors $\overline{\mathrm{unif}(\widetilde{\mathcal{Z}})}^{\mathrm{Zar}}$ est faiblement spéciale.*

Ax-Lindemann hyperbolique est précisément le même énoncé lorsque la variété de Shimura mixte ambiante S est pure. Le théorème d'Ax-Lindemann mixte implique Ax-Lindemann plat et Ax-Lindemann hyperbolique [29]. De plus il est vraiment une version en famille. Pour le démontrer, nous utilisons un résultat de comptage pour Ax-Lindemann hyperbolique [29, Theorem 1.3].

Une esquisse de la démonstration d'Ax-Lindemann mixte sera donnée dans la prochaine section. Avant de passer à la démonstration, donnons ici un autre théorème assez proche d'Ax. Rappelons que nous avons une variété algébrique $\mathcal{X}^\vee$ telle que $\mathcal{X}^+ \hookrightarrow \mathcal{X}^\vee$.

Théorème (Ax de type log[5]). *Soient Y une sous-variété algébrique irréductible de S et $\widetilde{Y}$ une composante complexe analytique irréductible de* $\mathrm{unif}^{-1}(Y)$. *Définissons*

$$\overline{\widetilde{Y}}^{\mathrm{Zar}} := \textit{la composante complexe analytique irréductible de l'intersection de } \mathcal{X}^+ \textit{ avec l'adhérence de Zariski de } \widetilde{Y} \textit{ dans } \mathcal{X}^\vee \textit{ qui contient } \widetilde{Y}.$$

Alors $\overline{\widetilde{Y}}^{\mathrm{Zar}}$ est faiblement spéciale.

Ceci est aussi un résultat de cette thèse et sa version plus détaillée est le Theorem 2.3.1, où l'existence de $\overline{\widetilde{Y}}^{\mathrm{Zar}}$ (qui n'est pas claire à priori) est aussi démontrée. Si S est une variété de Shimura pure, ce théorème peut se déduire d'un résultat de Moonen [39, 3.6, 3.7]. Dans un article d'Ullmo-Yafaev à venir, sa version pure dans le cadre de la bi-algébricité sera expliquée avec plus de détails.

L'esquisse de la démonstration d'Ax-Lindemann mixte

Dans cette section nous donnons une esquisse de la démonstration du théorème d'Ax-Lindemann mixte. Pour simplifier, nous considérons seulement la famille universelle $\mathfrak{A}_g$, c'est-à-dire $(P,\mathcal{X}^+) = (P_{2g,\mathrm{a}}, \mathcal{X}^+_{2g,\mathrm{a}})$, $S = \mathfrak{A}_g$, $(G,\mathcal{X}^+_G) = (\mathrm{GSp}_{2g}, \mathbb{H}^+_g)$ et $S_G = \mathcal{A}_g$ avec Γ net. Supposons maintenant que $\widetilde{\mathcal{Z}} \subset \mathcal{X}^+_{2g,\mathrm{a}}$ est un sous-ensemble semi-algébrique et complexe analytique irréductible. Le diagramme suivant sera utile :

$$\begin{array}{ccc}
(P,\mathcal{X}^+) & \xrightarrow{\pi} & (G,\mathcal{X}^+_G) \\
\downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}_G} \\
S = \Gamma\backslash\mathcal{X}^+ & \xrightarrow{[\pi]} & S_G = \Gamma_G\backslash\mathcal{X}^+_G
\end{array}$$

La démonstration sera divisée en 6 étapes.

[5]Le fait que cet énoncé est assez proche d'Ax m'a été signalé par Bertrand, ainsi que le nom « Ax de type log ».

Étape 1 Définissons $Y := \overline{\mathrm{unif}(\widetilde{\mathcal{Z}})}^{\mathrm{Zar}}$. Soit $\widetilde{Z}$ un sous-ensemble maximal parmi tous les sous-ensembles semi-algébriques et complexes analytiques irréductibles de $\mathcal{X}^+$, qui à la fois contiennent $\widetilde{\mathcal{Z}}$ et à la fois sont contenus dans $\mathrm{unif}^{-1}(Y)$. L'existence d'un tel $\widetilde{Z}$ découle d'un argument de dimension. Alors $\widetilde{Z}$ est algébrique irréductible au sens de Definition 1.3.5, c'est-à-dire que $\widetilde{Z}$ est une composante complexe analytique irréductible de l'intersection de son adhérence de Zariski dans $\mathcal{X}^\vee$ et $\mathcal{X}^+$. Remplaçons S par la plus petite sous-variété de Shimura connexe mixte de S contenant Y et remplaçons $(P, \mathcal{X}^+)$, Γ, $(G, \mathcal{X}_G^+)$ et Γ_G respectivement. Remarquons que pour des raisons évidentes ces remplacements ne changent ni l'hypothèse ni la conclusion d'Ax-Lindemann. Il suffit alors de démontrer que $\widetilde{Z}$ est faiblement spéciale par la bi-algébricité des sous-variétés faiblement spéciales.

Notons N le groupe de monodromie algébrique connexe de Y^{sm}, c'est-à-dire

$$N = (\overline{\mathrm{Im}\big(\pi_1(Y^{\mathrm{sm}}) \to \pi_1(S) = \Gamma\big)}^{\mathrm{Zar}})^\circ.$$

Alors par les résultats d'André [1, Theorem 1] et de Wildeshaus [71, Theorem 2.2], $N \lhd P$. Voir la démonstration du Théorème 2.3.1(1).

Étape 2 Définissons le $\mathbb{Q}$-stabilisateur de $\widetilde{Z}$

$$H_{\widetilde{Z}} := (\overline{\mathrm{Stab}_{P(\mathbb{R})}(\widetilde{Z}) \cap \Gamma}^{\mathrm{Zar}})^\circ.$$

Alors *Ax de type log* implique $H_{\widetilde{Z}} \lhd N$. Voir Lemma 3.2.3.

Étape 3 Trouvons un ensemble fondamental $\mathcal{F}$ pour l'action de Γ sur $\mathcal{X}^+$ tel que $\mathrm{unif}|_{\mathcal{F}}$ est définissable dans la théorie o-minimale $\mathbb{R}_{an,\exp}$.

Pour la théorie o-minimale nous nous référons à [67, Section 3] (pour une version concise) et [48, Section 2, Section 3] (pour une version détaillée). Expliquons ici brièvement pourquoi et comment la théorie o-minimale est utile pour la démonstration. D'après l'énoncé d'Ax-Lindemann, c'est un théorème géométrique. Donc on souhaite chercher une démonstration géométrique. Pourtant il ne suffit pas d'utiliser uniquement la géométrie algébrique parce que le morphisme unif est transcendant. Pour résoudre ce problème, une façon possible est de « raffiner la topologie de Zariski » : à part des ($\mathbb{R}$-)polynômes, on permet à d'autres fonctions de définir les ensembles constructibles. La théorie o-minimale $\mathbb{R}_{an,\exp}$ est par définition la collection de tous les sous-ensembles de $\mathbb{R}^m$ ($\forall m \in \mathbb{N}$) qui sont définis par des équations et des inégalités des $\mathbb{R}$-polynômes, de la fonction $\mathbb{R}$-exponentielle et des fonctions réellement analytiques restreintes. Les sous-ensembles ci-dessus sont appelés **ensembles définissables dans** $\mathbb{R}_{an,\exp}$, et les applications dont les graphes sont définissables sont appelées **applications définissables dans** $\mathbb{R}_{an,\exp}$. Bien que $\mathbb{R}_{an,\exp}$ ne soit pas une topologie, les ensembles définissables jouent un rôle de même nature que les ensembles constructibles dans la topologie de Zariski. La théorie o-minimale $\mathbb{R}_{an,\exp}$ satisfait les propriétés suivantes :

1. $\mathbb{R}_{an,\exp}$ est une algèbre de Boole;

2. (Théorème de Chevalley) pour tout ensemble définissable A et toute application définissable $f\colon A \to B$, l'image $f(A)$ est aussi définissable;

3. (Décomposition connexe finie) tout ensemble définissable A peut s'écrire comme une union finie des ensembles définissables connexes.

4. (Décomposition cellulaire, voir [69, 2.11]) La décomposition connexe finie peut être renforcée : pour tout ensemble définissable A dans $\mathbb{R}^m$, il existe une décomposition cellulaire $\mathcal{D}$ de $\mathbb{R}^m$ telle que A est une union finie d'éléments de $\mathcal{D}$.

Si on peut trouver un ensemble fondamental $\mathcal{F}$ pour l'action de Γ sur $\mathcal{X}^+$ tel que unif $|_{\mathcal{F}}$ est définissable dans $\mathbb{R}_{an,\exp}$, alors on peut utiliser les outils de la théorie o-minimale pour étudier unif : $\mathcal{X}^+ \to S$. Finalement on souhaite récupérer des informations algébriques puisque, comme expliqué avant, la conclusion d'Ax-Lindemann est de trouver une collection des objets bi-algébriques. Les théorèmes de comptage de Pila-Wilkie serviront à cette fin. L'utilisation de la théorie o-minimale pour la démonstration sera expliquée dans l'*Étape 4*.

L'existence d'un tel $\mathcal{F}$ a été démontrée par Peterzil-Starchenko pour $\mathfrak{A}_g$ [47, Theorem 1.3] (dans leur preuve chaque fonction thêta est écrite en terme de $\mathbb{R}$-polynômes, de $\mathbb{R}$-exp et des fonctions réellement analytiques restreintes) et Klingler-Ullmo-Yafaev pour toutes les variétés de Shimura connexes pures [29, Theorem 1.2] (la preuve exploite les outils développés pour les compactifications toroïdales des variétés de Shimura pures [4]). Il est bon de remarquer que l'ensemble fondamental $\mathcal{F}$ construit par Peterzil-Starchenko est le plus naturel possible (voir Remark 1.3.4). En combinant ces deux théorèmes et quelques résultats supplémentaires, l'existence d'un tel $\mathcal{F}$ pour toutes les variétés de Shimura mixtes sera démontrée dans cette thèse §3.3.1.

Remarque. *Dans les trois premières étapes, la démonstration d'Ax-Lindemann mixte et celle d'Ax-Lindemann hyperbolique [29] ne sont pas essentiellement différentes : il suffit d'utiliser et de démontrer les résultats respectifs pour chaque cas. Mais à partir de l'Étape 4, les deux démonstrations diffèrent beaucoup.*

Étape 4 Pour le cas hyperbolique (c'est-à-dire pur), on souhaite démontrer $\dim(H_{\widetilde{Z}}) > 0$ dans cette étape. Ceci est fait par Klingler-Ullmo-Yafaev [29] en calculant les volumes des courbes algébriques dans l'espace d'uniformisation près de la frontière. Notons que c'est presque la dernière étape pour la démonstration du cas pur parce que l'on en déduira $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})\widetilde{z}$ (pour un point $\widetilde{z} \in \widetilde{Z}$) par une récurrence assez simple.

Pour le cas mixte, il ne suffit pas de démontrer uniquement $\dim(H_{\widetilde{Z}}) > 0$. Voici un cas qui est évidement impossible d'après Ax-Lindemann mixte et que la condition $\dim(H_{\widetilde{Z}}) > 0$ toute seule ne suffit pas à exclure : $\dim \pi(\widetilde{Z}) > 0$

mais $H_{\widetilde{Z}} < V_{2g}$. Dans ce cas, il est clair que $\widetilde{Z}$ ne peut pas être une orbite sous $H_{\widetilde{Z}}(\mathbb{R})^+$, pourtant il est possible que $\dim(H_{\widetilde{Z}})$ soit strictement positive.

Pour résoudre ce problème, nous démontrons dans cette étape (Proposition 3.2.6)

$$\pi(H_{\widetilde{Z}}) = (\overline{\mathrm{Stab}_{G(\mathbb{R})}\big(\pi(\widetilde{Z})\big) \cap \Gamma_G}^{\mathrm{Zar}})^{\circ}.$$

Il est évident que $\pi(H_{\widetilde{Z}})$ est contenu dans le membre droit de l'équation. Donc cette égalité révèle que $\pi(H_{\widetilde{Z}})$ est le plus grand possible.

C'est au cours de la démonstration de cette égalité que l'on doit utiliser la théorie o-minimale et le théorème de comptage de Pila-Wilkie. De plus, comparé à l'estimation de Klingler-Ullmo-Yafaev, on doit exploiter toutes les conclusions de la version en famille de Pila-Wilkie. Voir §3.3.2 pour la démonstration complète. Ici dans l'introduction, nous expliquons brièvement comment démontrer

$$\dim \pi(H_{\widetilde{Z}}) > 0$$

si $\dim \pi(\widetilde{Z}) > 0$.

Rappelons que $Y = \overline{\mathrm{unif}(\widetilde{Z})}^{\mathrm{Zar}}$. Définissons

$$\Sigma(\widetilde{Z}) := \{p \in P(\mathbb{R}) |\ \dim(p\widetilde{Z} \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F}) = \dim \widetilde{Z}\} \subset P(\mathbb{R}),$$

alors par le prolongement analytique,

$$\Sigma(\widetilde{Z}) = \{p \in P(\mathbb{R}) |\ p\widetilde{Z} \subset \mathrm{unif}^{-1}(Y),\ p\widetilde{Z} \cap \mathcal{F} \neq \emptyset\}.$$

Les faits suivants sur $\Sigma(\widetilde{Z})$ ne sont pas difficiles à démontrer :

1. $\Sigma(\widetilde{Z})$ et $\pi(\Sigma(\widetilde{Z}))$ sont tous les deux définissable dans $\mathbb{R}_{an,\exp}$ (par la première écriture de $\Sigma(\widetilde{Z})$ parce que $\mathrm{unif}|_{\mathcal{F}}$ est définissable et la fonction dim l'est aussi);

2. $\Sigma(\widetilde{Z}) \cdot \widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ (par la deuxième écriture de $\Sigma(\widetilde{Z})$);

3. $\pi\big(\Sigma(\widetilde{Z}) \cap \Gamma\big) = \pi\big(\Sigma(\widetilde{Z})\big) \cap \Gamma_G$ (voir Lemma 3.3.2[6]).

Pour démontrer $\dim \pi(H_{\widetilde{Z}}) > 0$, il suffit de prouver $|\pi(H_{\widetilde{Z}})(\mathbb{R}) \cap \Gamma_G)| = \infty$. Et donc il suffit de trouver deux nombres réels $c' > 0$ et $\delta > 0$ tels que pour tout $T \gg 0$,

$$|\{\gamma_G \in \pi(H_{\widetilde{Z}})(\mathbb{R}) \cap \Gamma_G |\ H(\gamma_G) \leqslant T\}| \geqslant c' T^{\delta}.$$

Donc il suffit de démontrer qu'il existe deux nombres réels $c' > 0$ et $\delta > 0$ et, pour chaque T assez grand, un bloc[7] $B(T) \subset \Sigma(\widetilde{Z})$ tel que

$$|\{\gamma_G \in \pi(B(T)) \cap \Gamma_G |\ H(\gamma_G) \leqslant T\}| \geqslant c' T^{\delta}.$$

[6] Ici il faut modifier un peu l'ensemble fondamental $\mathcal{F}$ choisi auparavant, mais ceci est faisable par quelques opérations simples. Voir la fin de §3.3.1.

[7] Un bloc est un ensemble définissable connexe tel que sa dimension coïncide avec la dimension de son adhérence dans la topologie de $\mathbb{R}$-Zariski.

Voir la fin de §3.3 pour plus de détails.

Maintenant nous utilisons un résultat de comptage dû à Klingler-Ullmo-Yafaev [29, Theorem 1.3] qui dit : il existe un nombre réel $\varepsilon > 0$ tel que pour tout $T \gg 0$,

$$|\{\gamma_G \in \pi(\Sigma(\widetilde{Z})) \cap \Gamma_G |\ H(\gamma_G) \leqslant T\}| \geqslant T^{\varepsilon}.$$

Mais d'après le théorème de Pila-Wilkie [48, cas $\mu = 0$ de 3.6] (ou tout simplement [29, Theorem 6.1]), il existe un nombre réel $c = c(\varepsilon) > 0$ tel que l'ensemble

$$\{\gamma_G \in \pi(\Sigma(\widetilde{Z})) \cap \Gamma_G |\ H(\gamma_G) \leqslant T\}$$

est contenu dans une union d'au plus $cT^{\varepsilon/2}$ blocs. Ceci implique qu'il existe deux nombres réels $c' > 0$, $\delta > 0$ tels que pour tout T assez grand, il existe un bloc $B_G(T) \subset \pi(\Sigma(\widetilde{Z}))$ avec

$$|\{\gamma_G \in B_G(T) \cap \Gamma_G |\ H(\gamma_G) \leqslant T\}| \geqslant c'T^{\delta}.$$

Remarquons que cette inégalité est exactement ce que nous souhaitons pour conclure cette étape de la démonstration d'Ax-Lindemann hyperbolique (c'est-à-dire pur). Mais pour démontrer Ax-Lindemann mixte, nous sommes obligés d'utiliser le fait que ces blocs $B_G(T)$ (pour tout T assez grand) viennent d'un nombre fini de familles de blocs ! Plus concrètement, au delà du fait que l'ensemble $\{\gamma_G \in \pi(\Sigma(\widetilde{Z})) \cap \Gamma_G |\ H(\gamma_G) \leqslant T\}$ est contenu dans une union d'au plus $cT^{\varepsilon/2}$ blocs, [48, cas $\mu = 0$ de 3.6] nous assure qu'il existe un entier $J > 0$ et J familles de blocs $B^j \subset \Sigma(\widetilde{Z}) \times \mathbb{R}^l$ ($j = 1, ...J$) tels que chacun de ces (au plus) $cT^{\varepsilon/2}$ blocs, en particulier les $B_G(T)$ pour tout T assez grand, est B^j_y pour certains j et $y \in \mathbb{R}^l$.

Pour chaque T assez grand, regardons $\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z})$. Parce que $B_G(T) = B^j_y$ pour certains j et $y \in \mathbb{R}^l$, $\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z})$ est la fibre de $(\pi \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma(\widetilde{Z}) \times \mathbb{R}^l)$ sur $y \in \mathbb{R}^l$. L'ensemble $(\pi \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma(\widetilde{Z}) \times \mathbb{R}^l)$ étant une famille définissable sur un sous-ensemble de $\mathbb{R}^l$, la décomposition cellulaire de $\mathbb{R}_{an,\exp}$ implique qu'il existe un entier $n_0 > 0$ tel que chaque fibre de $(\pi \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma(\widetilde{Z}) \times \mathbb{R}^l)$, en particulier chaque $\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z})$ pour T assez grand, a au plus n_0 composantes connexes (voir [69, 3.6]). Par conséquent, $\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z})$ a au plus n_0 composantes connexes. Par ailleurs,

$$\begin{aligned}
&\pi\big(\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z}) \cap \Gamma\big) \\
&= B_G(T) \cap \pi\big(\Sigma(\widetilde{Z}) \cap \Gamma\big) \\
&= B_G(T) \cap \pi\big(\Sigma(\widetilde{Z})\big) \cap \Gamma_G && \text{par le 3ème fait sur } \widetilde{Z} \text{ cité précédemment} \\
&= B_G(T) \cap \Gamma_G && \text{puisque } B_G(T) \subset \pi\big(\Sigma(\widetilde{Z})\big).
\end{aligned}$$

Donc il existe une composante connexe $B(T)$ de $\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z})$ telle que

$$|\{\gamma_G \in \pi(B(T)) \cap \Gamma_G |\ H(\gamma_G) \leqslant T\}| \geqslant \frac{c'}{n_0} T^{\delta}.$$

Mais par définition n_0 ne dépend pas de T. Donc cet ensemble $B(T)$ est ce que nous cherchons.

Remarque. *Par la démonstration, l'indépendance de n_0 vis-à-vis de T est cruciale. Mais le $B_G(T)$ que l'on obtient de Pila-Wilkie dépend de la hauteur choisie T et par conséquent, n_0 aussi dépend de T à priori. C'est pour surmonter cette difficulté que nous sommes obligés d'utiliser le fait que tous les $B_G(T)$ viennent d'un nombre fini de familles de blocs pour le cas mixte.*

Étape 5 Démontrons que $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+\widetilde{z}$ pour un $\widetilde{z} \in \widetilde{Z}$.

Pour le cas hyperbolique (c'est-à-dire pur), ceci découle d'un argument de récurrence plutôt simple.

Pour le cas mixte, il faut étudier plus soigneusement la géométrie. Il faut utiliser le théorème d'Ax-Lindemann pour la fibre (qui est une variété abélienne pour $\mathfrak{A}_g \to \mathcal{A}_g$) et faire des calculs supplémentaires. Ceci sera fait dans Theorem 3.2.8(1). Remarquons que la structure complexe des fibres de $\mathcal{X}^+_{2g,\mathrm{a}} \xrightarrow{\pi} \mathbb{H}^+_g$ est utilisée à cette étape.

Remarque. *Pour une variété de Shimura mixte connexe arbitraire S associée à la donnée de Shimura mixte connexe $(P, \mathcal{X}^+)$, la fibre de $S \to S_G$, où S_G est la partie pure de S, n'a pas nécessairement une structure de groupe compatible à la loi de groupe de P (voir Lemma 2.1.1). En particulier le théorème d'Ax-Lindemann pour la fibre n'était pas connu jusqu'à présent en général. Sa démonstration, qui sera donnée dans §3.4, est aussi technique : nous devons répéter les arguments de l'Étape 4 à l'Étape 6 (Step I à Step IV dans §3.4), avec une « Étape 6 » assez différente (qui est Step IV dans §3.4).*

Étape 6 Démontrons $H_{\widetilde{Z}} \lhd P$.

Pour le cas hyperbolique (c'est-à-dire pur), ceci est une conséquence de la structure des groupes réductifs. Les faits que $H_{\widetilde{Z}} \lhd N \lhd P$ et que P est réductif impliquent directement $H_{\widetilde{Z}} \lhd P$.

Pour le cas mixte, cet argument n'est plus valable. En général, il est faux qu'un sous-groupe distingué d'un sous-groupe distingué soit encore un sous-groupe distingué du groupe de départ. Donc à part des arguments de la théorie de groupes (les résultats de §1.1.4 seront utilisés), il faut aussi étudier soigneusement la géométrie. Voir Theorem 3.2.8(2).

Ici expliquons seulement pourquoi $V_{H_{\widetilde{Z}}} := \mathcal{R}_u(H_{\widetilde{Z}}) = H_{\widetilde{Z}} \cap V_{2g}$ est distingué dans P. Pour cela, nous utilisons la structure complexe des fibres de $\pi\colon \mathcal{X}^+_{2g,\mathrm{a}} \to \mathbb{H}^+_g$: soit $\widetilde{z} \in \widetilde{Z}$ un point tel que $\pi(\widetilde{z})$ est Hodge-generique dans $\mathcal{X}^+_G$. Un tel $\widetilde{z}$ existe puisque l'on a supposé que S est la plus petite variété de Shimura connexe mixte qui contient $Y = \overline{\mathrm{unif}(\widetilde{Z})}^{\mathrm{Zar}}$. Donc le groupe de Mumford-Tate $\mathrm{MT}(\pi(\widetilde{z}))$ est égal à G. Mais $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+\widetilde{z}$ par l'*Étape 5*, donc la fibre de $\widetilde{Z}$ sur $\pi(\widetilde{z})$ est

$$\widetilde{Z}_{\pi(\widetilde{z})} = V_{H_{\widetilde{Z}}}(\mathbb{R})\widetilde{z}.$$

Comme $\widetilde{Z}$ est par définition un sous-ensemble complexe analytique de $\mathcal{X}^+$ (et donc de $\mathcal{X}^+_{2g,\mathrm{a}}$), $V_{H_{\widetilde{Z}}}(\mathbb{R})$ est un sous-espace complexe de $(\mathcal{X}^+_{2g,\mathrm{a}})_{\pi(\widetilde{z})} = V_{2g}(\mathbb{R})$. Mais la structure complexe de $(\mathcal{X}^+_{2g,\mathrm{a}})_{\pi(\widetilde{z})}$ est donnée par la structure de Hodge de type $\{(-1,0),(0,-1)\}$ sur V_{2g} dont le groupe de Mumford-Tate est $\mathrm{MT}(\pi(\widetilde{z})) = G$. Donc $V_{H_{\widetilde{Z}}}$ est un G-module. Donc $V_{H_{\widetilde{Z}}} \lhd P$ puisque $\mathcal{R}_u(P)$ est commutatif.

Conclusion Maintenant par les 6 étapes ci-dessus (surtout les conclusions de l'*Étape 5* et de l'*Étape 6*), $\mathrm{unif}(\widetilde{Z})$ est une sous-variété faiblement spéciale de $\mathfrak{A}_g$. Comme $Y = \overline{\mathrm{unif}(\widetilde{Z})}^{\mathrm{Zar}}$ par définition et $\mathrm{unif}(\widetilde{Z})$, étant faiblement spéciale, est une sous-variété algébrique de $\mathfrak{A}_g$, $Y = \mathrm{unif}(\widetilde{Z})$. Mais $Y = \overline{\mathrm{unif}(\widetilde{\mathcal{Z}})}^{\mathrm{Zar}}$ par définition, donc $\overline{\mathrm{unif}(\widetilde{\mathcal{Z}})}^{\mathrm{Zar}}$ est faiblement spéciale.

D'Ax-Lindemann à André-Oort

Une des motivations principales pour étudier le théorème d'Ax-Lindemann est ses applications à la conjecture de Zilber-Pink. La conjecture d'André-Oort est le cas le plus connu de cette conjecture.

Conjecture (André-Oort)**.** *Soient S une variété de Shimura connexe mixte et Σ l'ensemble de ses points spéciaux. Soit Y une sous-variété irréductible de S. Si $\overline{Y \cap \Sigma}^{\mathrm{Zar}} = Y$, alors Y est une sous-variété de Shimura connexe mixte de S (ou, de manière équivalente, Y est faiblement spéciale[8]).*

Exemple. *Les points spéciaux de $\mathfrak{A}_g$ sont précisément les points correspondants aux points de torsion sur les variétés abéliennes CM. Donc la conjecture d'André-Oort recouvre partiellement la conjecture de Manin-Mumford.*

Cette conjecture a été démontrée, sous l'hypothèse de Riemann généralisée, pour toutes les variétés de Shimura pures par Klingler-Ullmo-Yafaev [66, 30]. Inspirés par la récente démonstration inconditionnelle d'André-Oort pour le cas $\mathcal{A}_1^N$ (faite par Pila [48]), des progrès ont été faits pour obtenir des preuves ne reposant pas sur GRH. Le cadre de la démonstration de Pila est la stratégie proposée par Pila-Zannier :

1. Démontrer le théorème d'Ax-Lindemann;

2. Déduire d'Ax-Lindemann la répartition[9] des sous-variétés (faiblement) spéciales de dimension strictement positive contenues dans une sous-variété;

3. Définir un paramètre (que l'on appelle la complexité) pour les points dans Σ et choisir un « bon » ensemble fondamental pour l'action de Γ sur $\mathcal{X}^+$ tel que $\mathrm{unif}|_{\mathcal{F}}$ est définissable dans $\mathbb{R}_{an,\exp}$;

[8] L'équivalence des deux conclusions découle de [54, Proposition 4.2, Proposition 4.15].
[9] Au sens du Théorème 4.1.3.

4. Démontrer une borne supérieure pour la hauteur d'un point arbitraire dans $\mathrm{unif}^{-1}(\Sigma) \cap \mathcal{F}$ par rapport à la complexité de son image dans Σ;

5. Démontrer une borne inférieure pour la taille des orbites sous Galois des points dans Σ par rapport à leurs complexités;

6. Conclure par le théorème d'Ax-Lindemann, le théorème de répartition dans (2), la borne supérieure dans (4) et la borne inférieure dans (5). Cette étape est une conséquence directe des étapes précédentes.

Le théorème d'Ax-Lindemann est démontré dans cette thèse sous la forme la plus générale. Le théorème de répartition dans (2) sera aussi démontré (Theorem 4.1.3). Remarquons que ce théorème pour les variétés de Shimura pures a été obtenu par Ullmo [64, Théorème 4.1] et aussi séparément par Pila-Tsimerman [50, Section 7] sans « faiblement ». Le choix de l'ensemble fondamental $\mathcal{F}$ et la définissabilité de $\mathrm{unif}|_{\mathcal{F}}$ dans (3) sont faits dans §3.3.1 et la complexité des points dans Σ est définie au cours de la démonstration du Théorème 4.3.1. La borne supérieure dans (4) a été démontrée par Pila-Tsimerman [49, Theorem 3.1] pour $\mathcal{A}_g$ et leur résultat peut être facilement généralisé aux variétés de Shimura mixtes de type abélien. Pour la borne inférieure dans (5), on ramènera le cas des variétés de Shimura mixtes au cas des variétés de Shimura pures dans §4.2. Le meilleur résultat pour les variétés de Shimura pures est donné par Tsimerman [62, Theorem 1.1] qui l'a démontré inconditionnellement pour tous les points spéciaux de $\mathcal{A}_6^N$ et sous GRH pour tous les points spéciaux de $\mathcal{A}_g$[10]. En combinant tous ces résultats, on a (Theorem 4.3.1)

Théorème. *La conjecture d'André-Oort est valable inconditionellement pour toute variété de Shimura mixte S dont la partie pure est une sous-variété de $\mathcal{A}_6^N$ (par exemple $\mathfrak{A}_6^N$). Elle est valable sous GRH pour toutes les variétés de Shimura mixtes de type abélien.*

Pour démontrer la conjecture d'André-Oort pour les variétés de Shimura mixtes qui ne sont pas de type abélien, il nous manque une bonne définition de la complexité pour les points dans Σ qui nous permet d'avoir la borne supérieure dans (4). Remarquons que par les arguments du Théorème 4.3.1, il suffit de l'avoir pour toutes les variétés de Shimura pures. Daw-Orr sont en train d'étudier ce problème.

D'André-Oort à André-Pink-Zannier

L'obstacle qui nous empêche de démontrer la conjecture d'André-Oort pour $\mathfrak{A}_g$ ($g \geqslant 7$) est la borne inférieure pour la taille des orbites sous Galois des

[10] La borne inférieure est conjecturée par Edixhoven [19]. L'étude de cette borne est initiée aussi par Edixhoven qui l'a démontré inconditionnellement pour les surfaces de Hilbert [18]. Des résultats similaires à celui de Tsimerman pour les points spéciaux de $\mathcal{A}_3^N$ ont été obtenus inconditionnellement par Ullmo-Yafaev séparément et ils ont aussi démontré la borne inférieure pour toutes les variétés de Shimura pures sous GRH [68].

points spéciaux. On peut considérer une version plus faible d'André-Oort : remplaçons Σ par l'ensemble des points de torsion sur les variétés abéliennes CM **qui sont isogènes à une variété abélienne CM fixée**. Dans ce cas, l'obstacle ci-dessus a été surmonté par une série de travaux de Habegger-Pila [24, Section 6] et d'Orr [43]. Le point clé pour ce faire est un théorème de Masser-Wüstholz [35] et sa version effective donnée par Gaudron-Rémond [21].

Ce cas particulier d'André-Oort est contenu dans une autre conjecture que l'on appelle la conjecture d'André-Pink-Zannier.

Conjecture (André-Pink-Zannier)**.** *Soient S une variété de Shimura connexe mixte, s un point de S et Y une sous-variété irréductible de S. Soit Σ l'orbite de Hecke généralisée de s. Si $\overline{Y \cap \Sigma}^{\mathrm{Zar}} = Y$, alors Y est faiblement spéciale.*

Plusieurs cas de cette conjecture avaient déjà été étudiés par André avant que sa forme finale ait été donnée par Pink [54, Conjecture 1.6]. Elle est aussi liée à un problème proposé par Zannier. Voir §5.1.1 pour plus de détails. Pink a aussi démontré [54, Theorem 5.4] que cette conjecture implique la conjecture de Mordell-Lang.

La conjecture d'André-Pink-Zannier a été intensement étudiée par Orr [43, 42]. Dans cette thèse on considérera seulement la famille universelle $\mathfrak{A}_g$ pour la conjecture d'André-Pink-Zannier. Dans ce cas on peut calculer l'orbite de Hecke généralisée de s de manière explicite. On a (5.1.1)

$$\begin{aligned} \Sigma &= \text{points de division de l'orbite sous les isogénies polarisées de } s \\ &= \{t \in \mathfrak{A}_g \mid \exists n \in \mathbb{N} \text{ et une isogénie polarisée} \\ &\qquad f\colon (\mathfrak{A}_{g,\pi(s)}, \lambda_{\pi(s)}) \to (\mathfrak{A}_{g,\pi(t)}, \lambda_{\pi(t)}) \text{ tels que } nt = f(s)\}. \end{aligned}$$

Finalement nous démontrons (Theorem 4.3.2, Theorem 5.1.4 et Theorem 5.1.5)

Théorème. *La conjecture d'André-Pink-Zannier est valable pour $\mathfrak{A}_g$ et Y dans chacune des trois situations suivantes :*

1. *s est un point de torsion de $\mathfrak{A}_{g,\pi(s)}$ et $\mathfrak{A}_{g,\pi(s)}$ est une variété abélienne CM (ce qui est un cas spécifique de la version faible de la conjecture d'André-Oort mentionnée auparavant);*

2. *s est un point de torsion de $\mathfrak{A}_{g,\pi(s)}$ et $\dim \pi(Y) \leqslant 1$;*

3. *$s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$ et $\dim(Y) = 1$.*

La première partie de ce théorème est une généralisation des anciens résultats de Edixhoven-Yafaev [72, 20] (pour les courbes dans les variétés de Shimura pures) et Klingler-Ullmo-Yafaev [66, 30] (pour les variétés de Shimura pures) et sa version p-adique a été démontrée par Scanlon [58].

Nous consacrerons la dernière section de cette thèse §5.5 à expliquer que le même énoncé d'André-Pink-Zannier en remplaçant s par un sous-groupe finiment engendré d'une fibre de $\mathfrak{A}_g \to \mathcal{A}_g$ (qui est une variété abélienne) et en remplaçant l'orbite sous les isogénies polarisées par l'orbite sous les isogénies (pas nécessairement polarisées) se déduit en fait de la conjecture d'André-Pink-Zannier.

Zilber-Pink

Finalement abordons la conjecture de Zilber-Pink [54, 73, 57].

Conjecture (Zilber-Pink)**.** *Soit S une variété de Shimura connexe mixte. Soit Y une sous-variété Hodge-générique de S. Alors*

$$\bigcup_{\substack{S' \text{ spéciale}, \\ \operatorname{codim}(S') > \dim(Y)}} S' \cap Y$$

n'est pas Zariski dense dans Y.

Cette conjecture est une généralisation commune de la conjecture d'André-Oort et la conjecture d'André-Pink-Zannier (voir [52, Theorem 3.3]). Habegger-Pila ont démontré récemment plusieurs résultats pour la conjecture de Zilber-Pink pour $\mathcal{A}_1^N$ [23] (dans le même article ils ont aussi démontré la conjecture de Zilber-Pink pour toutes les courbes sur $\overline{\mathbb{Q}}$ dans les variétés abéliennes), notamment un résultat inconditionel pour une grande classe de courbes [24]. Nous ne parlerons pas du cas des groupes algébriques (voir l'éxposé Bourbaki de Chambert-Loir [14] pour un résumé avant les travaux de Habegger-Pila).

Pour les variétés de Shimura mixtes, il n'y a pas beaucoup de résultats pour cette conjecture. Á part des résultats de cette thèse, Bertrand, Bertrand-Edixhoven, Bertrand-Pillay et Bertrand-Masser-Pillay-Zannier ont étudié récemment les biextentions de Poincaré [7, 11, 8, 9, 10]. Ils ont obtenu plusieurs résultats dont certains fournissent des exemples reliés à cette thèse.

Structure de la thèse

Le Chapitre 1 introduit les préliminaires de cette thèse. La section §1.1 fait un résumé de la théorie des variétés de Shimura mixtes, se concentrant sur les aspects traités dans la thèse. En particulier, la section §1.1.1 fait un résumé de la théorie des structures de Hodge mixtes qui conduit naturellement à la définition des variétés de Shimura mixtes dans §1.1.2. D'autres propriétés élémentaires seront aussi données dans §1.1.2. La section §1.1.3 introduit les variétés de Shimura mixtes de type Siegel (en particulier la famille universelle des variétés abéliennes) et se termine en un « reduction lemma ». Toutes ces sous-sections sont des faits connus et la référence principale est [53, Chapitre 1-Chapitre 3]. Dans §1.1.4 nous démontrons une proposition de la théorie des groupes algébriques qui sera utilisée dans la thèse par la suite. La section §1.2 fait un résumé des propriétés élémentaires des sous-variétés faiblement spéciales et donne la description géométrique des sous-variétés faiblement spéciales des variétés de Shimura mixtes de type Kuga. La section §1.3 concerne la situation bi-algébrique pour les variétés de Shimura mixtes.

Le Chapitre 2 démontre le théorème d'Ax de type log. La section §2.1 concerne des résultats sur la partie unipotente, c'est-à-dire la fibre de la projection d'une variété de Shimura connexe mixte vers sa partie pure. La section §2.2 comporte plusieurs résultats connus pour les groupes de monodromie des variations admissibles des structures de Hodge. Après ces préliminaires, le théorème d'Ax de type log sera démontré dans §2.3.

Le Chapitre 3 démontre le théorème d'Ax-Lindemann mixte. La section §3.1 donne quatre énoncés équivalents pour ce théorème. La section §3.2 esquisse la démonstration et prouve en détails l'*Étape 1*, l'*Étape 2*, l'*Étape 5* et l'*Étape 6*. La section §3.3 traite l'estimation en utilisant la théorie o-minimale. Ceci correspond à l'*Étape 3* et à l'*Étape 4*. La section §3.4 traite la partie unipotente et répond à une question restante pour l'*Étape 5*. Dans l'appendice de ce chapitre nous discutons de deux aspects: §3.5.1 présente plus de détails sur un fait simple que nous admettons à propos de la définissabilité dans §3.3.1 et §3.5.2 esquisse une démonstration simplifiée du théorème d'Ax-Lindemann plat.

Le Chapitre 4 concerne plusieurs aspects pour passer d'Ax-Lindemann à André-Oort. La section §4.1 démontre le théorème de répartition comme une conséquence du théorème d'Ax-Lindemann mixte. La section §4.2 ramène la borne inférieure pour les orbites sous Galois des points spéciaux des variétés de Shimura mixtes à la borne inférieure pour les variétés de Shimura pures. En combinant ces deux résultats, Ax-Lindemann et la borne supérieure étudiée par Pila-Tsimerman, nous démontrons le résultat principal pour la conjecture d'André-Oort dans §4.3. La démonstration de la version faible d'André-Oort sera aussi donnée dans §4.3. L'appendice de ce chapitre résume les estimées des orbites sous Galois des points spéciaux des variétés de Shimura pures obtenue par Ullmo-Yafaev [66, Section 2].

Le Chapitre 5 concerne la conjecture d'André-Pink-Zannier. La section §5.1 donne le contexte et énonce les résultats principaux. La section §5.2 calcule les orbites de Hecke généralisées dans $\mathfrak{A}_g$. La section §5.3 démontre le cas de torsion et §5.4 démontre le cas de non-torsion. Chaque démonstration contient la définition des complexités des point dans l'orbite de Hecke généralisée, la borne supérieure pour les hauteurs et la borne inférieure pour les orbites sous Galois. Les estimations pour les deux cas sont légèrement différentes. La section §5.5 discute des variantes de la conjecture d'André-Pink-Zannier.

Introduction (English)

The goal of this dissertation is to study the Diophantine geometry of mixed Shimura varieties. A main result is the mixed Ax-Lindemann theorem. Then we shall deduce a distribution theorem from it and use both results to study the Zilber-Pink conjecture. We will focus on two aspects of this conjecture: the André-Oort conjecture and the André-Pink-Zannier conjecture.

Subvarieties of algebraic varieties are always assumed to be closed unless stated otherwise.

Universal family of abelian varieties

Consider the pair $(\mathrm{GSp}_{2g}, \mathbb{H}_g^+)$, where

- GSp_{2g} is the $\mathbb{Q}$-group

$$\mathrm{GSp}_{2g} := \left\{ h \in \mathrm{GL}_{2g} \mid h \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} h^t = \nu(h) \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \text{ with } \nu(h) \in \mathbb{G}_m \right\}.$$

- $\mathbb{H}_g^+ := \{Z = X + iY \in M_g(\mathbb{C}) |\ Z = Z^t,\ Y > 0\}$.

A basic fact about this pair is that $\mathrm{GSp}_{2g}(\mathbb{R})^+$, the connected component of $\mathrm{GSp}_{2g}(\mathbb{R})$ containing 1 in the archimedean topology, acts transitively on $\mathbb{H}_g^+$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

Moreover, the inclusion $\mathbb{H}_g^+ \subset M_g(\mathbb{C}) \simeq \mathbb{C}^{g^2}$ induces a complex structure on $\mathbb{H}_g^+$. In classical theory, this pair corresponds to the moduli space of principally polarized abelian varieties.

In order to get another pair corresponding the universal family, we shall enlarge $(\mathrm{GSp}_{2g}, \mathbb{H}_g^+)$. Define now a pair $(P_{2g,\mathrm{a}}, \mathcal{X}_{2g,\mathrm{a}}^+)$[1] as follows:

- $P_{2g,\mathrm{a}}$ is the $\mathbb{Q}$-group $V_{2g} \rtimes \mathrm{GSp}_{2g}$, where V_{2g} is the $\mathbb{Q}$-vector group of dimension $2g$ and GSp_{2g} acts on V_{2g} by the natural representation;

- $\mathcal{X}_{2g,\mathrm{a}}^+$ is $\mathbb{R}^{2g} \times \mathbb{H}_g^+$ as sets, with the action of $P_{2g,\mathrm{a}}(\mathbb{R})^+$ on $\mathcal{X}_{2g,\mathrm{a}}^+$ defined by
$$(v, h) \cdot (v', x) := (v + hv', hx)$$
for $(v, h) \in P_{2g,\mathrm{a}}(\mathbb{R})^+$ and $(v', x) \in \mathcal{X}_{2g,\mathrm{a}}^+$. One can check that this action is also transitive. Besides, this action is algebraic.

[1] The subscript "a", being the initial of "abelian", is written here in order to indicate that this pair corresponds to the universal family of abelian varieties. We do not use $(P_{2g}, \mathcal{X}_{2g}^+)$ because the latter notation is used for another pair corresponding to the canonical ample $\mathbb{G}_m$-torsor over the universal family.

Defining the complex structure on $\mathcal{X}_{2g,\mathrm{a}}^{+}$ is more tricky: first of all by the transitivity of the action of $P_{2g,\mathrm{a}}(\mathbb{R})^{+}$ on $\mathcal{X}_{2g,\mathrm{a}}^{+}$, we have (for a point $x_0 \in \mathcal{X}_{2g,\mathrm{a}}^{+}$)

$$\mathcal{X}_{2g,\mathrm{a}}^{+} = P_{2g,\mathrm{a}}(\mathbb{R})^{+} \cdot x_0.$$

Next recall that the $P_{2g,\mathrm{a}}(\mathbb{R})^{+}$-set $\mathcal{X}_{2g,\mathrm{a}}^{+}$ embeds equivariantly into a $P_{2g,\mathrm{a}}(\mathbb{C})$-set[2]. Hence we have

$$\mathcal{X}_{2g,\mathrm{a}}^{+} = P_{2g,\mathrm{a}}(\mathbb{R})^{+} \cdot x_0 \hookrightarrow P_{2g,\mathrm{a}}(\mathbb{C}) \cdot x_0 = P_{2g,\mathrm{a}}(\mathbb{C})/\mathrm{Stab}_{P_{2g,\mathrm{a}}(\mathbb{C})}(x_0) =: \mathcal{X}^{\vee}.$$

Then $\mathcal{X}^{\vee}$ is a complex algebraic variety. The inclusion above realizes $\mathcal{X}_{2g,\mathrm{a}}^{+}$ as a semi-algebraic open subset (w.r.t. the archimedean topology) of $\mathcal{X}^{\vee}$, and hence induces a complex structure on $\mathcal{X}_{2g,\mathrm{a}}^{+}$.

Remark. *A more concrete way to see this complex structure on $\mathcal{X}_{2g,\mathrm{a}}^{+}$ is (essentially) as follows (take the case $g = 1$): over each point $\tau \in \mathbb{H}^{+}$, the fiber of the projection $\mathcal{X}_{2,\mathrm{a}}^{+} \to \mathbb{H}^{+}$ is*

$$(\mathcal{X}_{2,\mathrm{a}}^{+})_{\tau} = \begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\sim} & \mathbb{C} \\ (a,b) & \mapsto & a + b\tau \end{array}.$$

Higher dimensional analogue for this identification still holds. See Remark 1.3.4.

Now take a neat congruence group $\Gamma := \mathbb{Z}^{2g} \rtimes \Gamma_G < P_{2g}(\mathbb{Z})$, we have then

$$\mathfrak{A}_g := \Gamma \backslash \mathcal{X}_{2g}^{+} \xrightarrow{[\pi]} \mathcal{A}_g := \Gamma_G \backslash \mathbb{H}_g^{+}.$$

The fiber of $[\pi]$ over a point $[x] \in \mathcal{A}_g$ is $\mathbb{Z}^{2g} \backslash \mathbb{R}^{2g}$ with the complex structure of $(\mathcal{X}_{2g,\mathrm{a}}^{+})_x$. In dimension 1 ($g = 1$ and $x = \tau \in \mathbb{H}$) this is just $\mathbb{R}^2 \simeq \mathbb{C}$, $(a,b) \mapsto a + b\tau$ by the discussion above.

Theorem (Kuga, Brylinski, Pink)**.** $\mathfrak{A}_g \xrightarrow{[\pi]} \mathcal{A}_g$ *is the universal family of principally polarized abelian varieties with the level structure Γ_G over the fine moduli space $\mathcal{A}_g$. Both $\mathfrak{A}_g$ and $\mathcal{A}_g$ are algebraic varieties.*

Arbitrary connected mixed Shimura variety

The universal family $\mathfrak{A}_g$ is an example of connected mixed Shimura varieties. Other examples include:

1. The canonical ample $\mathbb{G}_m$-torsor over $\mathfrak{A}_g$;
2. The Poincaré bi-extension over $\mathcal{A}_g$.

[2]For readers who are more familiar with Hodge theory, this new set will be the set of all mixed $\mathbb{Q}$-Hodge structure of type $\{(-1,0),(0,-1),(-1,-1)\}$ on the $\mathbb{Q}$-vector space of dimension $2g+1$. We shall not go into detail for this in the Introduction. See the beginning of §1.3.1 for more details.

The precise definitions of connected mixed Shimura data and connected mixed Shimura varieties will be given in §1.1.2.1. Here we just say that a connected mixed Shimura datum is a pair $(P, \mathcal{X}^+)$ which "behaves" like $(P_{2g,\mathrm{a}}, \mathcal{X}^+_{2g,\mathrm{a}})$, e.g. P is a $\mathbb{Q}$-group and $P(\mathbb{R})^+U(\mathbb{C})^3$ acts transitively on $\mathcal{X}^+$ and this action is algebraic. A connected mixed Shimura variety S associated with $(P, \mathcal{X}^+)$ is then defined to be $\Gamma\backslash\mathcal{X}^+$ for a congruence subgroup of $P(\mathbb{Q})$. The fact that S has a canonical structure of algebraic variety is a theorem of Pink, generalizing the result of Baily-Borel for pure Shimura varieties.

History of the Ax-Lindemann theorem

In this subsection, we briefly review the history of the Ax-Lindemann theorem and see how it is a natural generalization of the functional analogue of the classical Lindemann-Weierstrass theorem. We start with the classical Lindemann-Weierstrass theorem

Theorem (Lindemann-Weierstrass)**.** *Let $\alpha_1, ..., \alpha_n \in \overline{\mathbb{Q}}$. If they are linearly independent over $\mathbb{Q}$, then $\exp(\alpha_1), ..., \exp(\alpha_n)$ are algebraically independent over $\overline{\mathbb{Q}}$.*

The analogue of this theorem for the functional case says the follows:

Theorem (Analogue for functional case, proved by Ax [5, 6])**.** *Let $\mathcal{Z}$ be an irreducible algebraic variety over $\mathbb{C}$ and let $f_1, ..., f_n \in \mathbb{C}[\mathcal{Z}]$ be regular functions on $\mathcal{Z}$. If the functions $f_1, ..., f_n$ are $\mathbb{Q}$-linearly independent modulo constants, i.e. there do not exist $a_1, ..., a_n \in \mathbb{Q}$ (not all zero) such that $a_1 f_1 + ... + a_n f_n \in \mathbb{C}$, then the functions*

$$\exp(f_1), ..., \exp(f_n) \colon \mathcal{Z} \to \mathbb{C}$$

are algebraically independent over $\mathbb{C}$.

This functional analogue can be rewritten in the following geometric form (reformulated by Pila-Zannier). This is the form which is easier to generalize to any connected mixed Shimura variety.

Theorem (Ax-Lindemann for algebraic tori over $\mathbb{C}$)**.** *Let* $\mathrm{unif} = (\exp, ..., \exp) \colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ *and let $\mathcal{Z}$ be an irreducible algebraic subvariety of $\mathbb{C}^n$. Then $\overline{\mathrm{unif}(\mathcal{Z})}^{\mathrm{Zar}}$ is the translate of a subtorus of $(\mathbb{C}^*)^n$.*

By the statement of this Ax-Lindemann theorem, we are in the following **bi-algebraic situation**: Both $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ are algebraic varieties, however the morphism $\mathrm{unif} \colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ is transcendental. Hence a priori, there is no obvious relation between the two algebraic structures on $\mathbb{C}^n$ and on $(\mathbb{C}^*)^n$. Nevertheless, we have found by Ax-Lindemann a class of subvarieties, i.e.

[3] Here U is a normal subgroup of P which is a vector group. It is uniquely determined by P (see Definition 1.1.12). For $\mathfrak{A}_g$ it is trivial.

$\overline{\mathrm{unif}(\mathcal{Z})}^{\mathrm{Zar}}$ with $\mathcal{Z}$ algebraic in $\mathbb{C}^n$, which are all bi-algebraic. Here a subset V of $\mathbb{C}^n$ is said to be **bi-algebraic for** $\mathbb{C}^n \xrightarrow{\mathrm{unif}} (\mathbb{C}^*)^n$ if V is closed irreducible algebraic and its image under unif is also closed irreducible algebraic. A subset V' of $(\mathbb{C}^*)^n$ is said to be **bi-algebraic for** $\mathbb{C}^n \xrightarrow{\mathrm{unif}} (\mathbb{C}^*)^n$ if it is the image of a bi-algebraic subset of $\mathbb{C}^n$. Remark that Ax-Lindemann has the following direct corollary: the bi-algebraic subvarieties of $(\mathbb{C}^*)^n$ are precisely the translates of the subtori of $(\mathbb{C}^*)^n$.

A similar result holds for complex abelian varieties:

Theorem (Ax-Lindemann for complex abelian varieties, proved by Ax [5, 6]). *Let A be a complex abelian variety, let* $\mathrm{unif}\colon \mathbb{C}^n \to A$ *and let $\mathcal{Z}$ be an irreducible subvariety of* $\mathbb{C}^n$. *Then* $\overline{\mathrm{unif}(\mathcal{Z})}^{\mathrm{Zar}}$ *is the translate of an abelian subvariety of A.*

For this case, we are in a similar **bi-algebraic situation**: Both $\mathbb{C}^n$ and A are algebraic, however the morphism $\mathrm{unif}\colon \mathbb{C}^n \to A$ is transcendental. Hence a priori, there is no obvious relation between the two algebraic structures on $\mathbb{C}^n$ and on A. Nevertheless, we have found by Ax-Lindemann a class of subvarieties, i.e. $\overline{\mathrm{unif}(\mathcal{Z})}^{\mathrm{Zar}}$ with $\mathcal{Z}$ algebraic in $\mathbb{C}^n$, which are all bi-algebraic. Here a subset V of $\mathbb{C}^n$ is said to be **bi-algebraic for** $\mathbb{C}^n \xrightarrow{\mathrm{unif}} A$ if V is closed irreducible algebraic and its image under unif is also closed irreducible algebraic. A subset V' of A is said to be **bi-algebraic for** $\mathbb{C}^n \xrightarrow{\mathrm{unif}} A$ if it is the image of a bi-algebraic subset of $\mathbb{C}^n$. In the case Ax-Lindemann also implies the description of the bi-algebraic subvarieties: the bi-algebraic subvarieties of A are precisely the translates of the abelian subvarieties of A.

Both Ax-Lindemann for algebraic tori over $\mathbb{C}$ and Ax-Lindemann for complex abelian varieties are proved by Ax [5, 6]. Proofs via o-minimal theory have been found by Pila-Zannier [51] and Peterzil-Starchenko [46]. We call these two cases the **flat Ax-Lindemann** theorems. Later, different cases of the **hyperbolic Ax-Lindemann** theorem (i.e. Ax-Lindemann for connected pure Shimura varieties)[4] have been studied and proved by Pila [48] (for $\mathcal{A}_1^N$), Ullmo-Yafaev [67] (for compact pure Shimura varieties), Pila-Tsimerman [50] (for $\mathcal{A}_g$). The result of Pila, being a breakthrough for this theorem, led to an unconditional proof of the André-Oort conjecture for $\mathcal{A}_1^N$, which is the second unconditional proof for special cases of this conjecture after the work of André himself for $\mathcal{A}_1^2$ [2]. The full version of the hyperbolic Ax-Lindemann has recently been proved by Klingler-Ullmo-Yafaev [29]. The hyperbolic Ax-Lindemann is also a bi-algebraic statement in a bi-algebraic situation similar to the flat Ax-Lindemann.

Having all these results, one may ask the following questions:

[4]Instead of giving the precise statement of the hyperbolic Ax-Lindemann theorem, we will explain in detail the mixed Ax-Lindemann theorem in the next section and point out to which case hyperbolic Ax-Lindemann corresponds.

Question. • *Is there a result which contains both the flat and the hyperbolic Ax-Lindemann theorems?*

• *Furthermore, is there a family version?*

The answers to these questions are yes. One of the main results of this dissertation is to prove the mixed Ax-Lindemann theorem, which is the desired result.

Before proceeding to the next subsection, let us do the following remark:

Remark. *In both cases of the flat Ax-Lindemann theorem, the conclusion does not change if we only assume Z to be **semi-algebraic and complex analytic irreducible**. This follows from a result of Pila-Tsimerman [49, Lemma 4.1].*

The statement of the mixed Ax-Lindemann theorem

In this part, let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S$ be its uniformization. First of all, recall that the Ax-Lindemann theorem is a theorem of bi-algebraicity. Hence we should first explain the bi-algebraic situation for this case. The variety S has a natural algebraic structure, however the uniformizing space $\mathcal{X}^+$ is almost never an algebraic variety. Nevertheless we have:

Proposition. *For any connected mixed Shimura datum $(P, \mathcal{X}^+)$, there exists a complex algebraic variety $\mathcal{X}^\vee$ defined in terms of $(P, \mathcal{X}^+)$ and an inclusion $\mathcal{X}^+ \hookrightarrow \mathcal{X}^\vee$ which realizes $\mathcal{X}^+$ as a semi-algebraic open subset of $\mathcal{X}^\vee$ (w.r.t. the archimedean topology).*

By the remark of the last subsection, it suffices to consider the following "bi-algebraic situation": consider the semi-algebraic and complex analytic irreducible subsets of $\mathcal{X}^+$ and the natural algebraic structure of S. Recall that $\mathrm{unif}\colon \mathcal{X}^+ \to S$ is transcendental. As before we want to find "bi-algebraic" objects.

Question. *What are the bi-algebraic objects (i.e. semi-algebraic and complex analytic irreducible subsets of $\mathcal{X}^+$ whose images are algebraic in S)?*

To answer this question, we use the notion of weakly special subvarieties introduced by Pink (see Definition 1.2.2).

Definition. *1. A subset $\widetilde{F} \subset \mathcal{X}^+$ is called **weakly special** if there exist a connected mixed Shimura subdatum $(Q, \mathcal{Y}^+)$ of $(P, \mathcal{X}^+)$, a connected normal subgroup N of Q and a point $\widetilde{y} \in \mathcal{Y}^+$ such that*

$$\widetilde{F} = N(\mathbb{R})^+ U_N(\mathbb{C})\widetilde{y},$$

where $U_N := N \cap U$ (recall that U is a normal vector subgroup of P which is determined by P). If $(P, \mathcal{X}^+) = (P_{2g,\mathrm{a}}, \mathcal{X}^+_{2g,\mathrm{a}})$ (which is the case we will focus on in the Introduction), then U is trivial.

2. *A subvariety F of S is called* ***weakly special*** *if $F = \mathrm{unif}(\widetilde{F})$ for some $\widetilde{F} \subset \mathcal{X}^+$ weakly special.*

For pure Shimura varieties, Moonen proved that weakly special subvarieties of a pure Shimura variety are precisely its totally geodesic subvarieties [39, 4.3]. For mixed Shimura varieties, let us give an example:

Example 0.0.1 (See Proposition 1.2.14). *Let $\mathfrak{A} \to B$ be a family of principally polarized abelian varieties of dimension g over an algebraic curve B. Let $\mathcal{C}$ be its isotrivial part, i.e. the largest isotrivial abelian subscheme of $\mathfrak{A} \to B$. Then up to taking finite covers of B, we may assume that $\mathcal{C}$ is a constant family and that there exists a cartesian diagram*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{i} & \mathfrak{A}_g \\ \downarrow & & \downarrow{\scriptstyle [\pi]} \\ B & \xrightarrow{i_B} & \mathcal{A}_g \end{array}$$

where i_B is either constant or quasi-finite, in which case i is also quasi-finite. Then

$$\{i^{-1}(E)|\ E \text{ weakly special in } \mathfrak{A}_g\} = \{\text{translates of abelian subschemes of } \mathfrak{A} \to B \text{ by a torsion section and then by a constant section of } \mathcal{C} \to B\}$$

We will prove in this dissertation (see Remark 1.3.7, pure case by Ullmo-Yafaev [65]):

Theorem. *A subset $F \subset S$ is weakly special iff $\widetilde{F}$ (a complex analytic irreducible component of $\mathrm{unif}^{-1}(F)$) is semi-algebraic in $\mathcal{X}^+$ and F is irreducible algebraic in S.*

Now we are ready to give the statement of the mixed Ax-Lindemann theorem, which will be proved in Chapter 3 of this dissertation (from §3.1 to §3.4).

Theorem (mixed Ax-Lindemann). *Let $\widetilde{\mathcal{Z}}$ be a semi-algebraic and complex analytic irreducible subset of $\mathcal{X}^+$. Then $\overline{\mathrm{unif}(\widetilde{\mathcal{Z}})}^{\mathrm{Zar}}$ is weakly special.*

The hyperbolic Ax-Lindemann is precisely the same statement when the ambient mixed Shimura variety S is pure. The mixed Ax-Lindemann theorem contains both the flat and the hyperbolic Ax-Lindemann theorem [29] and is indeed a family version. A counting result for hyperbolic Ax-Lindemann [29, Theorem 1.3] is used for its proof.

The sketch of the proof for mixed Ax-Lindemann will be given in the next section. Before proceeding to the proof, I would like to state another theorem which is of Ax's type. Recall that there exists an algebraic variety $\mathcal{X}^\vee$ such that $\mathcal{X}^+ \hookrightarrow \mathcal{X}^\vee$.

Theorem (Ax of log type[5])**.** *Let Y be an irreducible algebraic subvariety of S and let $\widetilde{Y}$ be a complex analytic irreducible component of* $\mathrm{unif}^{-1}(Y)$*. Define*

$$\overline{\widetilde{Y}}^{\mathrm{Zar}} := \text{the complex analytic irreducible component of the intersection of } \mathcal{X}^+ \text{ and the Zariski closure of } \widetilde{Y} \text{ in } \mathcal{X}^\vee \text{ which contains } \widetilde{Y}.$$

Then $\overline{\widetilde{Y}}^{\mathrm{Zar}}$ is weakly special.

This is also a result of this dissertation and a more refined version is Theorem 2.3.1, where the existence of $\overline{\widetilde{Y}}^{\mathrm{Zar}}$ (which is not obvious) is also proved. When S is a pure Shimura variety, this theorem follows from a result of Moonen [39, 3.6, 3.7]. In a forthcoming article of Ullmo-Yafaev, its pure version in the framework of the bi-algebraicity will be explained with more details.

Sketch of the proof for mixed Ax-Lindemann

In this section we give a sketch of the proof for the mixed Ax-Lindemann theorem. For simplification we will focus on the universal family $\mathfrak{A}_g$, i.e. $(P,\mathcal{X}^+) = (P_{2g,\mathrm{a}},\mathcal{X}^+_{2g,\mathrm{a}})$, $S=\mathfrak{A}_g$, $(G,\mathcal{X}^+_G) = (\mathrm{GSp}_{2g},\mathbb{H}^+_g)$ and $S_G=\mathcal{A}_g$ with Γ neat. Assume that $\widetilde{\mathcal{Z}} \subset \mathcal{X}^+_{2g,\mathrm{a}}$ is a semi-algebraic and complex analytic irreducible subset. The following diagram will be useful:

$$\begin{array}{ccc} (P,\mathcal{X}^+) & \xrightarrow{\pi} & (G,\mathcal{X}^+_G) \\ \downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}_G} \\ S=\Gamma\backslash\mathcal{X}^+ & \xrightarrow{[\pi]} & S_G=\Gamma_G\backslash\mathcal{X}^+_G \end{array}$$

The proof will be divided into 6 steps.

Step 1 Let $Y := \overline{\mathrm{unif}(\widetilde{\mathcal{Z}})}^{\mathrm{Zar}}$. Let $\widetilde{Z}$ be a semi-algebraic and complex analytic irreducible subset of $\mathcal{X}^+$ which contains $\widetilde{\mathcal{Z}}$ and is contained in $\mathrm{unif}^{-1}(Y)$, maximal for these properties. The existence of such a $\widetilde{Z}$ follows from a dimension argument. Then $\widetilde{Z}$ is irreducible algebraic in the sense of Definition 1.3.5, i.e. $\widetilde{Z}$ is a complex analytic irreducible component of the intersection of its Zariski closure in $\mathcal{X}^\vee$ and $\mathcal{X}^+$. Replace S by the smallest connected mixed Shimura subvariety of S containing Y and replace $(P,\mathcal{X}^+)$, Γ, $(G,\mathcal{X}^+_G)$ and Γ_G accordingly. Remark that for obvious reasons this does not change the assumption or the conclusion of Ax-Lindemann. It then suffices to prove that $\widetilde{Z}$ is weakly special by the bi-algebraicity of weakly special subvarieties.

Let N be the connected algebraic monodromy group of Y^{sm}, i.e.

$$N = (\overline{\mathrm{Im}\big(\pi_1(Y^{\mathrm{sm}}) \to \pi_1(S)=\Gamma\big)}^{\mathrm{Zar}})^\circ.$$

[5] The fact that this is a statement of Ax's type, as well as the name "Ax of log type", is pointed out to me by Bertrand.

Then by results of André [1, Theorem 1] and Wildeshaus [71, Theorem 2.2], $N \lhd P$. See the proof of Theorem 2.3.1(1).

Step 2 Define the $\mathbb{Q}$-stabilizer of $\widetilde{Z}$

$$H_{\widetilde{Z}} := (\overline{\mathrm{Stab}_{P(\mathbb{R})}(\widetilde{Z}) \cap \Gamma}^{\mathrm{Zar}})^{\circ}.$$

Then *Ax of log type* implies $H_{\widetilde{Z}} \lhd N$. See Lemma 3.2.3.

Step 3 Find a fundamental set $\mathcal{F}$ for the action of Γ on $\mathcal{X}^+$ such that $\mathrm{unif}|_{\mathcal{F}}$ is definable in the o-minimal theory $\mathbb{R}_{an,\exp}$.

For basic knowledge of the o-minimal theory we refer to [67, Section 3] (for a concise version) and [48, Section 2, Section 3] (for a more detailed version). Here we briefly explain why and how o-minimal theory is useful for the proof. By the statement of Ax-Lindemann, it is a geometric theorem. Therefore we wish to find a geometric proof. However it is not enough to use merely algebraic geometry because the morphism unif is transcendental. To solve this problem, one possible way is to "refine the Zariski topology": apart from the ($\mathbb{R}$-)polynomials, we also allow other functions to define the constructible sets. The o-minimal theory $\mathbb{R}_{an,\exp}$ is defined to be the collection of all subsets of $\mathbb{R}^m$ ($\forall m \in \mathbb{N}$) which are defined by equalities and inequalities of $\mathbb{R}$-polynomials, the $\mathbb{R}$-exponential function and all restricted real analytic functions. The subsets of $\mathbb{R}^m$ above are called **definable sets in** $\mathbb{R}_{an,\exp}$, and the morphisms whose graphs are definable sets are called **definable maps in** $\mathbb{R}_{an,\exp}$. Although $\mathbb{R}_{an,\exp}$ is not a topology, definable sets play a similar role of constructible sets for the Zariski topology. The o-minimal theory $\mathbb{R}_{an,\exp}$ behaves well for the following reasons:

1. $\mathbb{R}_{an,\exp}$ is a boolean algebra;
2. (Chevalley's theorem) for any definable set A and any definable map $f\colon A \to B$, the image $f(A)$ is also definable;
3. (finite connected decomposition) any definable set A can be written as a finite union of connected definable sets.
4. (Cell decomposition, see [69, 2.11]) The finite connected decomposition can be strengthened: for any definable set A in $\mathbb{R}^m$, there exists a cell decomposition $\mathcal{D}$ of $\mathbb{R}^m$ such that A is a finite union of elements of $\mathcal{D}$.

Now if we can find a fundamental set $\mathcal{F}$ for the action of Γ on $\mathcal{X}^+$ such that $\mathrm{unif}|_{\mathcal{F}}$ is definable in $\mathbb{R}_{an,\exp}$, then we can use tools from the o-minimal theory to study $\mathrm{unif}\colon \mathcal{X}^+ \to S$. Finally, we want to retrieve the algebraic information because, as discussed before, the conclusion of Ax-Lindemann is to find a class of bi-algebraic objects. The counting theorems of Pila-Wilkie will serve for this. The use of the o-minimal theory for the proof will be explained in *Step 4*.

The existence of such an $\mathcal{F}$ has been proved by different people in different cases: for $\mathfrak{A}_g$ by Peterzil-Starchenko [47] (in writting explicitely every theta function in terms of $\mathbb{R}$-polynomials, $\mathbb{R}$-exp and restricted real analytic functions), for any connected pure Shimura variety by Klingler-Ullmo-Yafaev [29, Theorem 1.2] (the proof exploited tools developped for the toroidal compactification of pure Shimura varieties [4]). It is good to remark that the fundamental set $\mathcal{F}$ constructed by Peterzil-Starchenko is the most natural possible (see Remark 1.3.4). Combining these two theorems with some extra work, the existence of such an $\mathcal{F}$ for any mixed Shimura variety is proved in this dissertation §3.3.1.

Remark. *In the first three steps, the proofs for mixed Ax-Lindemann and for hyperbolic Ax-Lindemann [29] are not essentially different: we just use and prove corresponding results for each case. However from Step 4, the two proofs will differ very much.*

Step 4 For the hyperbolic (i.e. pure) case, we want to prove $\dim(H_{\widetilde{Z}}) > 0$ in this step. This is done by Klingler-Ullmo-Yafaev [29] by calculating volumes of algebraic curves in the uniformizing space near the boundary. Note that this is almost the final step for the proof of the pure case, because an easy induction will then imply $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})\widetilde{z}$ for some $\widetilde{z} \in \widetilde{Z}$.

For the mixed case, it is not at all enough to prove merely $\dim(H_{\widetilde{Z}}) > 0$. A naive counterexample is as follows: $\dim \pi(\widetilde{Z}) > 0$ but $H_{\widetilde{Z}} < V_{2g}$. In this example, $\widetilde{Z}$ cannot be an $H_{\widetilde{Z}}(\mathbb{R})$-orbit, nevertheless $\dim(H_{\widetilde{Z}})$ can be positive.

In order to tackle this problem, we prove in this step (Proposition 3.2.6)

$$\pi(H_{\widetilde{Z}}) = (\overline{\mathrm{Stab}_{G(\mathbb{R})}\left(\pi(\widetilde{Z})\right) \cap \Gamma_G}^{\mathrm{Zar}})^{\circ}.$$

The group $\pi(H_{\widetilde{Z}})$ is contained in the right hand side. Hence the meaning of this equality is that $\pi(H_{\widetilde{Z}})$ is as large as possible.

It is in the proof of this equality that we use the o-minimal theory and the Pila-Wilkie counting theorem. Besides, compared to the estimate of Klingler-Ullmo-Yafaev, we have to exploit all the conclusions of the family version of Pila-Wilkie. See §3.3.2 for the whole proof. Here in the Introduction, we just briefly explain how to prove

$$\dim \pi(H_{\widetilde{Z}}) > 0$$

if $\dim \pi(\widetilde{Z}) > 0$.

Recall that $Y = \overline{\mathrm{unif}(\widetilde{Z})}^{\mathrm{Zar}}$. Define

$$\Sigma(\widetilde{Z}) := \{p \in P(\mathbb{R}) |\ \dim(p\widetilde{Z} \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F}) = \dim \widetilde{Z}\} \subset P(\mathbb{R}),$$

then by analytic continuation

$$\Sigma(\widetilde{Z}) = \{p \in P(\mathbb{R}) |\ p\widetilde{Z} \subset \mathrm{unif}^{-1}(Y),\ p\widetilde{Z} \cap \mathcal{F} \neq \emptyset\}.$$

There are some basic facts about $\Sigma(\widetilde{Z})$:

1. Both $\Sigma(\widetilde{Z})$ and $\pi(\Sigma(\widetilde{Z}))$ are definable in $\mathbb{R}_{an,\exp}$ (by the first form of $\Sigma(\widetilde{Z})$ because unif $|_{\mathcal{F}}$ is definable and the function dim is also definable);
2. $\Sigma(\widetilde{Z})\cdot\widetilde{Z}\subset \mathrm{unif}^{-1}(Y)$ (by the second form of $\Sigma(\widetilde{Z})$);
3. $\pi\big(\Sigma(\widetilde{Z})\cap\Gamma\big)=\pi\big(\Sigma(\widetilde{Z})\big)\cap\Gamma_G$ (see Lemma 3.3.2[6]).

In order to prove $\dim\pi(H_{\widetilde{Z}})>0$, it suffices to prove $|\pi(H_{\widetilde{Z}})(\mathbb{R})\cap\Gamma_G|=\infty$. Therefore it suffices to find two constants $c'>0$ and $\delta>0$ such that for any T large enough,

$$\{\gamma_G\in\pi(H_{\widetilde{Z}})(\mathbb{R})\cap\Gamma_G|\ H(\gamma_G)\leqslant T\}\geqslant c'T^{\delta}.$$

So it is enough to prove that there exist two constants $c'>0$ and $\delta>0$ and, for any T large enough, a block[7] $B(T)\subset\Sigma(\widetilde{Z})$ such that

$$|\{\gamma_G\in\pi(B(T))\cap\Gamma_G|\ H(\gamma_G)\leqslant T\}|\geqslant c'T^{\delta}.$$

See the end of §3.3 for more details.

Now we use a counting result of Klingler-Ullmo-Yafaev [29, Theorem 1.3], which says the following: there exists a constant $\varepsilon>0$ such that $\forall T\gg 0$,

$$|\{\gamma_G\in\pi(\Sigma(\widetilde{Z}))\cap\Gamma_G|\ H(\gamma_G)\leqslant T\}|\geqslant T^{\varepsilon}.$$

Then by the Pila-Wilkie theorem [48, 3.6, case $\mu=0$] (or a simply [29, Theorem 6.1]), there exists a constant $c=c(\varepsilon)>0$ such that the set

$$\{\gamma_G\in\pi(\Sigma(\widetilde{Z}))\cap\Gamma_G|\ H(\gamma_G)\leqslant T\}$$

is contained in a union of at most $cT^{\varepsilon/2}$ blocks. This implies that there exist two constants $c'>0$, $\delta>0$such that for any T large enough, there exists a block $B_G(T)\subset\pi(\Sigma(\widetilde{Z}))$ with

$$|\{\gamma_G\in B_G(T)\cap\Gamma_G|\ H(\gamma_G)\leqslant T\}|\geqslant c'T^{\delta}.$$

Remark that this inequality is exactly what we expect from this step for the proof of the hyperbolic (i.e. pure) Ax-Lindemann. However to prove the mixed Ax-Lindemann, we are obliged to use the fact that the blocks $B_G(T)$ (for $T\gg 0$) come from finitely many block families! More concretely, apart from the fact that $\{\gamma_G\in\pi(\Sigma(\widetilde{Z}))\cap\Gamma_G|\ H(\gamma_G)\leqslant T\}$ is contained in a union of at most $cT^{\varepsilon/2}$ blocks, [48, 3.6] also concludes that there exist an integer $J>0$ and J block families $B^j\subset\Sigma(\widetilde{Z})\times\mathbb{R}^l$ ($j=1,...,J$) such that each of the (at most) $cT^{\varepsilon/2}$ blocks, in particular all $B_G(T)$ for T large enough, is B^j_y for some j and $y\in\mathbb{R}^l$.

[6] Here we should modify a bit the fundamental set $\mathcal{F}$ chosen before, but this can be done by some easy operation. See the end of §3.3.1.

[7] A block is a connected definable set whose dimension equals the dimension of its closure in the $\mathbb{R}$-Zariski topology.

For any $T \gg 0$, let us look at $\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z})$. Because $B_G(T) = B^j_y$ for some j and $y \in \mathbb{R}^l$, $\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z})$ is the fiber of $(\pi \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma(\widetilde{Z}) \times \mathbb{R}^l)$ over $y \in \mathbb{R}^l$. But $(\pi \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma(\widetilde{Z}) \times \mathbb{R}^l)$ being a definable family over a subset of $\mathbb{R}^l$, the cell decomposition implies that there exists an integer $n_0 > 0$ such that each fiber of $(\pi \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma(\widetilde{Z}) \times \mathbb{R}^l)$, in particular $\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z})$, has at most n_0 connected component (see [69, 3.6]). On the other hand,

$$\begin{aligned}
\pi\big(\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z}) \cap \Gamma\big) &= B_G(T) \cap \pi\big(\Sigma(\widetilde{Z}) \cap \Gamma\big) \\
&= B_G(T) \cap \pi\big(\Sigma(\widetilde{Z})\big) \cap \Gamma_G \text{ (by the 3rd fact about } \widetilde{Z} \text{ listed above)} \\
&= B_G(T) \cap \Gamma_G \qquad (\text{since } B_G(T) \subset \pi\big(\Sigma(\widetilde{Z})\big)).
\end{aligned}$$

Hence there exists a connected component $B(T)$ of $\pi^{-1}(B_G(T)) \cap \Sigma(\widetilde{Z})$ such that

$$|\{\gamma_G \in \pi(B(T) \cap \Gamma) | \ H(\gamma_G) \leqslant T\}| \geqslant \frac{c'}{n_0} T^{\delta}.$$

But n_0 does not depend on T as explained above. So this B is what we desire.

Remark. *For the proof, the independence of n_0 on T is crucial. But the $B_G(T)$ we get from Pila-Wilkie depends on the choice of T and hence n_0 also depends on T a priori. This explains why the fact that all the $B_G(T)$ come from finitely many block families is crucial for the proof of the mixed case.*

Step 5 Prove that $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})\widetilde{z}$ for some $\widetilde{z} \in \widetilde{Z}$.

For the hyperbolic (i.e. pure) case, this follows from an easy induction argument.

For the mixed case, we should study more carefully the geometry. Here we should use the Ax-Lindemann theorem for the fiber (which for $\mathfrak{A}_g \to \mathcal{A}_g$ is an abelian variety) and some extra computation. This is done in Theorem 3.2.8(1). Remark that the complex structure of fibers of $\mathcal{X}^+_{2g,\mathrm{a}} \xrightarrow{\pi} \mathbb{H}^+_g$ is used in this step.

Remark. *For an arbitrary connected mixed Shimura variety S associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$, the fiber of $S \to S_G$, where S_G is its pure part, does not necessarily have a group structure compatible with the group law of P (see Lemma 2.1.1). In particular the Ax-Lindemann theorem for the fiber was not known before except some special cases. The proof of it, which will be given in §3.4, is again quite technical: one should repeat the argument from Step 4 to Step 6 (Step I to Step IV in §3.4), with a very different "Step 6" (which is Step IV in §3.4).*

Step 6 Prove that $H_{\widetilde{Z}} \lhd P$.

For the hyperbolic (i.e. pure) case, this follows from the structure of reductive groups. The facts $H_{\widetilde{Z}} \lhd N \lhd P$ and that P is reductive imply directly $H_{\widetilde{Z}} \lhd P$.

For the mixed case, it is obvious that this argument is no longer sufficient. In general, a normal subgroup of a normal subgroup of a given group is no longer normal. So apart from some group-theoretical argument (results of §1.1.4 will be used), we should also study carefully the geometry. See Theorem 3.2.8(2).

Here we just explain why $V_{H_{\widetilde{Z}}} := \mathcal{R}_u(H_{\widetilde{Z}}) = H_{\widetilde{Z}} \cap V_{2g}$ is normal in P. In order to do this, we should use the complex structure of the fibers of $\pi\colon \mathcal{X}^+_{2g,\mathrm{a}} \to \mathbb{H}^+_g$: let $\widetilde{z} \in \widetilde{Z}$ be any point such that $\pi(\widetilde{z})$ is Hodge generic in $\mathcal{X}^+_G$. Such a $\widetilde{z}$ exists since we have assumed that S is the smallest connected mixed Shimura variety containing $Y = \overline{\mathrm{unif}(\widetilde{Z})}^{\mathrm{Zar}}$. Therefore the Mumford-Tate group $\mathrm{MT}(\pi(\widetilde{z})) = G$. But $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})\widetilde{z}$ by *Step 5*, so the fiber of $\widetilde{Z}$ over $\pi(\widetilde{z})$ is

$$\widetilde{Z}_{\pi(\widetilde{z})} = V_{H_{\widetilde{Z}}}(\mathbb{R})\widetilde{z}.$$

Since $\widetilde{Z}$ is defined to be a complex analytic subset of $\mathcal{X}^+$ (and hence of $\mathcal{X}^+_{2g,\mathrm{a}}$), $V_{H_{\widetilde{Z}}}(\mathbb{R})$ is a complex subspace of $(\mathcal{X}^+_{2g,\mathrm{a}})_{\pi(\widetilde{z})} = V_{2g}(\mathbb{R})$. But the complex structure on $(\mathcal{X}^+_{2g,\mathrm{a}})_{\pi(\widetilde{z})}$ is given by the Hodge structure of type $\{(-1,0),(0,-1)\}$ on V_{2g} whose Mumford-Tate group is $\mathrm{MT}(\pi(\widetilde{z})) = G$. Hence $V_{H_{\widetilde{Z}}}$ is a G-module. Therefore $V_{H_{\widetilde{Z}}} \lhd P$ since $\mathcal{R}_u(P)$ is commutative.

Conclusion Now by the 6 steps above (especially the conclusions of *Step 5* and *Step 6*), $\mathrm{unif}(\widetilde{Z})$ is a weakly special subvariety of $\mathfrak{A}_g$. Since $Y = \overline{\mathrm{unif}(\widetilde{Z})}^{\mathrm{Zar}}$ by definition and $\mathrm{unif}(\widetilde{Z})$, being weakly special, is an algebraic subvariety of $\mathfrak{A}_g$, $Y = \mathrm{unif}(\widetilde{Z})$. But $Y = \overline{\mathrm{unif}(\widetilde{\mathcal{Z}})}^{\mathrm{Zar}}$ by definition, hence $\overline{\mathrm{unif}(\widetilde{\mathcal{Z}})}^{\mathrm{Zar}}$ is weakly special.

From Ax-Lindemann to André-Oort

A main motivation to study the Ax-Lindemann theorem is its application to the Zilber-Pink conjecture, and the André-Oort conjecture is the best-known subconjecture of Zilber-Pink. The conjecture says the follows:

Conjecture (André-Oort)**.** *Let S be a connected mixed Shimura variety and let Σ be the set of its special points. Let Y be an irreducible subvariety of S. If $\overline{Y \cap \Sigma}^{\mathrm{Zar}} = Y$, then Y is a connected mixed Shimura subvariety of S (or equivalently, Y is weakly special[8]).*

Example. *The special points of $\mathfrak{A}_g$ are precisely the points corresponding to torsion points of CM abelian varieties. Hence the André-Oort conjecture stated above contains part of the Manin-Mumford conjecture.*

This conjecture has been proved, under the generalized Riemann hypothesis, for all pure Shimura varieties by Klingler-Ullmo-Yafaev [66, 30]. Recent

[8] The equivalence follows from [54, Proposition 4.2, Proposition 4.15].

developments for this conjecture have been made in order to obtain proofs not relying on GRH since Pila's inspiring unconditional proof for $\mathcal{A}_1^N$ [48]. The framework of Pila's proof is the strategy proposed by Pila-Zannier:

1. Prove the Ax-Lindemann theorem;
2. Deduce from Ax-Lindemann the distribution[9] of positive-dimensional (weakly) special subvarieties of a given subvariety;
3. Define a good parameter (which we call the complexity) for points in Σ and choose a "good" fundamental set for the action of Γ on $\mathcal{X}^+$ such that $\mathrm{unif}_{\mathcal{F}}$ is definable in $\mathbb{R}_{an,\exp}$;
4. Prove an upper bound for the height of any point in $\mathrm{unif}^{-1}(\Sigma)\cap\mathcal{F}$ w.r.t. the complexity of its image in Σ;
5. Prove a lower bound for the size of the Galois orbits of points in Σ w.r.t. theirs complexities;
6. Conclude by the Ax-Lindemann theorem, the distribution theorem in (2), the upper bound in (4) and the lower bound in (5). This step follows immediately once we have proved all the previous steps.

The Ax-Lindemann theorem is proved in this dissertation in its most general form. The distribution theorem in (2) will also be proved as Theorem 4.1.3. Remark that this theorem for pure Shimura varieties has been obtained by Ullmo [64, Théorème 4.1] and, without "weakly", also by Pila-Tsimerman [50, Section 7] separately. The choice of $\mathcal{F}$ and the proof of the definability of $\mathrm{unif}|_{\mathcal{F}}$ in (3) are done in §3.3.1. The upper bound in (4) has been proved by Pila-Tsimerman [49, Theorem 3.1] for $\mathcal{A}_g$ and their result can be easily generalized to mixed Shimura varieties of abelian type. For the lower bound in (5), we will reduce the case of mixed Shimura varieties to the case of pure Shimura varieties in §4.2. Then for pure Shimura varieties, the best result is given by Tsimerman [62, Theorem 1.1] who proved it unconditionally for all special points of $\mathcal{A}_6^N$ and under GRH for all special points of $\mathcal{A}_g$[10]. Combining all these results, we have (Theorem 4.3.1)

Theorem. *The André-Oort conjecture holds unconditionally for any connected mixed Shimura variety S whose pure part is a subvariety of $\mathcal{A}_6^N$ (e.g. $\mathfrak{A}_6^N$). It holds under GRH for any connected mixed Shimura variety of abelian type.*

In order to prove the André-Oort conjecture for mixed Shimura varieties which are not of abelian type, we still need a good definition of the complexity

[9] In the sense of Theorem 4.1.3.

[10] The lower bound is conjected by Edixhoven [19], who also initiated the study of this lower bound and proved it unconditionaly for Hilbert modular surfaces [18]. Similar results to Tsimerman's for special points of $\mathcal{A}_3^N$ have been obtained unconditionally by Ullmo-Yafaev separately and they also proved the lower bound for all pure Shimura varieties under GRH [68].

for points in Σ which allows us to get the upper bound in (4). Remark that by the proof of Theorem 4.3.1, it is enough to define this complexity for all pure Shimura varieties. Daw-Orr are studying this problem.

From André-Oort to André-Pink-Zannier

The obstacle left to claim the André-Oort conjecture for $\mathfrak{A}_g$ ($g \geqslant 7$) is the lower bound of the size of Galois orbits of special points. we can consider a weaker version of André-Oort: replace Σ by the set of torsion points of CM abelian varieties **which are isogenious to a given CM abelian variety**. In this case, the obstacle is removed by a series of work of Habegger-Pila [24, Section 6] and Orr [43]. The key point to do this is a theorem of Masser-Wüstholz [35] and its effective version by Gaudron-Rémond [21].

This special case of André-Oort is contained in another conjecture, which we shall call the André-Pink-Zannier conjecture.

Conjecture (André-Pink-Zannier)**.** *Let S be a connected mixed Shimura variety, let s be a point of S and let Y be an irreducible subvariety of S. Let Σ be the generalized Hecke orbit of s. If $\overline{Y \cap \Sigma}^{\mathrm{Zar}} = Y$, then Y is weakly special.*

Several cases of this conjecture have already been studied by André before its final form was given by Pink [54, Conjecture 1.6]. It is also closely related to a problem proposed by Zannier. See §5.1.1 for more details. Pink also proved [54, Theorem 5.4] that this conjecture implies the Mordell-Lang conjecture.

The André-Pink-Zannier conjecture has been intensely studied by Orr [43, 42]. In this dissertation, we shall focus on $\mathfrak{A}_g$ for the André-Pink-Zannier conjecture. In this case the generalized Hecke orbit of s can be computed explicitly. We have (5.1.1)

$$\begin{aligned} \Sigma &= \text{division points of the polarized isogeny orbit of } s \\ &= \{t \in \mathfrak{A}_g \mid \exists n \in \mathbb{N} \text{ and a polarized isogeny} \\ &\qquad f\colon (\mathfrak{A}_{g,\pi(s)}, \lambda_{\pi(s)}) \to (\mathfrak{A}_{g,\pi(t)}, \lambda_{\pi(t)}) \text{ such that } nt = f(s)\}. \end{aligned}$$

Finally we prove (Theorem 4.3.2, Theorem 5.1.4 and Theorem 5.1.5)

Theorem. *The André-Pink-Zannier conjecture holds for $\mathfrak{A}_g$ and Y in each of the three following cases:*

1. *s is a torsion point of $\mathfrak{A}_{g,\pi(s)}$ and $\mathfrak{A}_{g,\pi(s)}$ is a CM abelian variety (this is a special case of the weak André-Oort conjecture we discussed before);*
2. *s is a torsion point on $\mathfrak{A}_{g,\pi(s)}$ and $\dim \pi(Y) \leqslant 1$;*
3. *$s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$ and $\dim(Y) = 1$.*

The first part of this theorem generalizes the previous work of Edixhoven-Yafaev [72, 20] (for curves in pure Shimura varieties) and Klingler-Ullmo-Yafaev [66, 30] (for pure Shimura varieties) and its p-adic version has been proved by Scanlon [58].

In the last part of this dissertation §5.5, we explain that the same statement as André-Pink-Zannier by replacing s by a finitely generated subgroup of a fiber of $\mathfrak{A}_g \to \mathcal{A}_g$ (which is an abelian variety) and replacing the polarized isogeny orbit by the isogeny orbit can be in fact deduced from the André-Pink-Zannier conjecture.

Zilber-Pink

Finally let us talk a bit about the more general Zilber-Pink Conjecture [54, 73, 57].

Conjecture (Zilber-Pink)**.** *Let S be a connected mixed Shimura variety. Let Y be a Hodge-generic irreducible subvariety of S. Then*

$$\bigcup_{\substack{S' \text{ special,} \\ \mathrm{codim}(S') > \dim(Y)}} S' \cap Y$$

is not Zariski dense in Y.

This conjecture contains the André-Oort conjecture and the André-Pink-Zannier conjecture (see [52, Theorem 3.3]). Habegger-Pila have proved recently many results about the Zilber-Pink conjecture for $\mathcal{A}_1^N$ [23] (in the same paper they also proved the Zilber-Pink conjecture for curves over $\overline{\mathbb{Q}}$ in abelian varieties), in particular an unconditional result for a large class of curves [24]. We will not talk more about the case of algebraic groups (see the Bourbaki talk of Chambert-Loir [14] for a summary before the work of Habegger-Pila).

For mixed Shimura varieties, there are not many results for this general conjecture. Apart from the results of this dissertation, Bertrand, Bertrand-Edixhoven, Bertrand-Pillay and Bertrand-Masser-Pillay-Zannier have recently been working on Poincaré biextensions [7, 11, 8, 9, 10]. They have got several interesting results, some of which provide good examples for this dissertation.

Structure of the dissertation

Chapter 1 introduces the preliminaries for this dissertation. Section §1.1 summarizes the theory of mixed Shimura varieties as they are used in this dissertation. In particular, §1.1.1 summarizes the theory of mixed Hodge structures and naturally leads to the definition of mixed Shimura varieties in §1.1.2. Other basic properties will also be given in §1.1.2. Section §1.1.3 introduces mixed Shimura varieties of Siegel type (in particular the universal family of abelian varieties) and ends up with the reduction lemma. All these subsections are well-known facts and the main reference is [53, Chapter 1-Chapter 3]. In §1.1.4 we prove a group theoretical proposition which will be used later in the dissertation. Section §1.2 summarizes basic properties of weakly special subvarieties and gives the geometric description of weakly special subvarieties of mixed Shimura varieties of Kuga type. Section §1.3 concerns the bi-algebraic setting for the mixed Shimura varieties.

Chapter 2 proves Ax's theorem of log type. Section §2.1 concerns results for the unipotent part, i.e. the fiber of the projection of a connected mixed Shimura variety to its pure part. §2.2 collects some existing results for monodromy groups of admissible variations of Hodge structures. After these preliminaries, Ax's theorem of log type will be proved in §2.3.

Chapter 3 proves the mixed Ax-Lindemann theorem. Section §3.1 gives four equivalent statements for this theorem. Section §3.2 outlines the proof and gives *Step 1*, *Step 2*, *Step 5* and *Step 6*. Section §3.3 deals with the estimate using the o-minimal theory. This corresponds to *Step 3* and *Step 4*. Section §3.4 handles the unipotent part, which answers a question left for *Step 5*. In the Appendix of this chapter we will do two things: §3.5.1 gives more details on an easy fact we admit about the definability in §3.3.1 and §3.5.2 sketches a simplified proof for the flat Ax-Lindemann theorem.

Chapter 4 concerns several different aspects to pass from Ax-Lindemann to André-Oort. Section §4.1 proves the distribution theorem as a consequence of the mixed Ax-Lindemann theorem. Section §4.2 reduces the lower bound for Galois orbits of special points of mixed Shimura varieties to lower bound for pure Shimura varieties. Combining these two results together with Ax-Lindemann and the upper bound studied by Pila-Tsimerman, we prove our main result for the André-Oort conjecture in §4.3. The proof for the weak version of André-Oort will also be given in §4.3. The Appendix of this chapter summarizes the comparison of Galois orbits of special points of pure Shimura varieties obtained by Ullmo-Yafaev [66, Section 2].

Chapter 5 concerns the André-Pink-Zannier conjecture. Section §5.1 discusses the background and states the main results. Section §5.2 computes the generalized Hecke orbits in $\mathfrak{A}_g$. Section §5.3 proves the torsion case and §5.4 proves the non-torsion case. Each proof contains the definition of the complexity of points in the generalized Hecke orbit, the upper bound for heights and the lower bound for Galois orbits. The estimates for both cases are slightly different. Section §5.5 discusses some variants of the André-Pink-Zannier conjecture.

Chapter 1

Preliminaries

1.1 Mixed Shimura varieties

1.1.1 Mixed Hodge structure

In this section we recall some background knowledge about rational mixed Hodge structures. Most of this section is taken from [53, Chapter 1].

1.1.1.1 Definitions about mixed Hodge structures

We start by collecting some basic notions about Hodge structures. This subsection is taken from [53, 1.1 and 1.2]. In this subsection, $R = \mathbb{Z}$ or $\mathbb{Q}$.

Let M be a free R-module of finite rank. A **pure Hodge structure of weight** $n \in \mathbb{Z}$ on M is a decomposition $M_{\mathbb{C}} = \oplus_{p+q=n} M^{p,q}$ into $\mathbb{C}$-vector spaces such that for all $p, q \in \mathbb{Z}$ with $p+q=n$ one has $\overline{M^{q,p}} = M^{p,q}$. The associated (descending) **Hodge filtration** on $M_{\mathbb{C}}$ is defined by $F^p M_{\mathbb{C}} := \oplus_{p' \geqslant p} M^{p',q}$. It determines the Hodge structure uniquely, because $M^{p,q} = F^p M_{\mathbb{C}} \cap \overline{F^q M_{\mathbb{C}}}$.

A **mixed R-Hodge structure** on M is a triple $(M, \{W_n M\}_{n\in\mathbb{Z}}, \{F^p M_{\mathbb{C}}\}_{p\in\mathbb{Z}})$ consisting of an ascending exhausting separated filtration $\{W_n M\}_{n\in\mathbb{Z}}$ of M by R-modules of finite rank with each $M/W_n M$ free, called **weight filtration**, together with a descending exhausting separated filtration $\{F^p M_{\mathbb{C}}\}_{p\in\mathbb{Z}}$ of $M_{\mathbb{C}}$, called **Hodge filtration**, such that for all $n \in \mathbb{Z}$ the Hodge filtration induces a pure Hodge structure of weight n on $\mathrm{Gr}_n^W M := W_n M / W_{n-1} M$. A pure Hodge structure of weight n is considered a special case of a mixed Hodge structure by defining the weight filtration as $W_{n'} M = M$ for $n' \geqslant n$ and $W_{n'} M = 0$ for $n' < n$.

The **Hodge numbers** are defined as $h^{p,q} := \dim_{\mathbb{C}}(\mathrm{Gr}_{p+q}^W M)^{p,q}$. They satisfy $h^{q,p} = h^{p,q}$, almost all $h^{p,q}$ are zero, and their sum is equal to the dimension of M. If $A \subset \mathbb{Z} \oplus \mathbb{Z}$ is an arbitrary subset, then we say that the Hodge structure $(M, \{W_n M\}_{n\in\mathbb{Z}}, \{F^p M_{\mathbb{C}}\}_{p\in\mathbb{Z}})$ is **of type** A, if $h^{p,q} = 0$ for all $(p,q) \notin A$. The weights that occur in a mixed Hodge structure are the numbers $p+q$ for all pairs (p,q), for which $h^{p,q} \neq 0$. The notions **of weight** $\leqslant n$ and **of weight** $\geqslant n$ are defined in the obvious way.

A **morphism of mixed R-Hodge structures** is a homomorphism $f\colon M \to M'$ such that $f(W_n M) \subset W_n M'$ and $f(F^p M_{\mathbb{C}}) \subset F^p M'_{\mathbb{C}}$ for all $n, p \in \mathbb{Z}$. The rational mixed Hodge structures form an abelian category with these morphisms. Given mixed R-Hodge structures on M_1 and M_2, there are canonical rational mixed Hodge structures on $M_1 \oplus M_2$, on the dual $M_1^\vee$ and on $\mathrm{Hom}(M_1, M_2)$.

A mixed Hodge structure on M **splits over** $\mathbb{R}$, if there exists a decomposition $M_{\mathbb{C}} = \oplus_{p,q} M^{p,q}$ such that $W_n M_{\mathbb{C}} = \oplus_{p+q\leqslant n} M^{p,q}$, $F^p M_{\mathbb{C}} = \oplus_{p'\geqslant p} M^{p',q}$ and $\overline{M^{q,p}} = M^{p,q}$. This decomposition is then uniquely determined by these properties. Every pure Hodge structure splits over $\mathbb{R}$, but not every mixed Hodge structure does. If one weakens the requirements, however, one can still associate to every mixed Hodge structure a canonical decomposition $M_{\mathbb{C}} = \oplus_{p,q} M^{p,q}$, as in the following proposition.

Proposition 1.1.1 (Deligne)**.** *Fix a mixed R-Hodge structure on M.*

1. *There exists a decomposition $M_{\mathbb{C}} = \oplus_{p,q} M^{p,q}$ such that $W_n M_{\mathbb{C}} = \oplus_{p+q\leqslant n} M^{p,q}$ and $F^p M_{\mathbb{C}} = \oplus_{p'\geqslant p} M^{p',q}$.*
2. *The Hodge structure is uniquely determined by any such decomposition.*
3. *There exists a unique decomposition as in (1) which also satisfies*

$$\overline{M^{q,p}} \equiv M^{p,q} \mod \oplus_{p'<p,q'<q} M^{p',q'}.$$

Proof. [53, 1.2]. □

1.1.1.2 Equivariant families of mixed Hodge structures

The reference for this subsection is [53, 1.3-1.7]. In this section, $R = \mathbb{Z}$ or $\mathbb{Q}$.

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$. The torus $\mathbb{S}$ is called the **Deligne-torus**. Over $\mathbb{C}$ it is canonically isomorphic to $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$, but the action of complex conjugation is twisted by the automorphism c that interchanges the two factors. In particular $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ corresponds to the points of the form $(z, \overline{z})$ with $z \in \mathbb{C}^*$. While the character group of $\mathbb{G}_{m,\mathbb{C}}$ is $\mathbb{Z}$ in the standard way, we identify the character group of $\mathbb{S}$ with $\mathbb{Z} \oplus \mathbb{Z}$ such that the character (p, q) maps $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ to $z^{-p}\overline{z}^{-q} \in \mathbb{C}^*$. Under this identification the complex conjugation operates on $\mathbb{Z} \oplus \mathbb{Z}$ by interchanging the two factors. The following homomorphisms are important in the theory:

- the weight $\omega\colon \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$ induced by $\mathbb{R}^* \subset \mathbb{C}^*$;
- $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ sending $z \in \mathbb{C}^* \mapsto (z, 1) \in \mathbb{C}^* \times \mathbb{C}^* = \mathbb{S}(\mathbb{C})$;
- the norm $N\colon \mathbb{S} \twoheadrightarrow \mathbb{G}_{m,\mathbb{R}}$ sending $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^* \mapsto z\overline{z} \in \mathbb{R}^*$. The kernel $\mathbb{S}^1$ of N is anisotropic over $\mathbb{R}$, and we have a short exact sequence $1 \to \mathbb{S}^1 \to \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}} \to 1$.

Let M be a free R-module of finite rank. The choice of a representation $k\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$ is equivalent to the choice of a decomposition $M_{\mathbb{C}} = \oplus_{p,q} M^{p,q}$, where $M^{p,q}$ is the eigenspace in $M_{\mathbb{C}}$ to the character (p, q). As in the last subsection we call $W_n M_{\mathbb{C}} = \oplus_{p+q\leqslant n} M^{p,q}$ and $F^p M_{\mathbb{C}} = \oplus_{p'\geqslant p} M^{p',q}$ the associated weight filtration, respectively Hodge filtration, and define the notions "of type A", pure, etc. in the same way. These notions coincide with those of the last subsection, if the filtrations are those of a mixed R-Hodge structure on M. The following two propositions will tell us under which condition on k this is the case for $R = \mathbb{Q}$.

Proposition 1.1.2. *Let P be a connected $\mathbb{Q}$-linear algebraic group. Let $W := \mathcal{R}_u(P)$ be its unipotent radical, let $G := P/W$ and let $\pi\colon P \to G$ be the quotient map. Let $h\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ be a homomorphism such that the following conditions holds:*

- *$\pi \circ h\colon \mathbb{S}_{\mathbb{C}} \to G_{\mathbb{C}}$ is defined over $\mathbb{R}$;*
- *$\pi \circ h \circ \omega\colon \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is a cocharacter of the center of G, which is defined over $\mathbb{Q}$;*
- *Under the weight filtration on $(\mathrm{Lie}\, P)_{\mathbb{C}}$ defined by $\mathrm{Ad}_P \circ h$ we have $W_{-1}(\mathrm{Lie}\, P) = \mathrm{Lie}\, W$.*

Then

1. *For every ($\mathbb{Q}$-)representation $\rho\colon P \to \mathrm{GL}(M)$, the homomorphism $\rho \circ h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$ induces a rational mixed Hodge structure on M.*
2. *The weight filtration on M is stable under P.*
3. *For any $p \in P(\mathbb{R})W(\mathbb{C})$, the assertions (1) and (2) also hold for $\mathrm{int}(p) \circ h$ in place of h. The weight filtration and the Hodge numbers in any representation are the same for $\mathrm{int}(p) \circ h$ and for h.*

Proof. [53, 1.4]. □

Proposition 1.1.3. *Let M be a finite dimensional $\mathbb{Q}$-vector space. A representation $k\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$ defines a rational mixed Hodge structure on M iff there exist a connected $\mathbb{Q}$-linear algebraic group P, a representation $\rho\colon P \to \mathrm{GL}(M)$ and a homomorphism $h\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ such that $k = \rho \circ h$ and the conditions in Proposition 1.1.2 are satisfied. Moreover, every rational mixed Hodge structure on M is obtained by a unique representation $k\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$ with the property above.*

Proof. This is [53, 1.5] except the "Moreover" part, where the existence of k has been explained in the paragraph before Proposition 1.1.2 and the uniqueness of k follows from Proposition 1.1.1(3). □

Now we are ready to discuss equivariant families of Hodge structures, or more precisely homogeneous spaces parametrizing certain rational mixed Hodge structures.

Proposition 1.1.4. *Let P be a $\mathbb{Q}$-linear algebraic group and let $W := \mathcal{R}_u(P)$ be its unipotent radical. Let $\mathcal{X}_W$ be a $P(\mathbb{R})W(\mathbb{C})$-conjugacy class in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$. Assume that for one (and hence for all by Proposition 1.1.2(3)) $h \in \mathcal{X}_W$, the* *conditions in Proposition 1.1.2 holds. Let M be any faithful representation of P and let φ be the obvious map*

$$\mathcal{X}_W \to \{\text{rational mixed Hodge structures on } M\}.$$

Then:

1. *There exists a unique structure on $\varphi(\mathcal{X}_W)$ as a complex manifold such that the Hodge filtration on $M_{\mathbb{C}}$ depends analytically on $\varphi(h) \in \varphi(\mathcal{X}_W)$. This structure is $P(\mathbb{R})W(\mathbb{C})$-invariant and $W(\mathbb{C})$ acts analytically on $\varphi(\mathcal{X}_W)$.*

2. *For any other representation M' of P the analogous map*

$$\varphi' \colon \mathcal{X}_W \to \{\text{rational mixed Hodge structures on } M'\}$$

   factors through $\varphi(\mathcal{X}_W)$. The Hodge filtration on M' varies analytically with $\varphi(h) \in \varphi(\mathcal{X}_W)$.

3. *If in addition M' is faithful, then $\varphi(\mathcal{X}_W)$ and $\varphi'(\mathcal{X}_W)$ are canonically isomorphic and the isomorphism is compatible with the complex structure.*

Proof. [53, 1.7]. □

1.1.1.3 Mumford-Tate group and polarizations

In this subsection, $R = \mathbb{Z}$ or $\mathbb{Q}$. Also M will be a free R-module of finite rank equipped with a mixed R-Hodge structure $(M, \{W_n M\}_{n\in\mathbb{Z}}, \{F^p M_{\mathbb{C}}\}_{p\in\mathbb{Z}})$. By Proposition 1.1.3, the corresponding rational mixed Hodge structure on $M_{\mathbb{Q}}$ gives rises to a representation $k \colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$.

Definition 1.1.5. *The **Mumford-Tate group** of this mixed R-Hodge structure is defined to be the smallest $\mathbb{Q}$-subgroup P of $\mathrm{GL}(M_{\mathbb{Q}})$ such that $k(\mathbb{S}_{\mathbb{C}}) \subset P_{\mathbb{C}}$.*

Before defining the polarizations of pure Hodge structures, we introduce the **Tate Hodge structure**, which is defined to be the free R-module of rank 1 $R(1) := 2\pi\sqrt{-1}R$ with the pure R-Hodge structure of type $(-1,-1)$. For every $n \in \mathbb{Z}$, we get a pure R-Hodge structure of type $(-n,-n)$ on $R(n) := R(1)^{\otimes n}$.

Definition 1.1.6. *Suppose that the R-Hodge structure on M is pure of weight n. A **polarization** of this Hodge structure is a homomorphism of Hodge structures*

$$Q \colon M \otimes M \to R(-n)$$

which is $(-1)^n$-symmetric and such that the real-valued symmetric bilinear form

$$Q'(u,v) := (2\pi\sqrt{-1})^n Q(Cu, v)$$

is positive-definite on $M_{\mathbb{R}}$, where C acts on $M^{p,q}$ by $C|_{M^{p,q}} = (\sqrt{-1})^{p-q}$.

1.1.1.4 Variation of mixed Hodge structures

The reference for this subsection is [53, 1.9-1.13]. In this subsection, $R = \mathbb{Z}$ or $\mathbb{Q}$.

Definition 1.1.7. *([45, Definition 14.44]) Let S be a complex manifold. A **variation of mixed R-Hodge structures over** S is a triple $(\mathbb{V}, W_\cdot, \mathcal{F}^\cdot)$ with*

1. *a local system $\mathbb{V}$ of free R-modules of finite rank on S;*
2. *a finite increasing filtration $\{W_m\}$ of the local system $\mathbb{V}$ by local sub-systems with torsion free $\mathrm{Gr}_n^W \mathbb{V}$ for each n (this is called the weight filtration);*
3. *a finite decreasing filtration $\{\mathcal{F}^p\}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V} \otimes_{R_S} \mathcal{O}_S$, where R_S is the constant sheaf over S, by holomorphic sub-bundles (this is called the Hodge filtration).*

such that

1. *for each $s \in S$, the filtrations $\{\mathcal{F}^p(s)\}$ and $\{W_m\}$ of $\mathbb{V}(s) \simeq \mathbb{V}_s \otimes_R \mathbb{C}$ define a mixed Hodge structure on the R-module of finite rank $\mathbb{V}_s$;*
2. *the connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega_S^1$ whose sheaf of horizontal sections is $\mathbb{V}_\mathbb{C}$ satisfies the Griffiths' transversality condition*

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_S^1.$$

Definition 1.1.8. *A variation of mixed Hodge structures over S is said to be **graded-polarizable** if the induced variations of pure Hodge structure $\mathrm{Gr}_n^W \mathbb{V}$ are all polarizable, i.e. for each n, there exists a flat morphism of variations*

$$Q_n \colon \mathrm{Gr}_n^W \mathbb{V} \otimes \mathrm{Gr}_n^W \mathbb{V} \to R(-n)_S$$

which induces on each fibre a polarization of the corresponding Hodge structure of weight n.

Proposition 1.1.9. *Let P, $\mathcal{X}_W$, M and φ be as in Proposition 1.1.4. Then we have a variation of rational mixed Hodge structures on M over $\varphi(\mathcal{X}_W)$ iff for one (and hence for all) $h \in \mathcal{X}_W$ the Hodge structure on $\mathrm{Lie}\, P$ is of type*

$$\{(-1,1),(0,0),(1,-1),(-1,0),(0,-1),(-1,-1)\}.$$

Proof. [53, 1.10]. ∎

Proposition 1.1.10. *Let P, $\mathcal{X}_W$, M and φ be as in Proposition 1.1.4. Assume*

- *for one (and hence all) $h \in \mathcal{X}_W$, the conjugation by $h \circ \pi(\sqrt{-1})$ induces a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$ where $G := P/W$ and G^{ad} possesses no $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is compact;*
- *$P/P^{\mathrm{der}} = Z(G)$ is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$;*
- *M is an irreducible representation of P and the Hodge structure on M induced by one (and hence all) $h \in \mathcal{X}_W$ is pure of weight n.*

Then there exist a one dimensional representation of P on $\mathbb{Q}(-n)$ and a P-equivariant bilinear form $\Psi\colon M \times M \to \mathbb{Q}(-n)$ such that for all $h \in \mathcal{X}_W$ either Ψ or $-\Psi$ is a polarization of the corresponding Hodge structure on M.

Proof. [53, 1.12 and 1.13]. □

1.1.1.5 Replace $\mathcal{X}_W$ by a smaller orbit

The reference for this subsection is [53, 1.15 and 1.16].

Let P, $\mathcal{X}_W$, M and φ be as in Proposition 1.1.4. The aim of this subsection is to find a subgroup U of W such that the image of an orbit under $P(\mathbb{R})U(\mathbb{C})$ under φ is the same as $\varphi(\mathcal{X}_W)$.

Let $U < W$ be the unique connected subgroup such that $\mathrm{Lie}\, U = W_{-2}(\mathrm{Lie}\, W)$. By Proposition 1.1.2(3), it does not depend on $h \in \mathcal{X}_W$. Let π' be the quotient $P \to P/U$.

Proposition 1.1.11. *Under the notation as above. Let*

$$\mathcal{X} := \{h \in \mathcal{X}_W \mid \pi' \circ h\colon \mathbb{S}_{\mathbb{C}} \to (P/U)_{\mathbb{C}} \text{ is defined over } \mathbb{R}\}.$$

Then

1. *$\mathcal{X}$ is a non-empty $P(\mathbb{R})U(\mathbb{C})$-orbit in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$;*
2. *$\varphi(\mathcal{X}) = \varphi(\mathcal{X}_W)$;*
3. *If $F^0(\mathrm{Lie}\, U)_{\mathbb{C}} = 0$, then $\varphi(\mathcal{X}) \simeq \mathcal{X}$.*

Proof. [53, 1.16]. □

1.1.2 Mixed Shimura data and mixed Shimura varieties

1.1.2.1 Definitions and basic properties

Definition 1.1.12. *A **mixed Shimura datum** $(P, \mathcal{X})$ is a pair where*

- *P is a connected linear algebraic group over $\mathbb{Q}$ with unipotent radical W and with another algebraic subgroup $U \subset W$ which is normal in P and uniquely determined by $\mathcal{X}$ using condition (3) below;*

- *$\mathcal{X}$ is a left homogeneous space under the subgroup $P(\mathbb{R})U(\mathbb{C}) \subset P(\mathbb{C})$, and $\mathcal{X} \xrightarrow{h} \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ is a $P(\mathbb{R})U(\mathbb{C})$-equivariant map such that every fibre of h consists of at most finitely many points,*

such that for some (equivalently for all) $x \in \mathcal{X}$,

1. *the composite homomorphism $\mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} P_{\mathbb{C}} \to (P/U)_{\mathbb{C}}$ is defined over $\mathbb{R}$,*

2. *the adjoint representation induces on $\mathrm{Lie}\, P$ a rational mixed Hodge structure of type*
$$\{(-1,1),(0,0),(1,-1)\} \cup \{(-1,0),(0,-1)\} \cup \{(-1,-1)\},$$

3. *the weight filtration on $\mathrm{Lie}\, P$ is given by*
$$W_n(\mathrm{Lie}\, P) = \begin{cases} 0 & \text{if } n < -2 \\ \mathrm{Lie}\, U & \text{if } n = -2 \\ \mathrm{Lie}\, W & \text{if } n = -1 \\ \mathrm{Lie}\, P & \text{if } n \geqslant 0 \end{cases},$$

4. *the conjugation by $h_x(\sqrt{-1})$ induces a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$ where $G := P/W$, and G^{ad} possesses no $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is compact,*

5. *$P/P^{\mathrm{der}} = Z(G)$ is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$.*

If in addition P is reductive (resp. U is trivial), then $(P, \mathcal{X})$ is called a ***pure Shimura datum*** *(resp. a* ***mixed Shimura datum of Kuga type****).*

Remark 1.1.13. 1. *Let $\omega : \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$ be $t \in \mathbb{R}^* \mapsto t \in \mathbb{C}^*$. Conditions (2) and (3) together imply that the composite homomorphism $\mathbb{G}_{m,\mathbb{C}} \xrightarrow{\omega} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} P_{\mathbb{C}} \to (P/U)_{\mathbb{C}}$ is a co-character of the center of P/W defined over $\mathbb{R}$. This map is called the weight. Furthermore, condition (5) implies that the weight is defined over $\mathbb{Q}$.*

2. *Condition (5) also implies that every sufficiently small congruence subgroup Γ of $P(\mathbb{Q})$ is contained in $P^{\mathrm{der}}(\mathbb{Q})$ (cf [53, the proof of 3.3(a)]). Fix a Levi decomposition $P = W \rtimes G$ ([55, Theorem 2.3]), then $P^{\mathrm{der}} = W \rtimes G^{\mathrm{der}}$, and hence for any congruence subgroup $\Gamma < P^{\mathrm{der}}(\mathbb{Q})$, Γ is Zariski dense in P^{der} by condition (4) ([55, Theorem 4.10]).*

3. *Condition (5) in the definition is not too strict because we are only interested in a connected component of $\mathcal{X}$ ([53, 1.29]).*

Theorem 1.1.14. *Let $(P, \mathcal{X})$ be a mixed Shimura datum. Then $\mathcal{X}$ possesses a canonical $P(\mathbb{R})U(\mathbb{C})$-invariant complex structure and every connected component of $\mathcal{X}$ is isomorphic to a holomorphic vector bundle on a hermitian symmetric domain.*

Proof. The existence of the complex structure follows from Proposition 1.1.4 and Proposition 1.1.11. We will give the construction of this complex structure at the beginning of §1.3.1.

The second claim is [53, 2.19]. □

Definition 1.1.15. *Let $(P, \mathcal{X})$ be a mixed Shimura datum and let K be an open compact subgroup of $P(\mathbb{A}_f)$ where $\mathbb{A}_f$ is the ring of finite adèle of $\mathbb{Q}$. The corresponding* ***mixed Shimura variety*** *is defined as*

$$M_K(P, \mathcal{X}) := P(\mathbb{Q})\backslash\mathcal{X} \times P(\mathbb{A}_f)/K,$$

where $P(\mathbb{Q})$ acts diagonally on both factors on the left and K acts on $P(\mathbb{A}_f)$ on the right. The mixed Shimura variety $M_K(P, \mathcal{X})$ is said to be ***pure (resp. of Kuga type)*** *if $(P, \mathcal{X})$ is pure (resp. of Kuga type).*

In this article, we only consider connected mixed Shimura data and connected mixed Shimura varieties except in §4.2.

Definition 1.1.16.

1. *A* ***connected mixed Shimura datum*** *is a pair $(P, \mathcal{X}^+)$, where P is as in Definition 1.1.12, $\mathcal{X}^+ \xhookrightarrow{h} \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ is an orbit under the subgroup $P(\mathbb{R})^+U(\mathbb{C}) \subset P(\mathbb{C})$ such that for one (and hence for all) $x \in \mathcal{X}^+$ the conditions (1)-(5) in Definition 1.1.12 are satisfied.*

2. *A* ***connected mixed Shimura variety*** *S associated with $(P, \mathcal{X}^+)$ is of the form $\Gamma\backslash\mathcal{X}^+$ for some congruence subgroup $\Gamma \subset P(\mathbb{Q})_+ := P(\mathbb{Q}) \cap P(\mathbb{R})_+$, where $P(\mathbb{R})_+$ is the stabilizer in $P(\mathbb{R})$ of $\mathcal{X}^+ \subset \mathrm{Hom}_{\mathbb{C}}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$.*

Mixed Shimura varieties and connected mixed Shimura varieties are closely related. Their relationship is summarized in the following proposition.

Proposition 1.1.17. *Let $(P, \mathcal{X})$ be a mixed Shimura datum and let K be an open compact subgroup of $P(\mathbb{A}_f)$. Let $\mathcal{X}^+$ be a connected component of $\mathcal{X}$.*

1. *The pair $(P, \mathcal{X}^+)$ is a connected mixed Shimura datum.*

2. *The set $P(\mathbb{Q})_+\backslash P(\mathbb{A}_f)/K$ is a finite set.*

3. *For any $p_f \in P(\mathbb{A}_f)$, $\Gamma(p_f) := P(\mathbb{Q})_+ \cap p_f K p_f^{-1}$ is a congruence subgroup of $P(\mathbb{Q})_+$ depending only on $[p_f] \in P(\mathbb{Q})_+\backslash P(\mathbb{A}_f)/K$ and K.*

4. $$M_K(P, \mathcal{X}) = \coprod_{[p_f] \in P(\mathbb{Q})_+\backslash P(\mathbb{A}_f)/K} \Gamma(p_f)\backslash\mathcal{X}^+.$$

Proof. [53, 3.2] and [55, Theorem 8.1]. □

This proposition allows us to consider only connected mixed Shimura data and connected mixed Shimura varieties in this dissertation. One advantage of doing this is because of the notion which we introduce now: recall the following definition, which Pink calls "irreducible" in [53, 2.13].

Definition 1.1.18. *A connected mixed Shimura datum $(P, \mathcal{X}^+)$ is said to **have generic Mumford-Tate group** if P possesses no proper normal subgroup P' such that for one (equivalently all) $x \in \mathcal{X}^+$, h_x factors through $P'_{\mathbb{C}} \subset P_{\mathbb{C}}$. We shall denote this case by $P = \mathrm{MT}(\mathcal{X}^+)$. (This terminology will be explained in Remark 2.2.6).*

Proposition 1.1.19. *Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum, then*

1. *there exists a connected mixed Shimura datum $(P', \mathcal{X}'^+) \hookrightarrow (P, \mathcal{X}^+)$ such that $P' = \mathrm{MT}(\mathcal{X}'^+)$ and $\mathcal{X}'^+ = \mathcal{X}^+$;*

2. *if $(P, \mathcal{X}^+)$ has generic Mumford-Tate group, then P acts on U via a character. In particular, any subgroup of U is normal in P.*

Proof. [53, 2.13, 2.14]. □

Definition 1.1.20. *A **(Shimura) morphism of connected mixed Shimura data** $(Q, \mathcal{Y}^+) \to (P, \mathcal{X}^+)$ is a homomorphism $\varphi \colon Q \to P$ of algebraic groups over $\mathbb{Q}$ which induces a map $\mathcal{Y}^+ \to \mathcal{X}^+$, $y \mapsto \varphi \circ y$. A **Shimura morphism of connected mixed Shimura varieties** is a morphism of varieties induced by a Shimura morphism of connected mixed Shimura data.*

A very important result of the theory of Shimura varieties is that the category of connected mixed Shimura varieties is a subcategory of the category of algebraic varieties. More precisely,

Theorem 1.1.21.
1. *Let S be a connected mixed Shimura variety associated with $(P, \mathcal{X}^+)$ and let $\mathrm{unif} \colon \mathcal{X}^+ \to S = \Gamma \backslash \mathcal{X}^+$ be the uniformization. Then there is a canonical structure of a normal complex quasi-projective algebraic variety on S (the complex structure comes from the $P(\mathbb{R})^+ U(\mathbb{C})$-invariant complex structure of $\mathcal{X}^+$ given in Theorem 1.1.14). Moreover if Γ is neat, then S is smooth.*

2. *Every Shimura morphism between connected mixed Shimura varieties is algebraic.*

Proof. [53, 3.3 and 9.24]. □

1.1.2.2 Construction of new mixed Shimura data from a given one

Given a (connected) mixed Shimura datum $(P, \mathcal{X})$, we define in this section its quotient mixed Shimura data and its unipotent extensions.

Proposition 1.1.22 (Quotient mixed Shimura datum)**.** *Let $(P, \mathcal{X})$ be a mixed Shimura datum and let P_0 be a normal subgroup of P. Then there exist a quotient mixed Shimura datum $(P, \mathcal{X})/P_0$ and a morphism $(P, \mathcal{X}) \to (P, \mathcal{X})/P_0$, unique up to isomorphism, such that every Shimura morphism $(P, \mathcal{X}) \to (P', \mathcal{X}')$, where the homomorphism $P \to P'$ factors through P/P_0, factors in a unique way through $(P, \mathcal{X})/P_0$. In fact the underlying group for $(P, \mathcal{X})/P_0$ is P/P_0.*

Proof. This is [53, 2.9] except the "In fact" part, which is clear by the proof. □

Proposition 1.1.23 (Unipotent extension of a mixed Shimura datum)**.** *Let $(P,\mathcal{X})$ be a mixed Shimura datum and let $1 \to W_0 \to P_1 \to P \to 1$ be an extension of P by a unipotent group W_0. Let $G := P/\mathcal{R}_u(P)$. Assume that the Lie algebra of every irreducible subquotient of* $\mathrm{Lie}\, W_0$ *is of Hodge type $\{(-1,0),(0,-1),(-1,-1)\}$ as representation of G, and that the center of G acts on it through a torus that is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$. Then:*

1. *There exist a mixed Shimura datum $(P_1,\mathcal{X}_1)$ and a morphism $(P_1,\mathcal{X}_1) \to (P,\mathcal{X})$ that extends the given homomorphism $P_1 \to P$, with the property $(P_1,\mathcal{X}_1)/W_0 \simeq (P,\mathcal{X})$. They are uniquely determined up to isomorphism.*
2. *For every morphism $(P',\mathcal{X}') \to (P,\mathcal{X})$ and every factorization $P' \to P_1 \to P$, there exists exactly one extension $(P',\mathcal{X}') \to (P_1,\mathcal{X}_1) \to (P,\mathcal{X})$.*

Proof. This is [53, 2.17]. □

Example 1.1.24. *Let us see a particular example of the unipotent extensions of a given connected mixed Shimura datum. This is [54, Construction 2.9].*

Let $(P,\mathcal{X}^+)$ be a connected mixed Shimura datum and let V' be a finite dimensional representation of P. Then we can define the $\mathbb{Q}$-linear algebraic group $V' \rtimes P$. Assume that for one (and hence for all) $x \in \mathcal{X}^+$, the induced rational mixed Hodge structure on V' has type $\{(-1,0),(0,-1)\}$. Let

$$V'(\mathbb{R}) \rtimes \mathcal{X}^+ \subset \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, (V' \rtimes P)_{\mathbb{C}})$$

denote the conjugacy class under $V'(\mathbb{R}) \rtimes (P(\mathbb{R})^+U(\mathbb{C})) = (V' \rtimes P)(\mathbb{R})^+U(\mathbb{C})$ generated by $\mathcal{X}^+ \subset \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$. There is a natural bijection

$$V'(\mathbb{R}) \times \mathcal{X}^+ \xrightarrow{\sim} V'(\mathbb{R}) \rtimes \mathcal{X}^+, \qquad (v',x) \mapsto \mathrm{int}(v') \circ x.$$

Under this bejection the action of $(v,p) \in V'(\mathbb{R}) \rtimes (P(\mathbb{R})^+U(\mathbb{C}))$ corresponds to the twisted action $(v,p)\cdot(v',x) = (pv'+v, px)$. The complex structure of the fiber over $x \in \mathcal{X}^+$ of the projection

$$V'(\mathbb{R}) \rtimes \mathcal{X}^+ \to \mathcal{X}^+$$

is given by $V'(\mathbb{R}) \simeq V'(\mathbb{C})/F_x^0 V'(\mathbb{C})$.

The pair $(V' \rtimes P, V'(\mathbb{R}) \rtimes \mathcal{X}^+)$ is the extension of $(P,\mathcal{X}^+)$ by V'.

Notation 1.1.25. *For convenience, we fix some notation here. Given a connected mixed Shimura datum $(P,\mathcal{X}^+)$, we always denote by $W = \mathcal{R}_u(P)$ the unipotent radical of P, $G := P/W$ the reductive part, $U \lhd P$ the weight -2 part, $V := W/U$ the weight -1 part and $(P/U, \mathcal{X}^+_{P/U}) := (P,\mathcal{X}^+)/U$ (resp. $(G,\mathcal{X}^+_G) := (P,\mathcal{X}^+)/W$) the corresponding connected mixed Shimura datum* *whose weight* -2 *part is trivial (resp. pure Shimura datum). If we have several connected mixed Shimura data, say* $(P, \mathcal{X}^+)$ *and* $(Q, \mathcal{Y}^+)$*, then we distinguish the different parts associated with them by adding subscript* W_P, W_Q, G_P, G_Q, *etc. For a connected mixed Shimura variety* S*, we denote by* $S_{P/U}$ *(resp.* S_G*) its image under the Shimura morphism induced by* $(P, \mathcal{X}^+) \to (P/U, \mathcal{X}^+_{P/U})$ *(resp.* $(P, \mathcal{X}^+) \to (G, \mathcal{X}^+_G)$*). The pure Shimura datum* $(G, \mathcal{X}^+_G)$ *will be called the **pure part of*** $(P, \mathcal{X}^+)$ *and* S_G *will be called the **pure part of*** S.

1.1.2.3 Examples of Shimura morphisms

In this subsection, we discuss some Shimura morphisms. The first corresponds to families of abelian varieties. Then we define Shimura immersions, Shimura submersions and Shimura coverings.

Proposition 1.1.26. *Let* $S = \Gamma\backslash\mathcal{X}^+$ *be a connected mixed Shimura variety of Kuga type associated with* $(P, \mathcal{X}^+)$ *and let* S_G *be its pure part. Assume that* $\Gamma = \Gamma_V \rtimes \Gamma_G$ *and that* Γ_G *is neat. Then* $S \to S_G$ *is an abelian scheme.*

Proof. [53, 3.12(a) and 3.22(a)]. □

Proposition 1.1.27. *Let* $\varphi\colon (P, \mathcal{X}^+) \to (P', \mathcal{X}'^+)$ *be a Shimura morphism and let* $\Gamma \subset P(\mathbb{Q})_+$ *and* $\Gamma' \subset P'(\mathbb{Q})_+$ *be congruence subgroups such that* $\varphi(\Gamma) \subset \Gamma'$*. Then the map*

$$[\varphi]\colon \Gamma\backslash\mathcal{X}^+ \to \Gamma'\backslash\mathcal{X}'^+, \qquad [x] \mapsto [\varphi \circ x]$$

is well-defined and algebraic. Moreover, $[\varphi]$ *is*

1. *a finite morphism if* $\mathrm{Ker}(\varphi)^\circ$ *is a torus. In this case* $[\varphi]$ *is called a **Shimura immersion**.*
2. *surjective if Im*(φ) *contains* P'^{der}*. In this case* $[\varphi]$ *is called a **Shimura submersion**.*
3. *a (possibly ramified) covering if the conditions in (1) and (2) both hold. In this case* $[\varphi]$ *is called a **Shimura covering**.*

Proof. [53, 3.4 and 9.24]. □

At the end of this subsection, we state the following property for Shimura morphisms.

Proposition 1.1.28. *Let* $(Q, \mathcal{Y}) \xrightarrow{f} (P, \mathcal{X})$ *be a Shimura morphism, then* $f(W_Q) \subset W_P$ *(resp.* $f(U_Q) \subset f(U_P)$*), and hence* f *induces*

$$\overline{f} : (G_Q, \mathcal{Y}_{G_Q}) \to (G_P, \mathcal{X}_{G_P}) \qquad \textit{(resp. } \overline{f}' : (Q/Q_U, \mathcal{Y}_{Q/U_Q}) \to (P/U_P, \mathcal{X}_{P/U_P})).$$

Furthermore, if the underlying homomorphism of algebraic groups f *is injective, then so are* $\overline{f}$ *and* $\overline{f}'$.

Proof. Since

$$\mathrm{Lie}\,W_P = W_{-1}(\mathrm{Lie}\,P) \qquad \text{and} \qquad \mathrm{Lie}\,W_Q = W_{-1}(\mathrm{Lie}\,Q),$$

by the following commutative diagram

$$\begin{array}{ccc} \mathrm{Lie}\,W_Q & \longrightarrow & \mathrm{Lie}\,W_P \\ \wr\big\downarrow \exp & & \big\downarrow \exp \\ W_Q & \xrightarrow{\;f\;} & P \end{array}$$

(here exp is algebraic and is an isomorphism as a morphism between algebraic varieties because W_Q is unipotent), $f(W_Q) \subset W_P$.

Hence f induces a map $G_Q \to G_P$. Now the existence of $\overline{f}$ follows from the universal property of the quotient Shimura datum (Proposition 1.1.22).

Furthermore, suppose now that f is injective. By Levi decomposition, the exact sequence

$$1 \to W_Q \to Q \xrightarrow{\pi_Q} G_Q \to 1$$

splits. Choose a splitting $s_Q : G_Q \to Q$, then we have the following diagram whose solid arrows commute:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W_Q & \longrightarrow & Q & \underset{\pi_Q}{\overset{s_Q}{\rightleftarrows}} & G_Q & \longrightarrow & 1 \\ & & \big\downarrow & & f\big\downarrow & \overset{\lambda}{\swarrow} & \big\downarrow\overline{f} & & \\ 1 & \longrightarrow & W_P & \longrightarrow & P & \underset{\pi_P}{\longrightarrow} & G_P & \longrightarrow & 1 \end{array},$$

where $\lambda := f \circ s_Q$. Then λ is injective since f, s_Q are. And $\pi_P \circ \lambda = \pi_P \circ f \circ s_Q = \overline{f} \circ \pi_Q \circ s_Q = \overline{f}$, so we have

$$\mathrm{Ker}(\overline{f}) = G_Q \cap W_P$$

where the intersection is taken in P. $(G_Q \cap W_P)^\circ$ is smooth (since we are in the characteristic 0), connected unipotent (since it is in W_P) and normal in G_Q (since W_P is normal in P), so it is trivial since G_Q is reductive. So $G_Q \cap W_P$ is finite, hence trivial because W_P is unipotent over $\mathbb{Q}$. To sum it up, $\overline{f}$ is injective.

The proof for the statements with U's is similar. □

1.1.2.4 Generalized Hecke orbits

The reference for this subsection is [54, Section 3]. Let $S = \Gamma\backslash\mathcal{X}^+$ be a connected mixed Shimura variety associated with $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S$ be the uniformization.

Definition 1.1.29. *1. For any $\varphi \in \mathrm{Aut}\big((P, \mathcal{X}^+)\big)$, the diagram of Shimura coverings*

$$S = \Gamma\backslash\mathcal{X}^+ \xleftarrow{[\mathrm{id}]} \big(\Gamma \cap \varphi^{-1}(\Gamma)\big)\backslash\mathcal{X}^+ \xrightarrow{[\varphi]} \Gamma\backslash\mathcal{X}^+ = S$$

*is called a **generalized Hecke correspondence** on S and is denoted by T_φ. For any subset $Z \subset S$, the subset*

$$T_\varphi(Z) := [\varphi]\left([\mathrm{id}]^{-1}(Z)\right)$$

*is called the **translate of Z under T_φ**. We also abbreviate $T_\varphi(s) := T_\varphi(\{s\})$.*

2. *The generalized Hecke correspondence associated with an inner automorphism $\mathrm{int}(p)\colon p' \mapsto pp'p^{-1}$ for an element $p \in P(\mathbb{Q})_+$ is called a **(usual) Hecke correspondence** on S and is denoted by T_p.*

Definition 1.1.30. *Fix a point $s \in S$.*

1. *The union of $T_\varphi(s)$ for all $\varphi \in \mathrm{Aut}\left((P, \mathcal{X}^+)\right)$ is called the **generalized Hecke orbit of** s.*

2. *The union of $T_p(s)$ for all $p \in P(\mathbb{Q})_+$ is called the **(usual) Hecke orbit of** s.*

The following proposition, whose proof we omit, is very easy to check by definition.

Proposition 1.1.31. *Let s be a point of S. Let $\widetilde{s} \in \mathcal{X}^+$ be such that $\mathrm{unif}(\widetilde{s}) = s$. Then the generalized Hecke orbit of s equals*

$$\mathrm{unif}\Big(\mathrm{Aut}\left((P, \mathcal{X}^+)\right) \cdot \widetilde{s}\Big).$$

The generalized Hecke orbits in a particular connected mixed Shimura variety (the universal family of principally polarized abelian varieties) will be computed in the last chapter of this dissertation (5.1.1).

1.1.2.5 Structure of the underlying group

The reference for this subsection is [53, 2.15].

For a given connected mixed Shimura datum $(P, \mathcal{X}^+)$, we can associate to P a 4-tuple (G, V, U, Ψ) which is defined as follows:

- $G := P/\mathcal{R}_u(P)$ is the reductive part of P;

- U is the normal subgroup of P as in Definition 1.1.12 and $V := \mathcal{R}_u(P)/U$. Both of them are vector groups with an action of G induced by conjugation in P (which factors through G for reason of weight);

- The commutator on $W := \mathcal{R}_u(P)$ induces a G-equivariant alternating form $\Psi\colon V \times V \to U$ by reason of weight as explained by Pink in [53, 2.15]. Moreover, Ψ is given by a polynomial with coefficients in $\mathbb{Q}$.

On the other hand, P is uniquely determined up to isomorphism by this 4-tuple in the following sense:

- let W be the central extension of V by U defined by Ψ. More concretely, $W = U \times V$ as a $\mathbb{Q}$-variety and the group law on W is (this can be proved using the Baker-Campbell-Hausdorff formula)

$$(u, v)(u', v') = (u + u' + \frac{1}{2}\Psi(v, v'), v + v');$$

- define the action of G on W by $g((u, v)) := (gu, gv)$;
- define $P := W \rtimes G$.

1.1.3 Mixed Shimura varieties of Siegel type and the reduction lemma

The reference for this subsection is [53, 2.7, 2.25, 10.1-10.14].

Let $g \in \mathbb{N}_{>0}$. Let V_{2g} be a $\mathbb{Q}$-vector space of dimension $2g$ and let

$$\Psi \colon V_{2g} \times V_{2g} \to U_{2g} := \mathbb{G}_{a,\mathbb{Q}}$$

be a non-degenerate alternating form. Define

$$\mathrm{GSp}_{2g} := \{h \in \mathrm{GL}(V_{2g}) | \Psi(hv, hv') = \nu(h)\Psi(v, v') \text{ with } \nu(h) \in \mathbb{G}_m\},$$

and $\mathbb{H}_g$ the set of all homomorphisms

$$\mathbb{S} \to \mathrm{GSp}_{2g,\mathbb{R}}$$

which induce a pure Hodge structure of type $\{(-1, 0), (0, -1)\}$ on V_{2g} and for which either Ψ or $-\Psi$ defines a polarization. Let $\mathbb{H}_g^+$ be the set of all such homomorphisms such that Ψ defines a polarization.

GSp_{2g} acts on U_{2g} by the scalar ν, which induces a pure Hodge structure of type $(-1, -1)$ on U_{2g}. Let W_{2g} be the central extension of V_{2g} by U_{2g} defined by Ψ, then the action of GSp_{2g} on W_{2g} induces a Hodge structure of type $\{(-1, 0), (0, -1), (-1, -1)\}$ on $\mathrm{Lie}\, W_{2g}$.

By Proposition 1.1.23, there are connected mixed Shimura data $(P_{2g,\mathrm{a}}, \mathcal{X}_{2g,\mathrm{a}}^+)$ and $(P_{2g}, \mathcal{X}_{2g}^+)$, where $P_{2g,\mathrm{a}} := V_{2g} \rtimes \mathrm{GSp}_{2g}$ and $P_{2g} := W_{2g} \rtimes \mathrm{GSp}_{2g}$.

Definition 1.1.32. *The connected mixed Shimura data* $(\mathrm{GSp}_{2g}, \mathbb{H}_g^+)$, $(P_{2g,\mathrm{a}}, \mathcal{X}_{2g,\mathrm{a}}^+)$ *and* $(P_{2g}, \mathcal{X}_{2g}^+)$ *are called* ***of Siegel type*** *(of genus g).*

Next we introduce connected mixed Shimura varieties of Siegel type. They have very good modular interpretation ([53, 10.8-10.14]).

For $M \geqslant 4$ and even, define

$$\Gamma_{\mathrm{GSp}}(M) := \{h \in \mathrm{GSp}_{2g}(\mathbb{Z}) | h \equiv 1 \bmod M\} \tag{1.1.1}$$

and

$$\Gamma_W(M) := (M \cdot U_{2g}(\mathbb{Z})) \times (M \cdot V_{2g}(\mathbb{Z}))$$

under the identification $W \simeq U \times V$ in §1.1.2.5. $\Gamma_W(M)$ is indeed a subgroup of $W(\mathbb{Z})$ by the group operation (defined by Ψ). Let $\Gamma_V(M) := M \cdot V_{2g}(\mathbb{Z})$, and write

$$\mathcal{A}_g(M) := \Gamma_{\mathrm{GSp}}(M)\backslash\mathbb{H}_g^+ \tag{1.1.2}$$

$$\mathfrak{A}_g(M) := (\Gamma_V(M) \rtimes \Gamma_{\mathrm{GSp}}(M))\backslash\mathcal{X}_{2g,\mathrm{a}}^+ \tag{1.1.3}$$

$$\mathfrak{L}_g(M) := (\Gamma_W(M) \rtimes \Gamma_{\mathrm{GSp}}(M))\backslash\mathcal{X}_{2g}^+, \tag{1.1.4}$$

Definition 1.1.33. *The connected mixed Shimura varieties $\mathcal{A}_g(M)$, $\mathfrak{A}_g(M)$ and $\mathfrak{L}_g(M)$ are called of Siegel type of level M (and of genus g).*

Connected mixed Shimura varieties of Siegel type have very good moduli interpretation:

Theorem 1.1.34.
1. *$\mathfrak{A}_g(M)$ is the universal family of principally polarized abelian varieties of dimension g with a level-M-structure over the fine moduli space $\mathcal{A}_g(M)$.*
2. *$\mathfrak{L}_g(M) \to \mathfrak{A}_g(M)$ is a $\mathbb{G}_m$-torsor which is totally symmetric. Its inverse $\mathbb{G}_m$-torsor, i.e. replace the $\mathbb{G}_m$-action by its inverse, is relatively ample w.r.t. $\mathfrak{A}_g(M) \to \mathcal{A}_g(M)$. From now on, we replace the $\mathbb{G}_m$-torsor $\mathfrak{L}_g(M) \to \mathfrak{A}_g(M)$ by its inverse, but hence as a variety the "new" $\mathfrak{L}_g(M)$ is still equal to the "old" one.*
3. *Any point $a \in \mathcal{A}_g(M)$ represents the principally polarized abelian variety $(\mathfrak{A}_g(M)_a, \mathfrak{L}_g(M)_a)$ with some level-M-structure.*
4. *The varieties $\mathfrak{L}_g(M)$, $\mathfrak{A}_g(M)$ and $\mathcal{A}_g(M)$ are all canonically defined over $\overline{\mathbb{Q}}$.*
5. *$\mathfrak{A}_g(M) \to \mathcal{A}_g(M)$ can be compactified over $\overline{\mathbb{Q}}$ to smooth varieties $\overline{\mathfrak{A}_g(M)} \to \overline{\mathcal{A}_g(M)}$ such that any multiplication $[n]\colon \mathfrak{A}_g(M) \to \mathcal{A}_g(M)$ extends to the compactification.*
6. *$\mathfrak{L}_g(M)$ extends to an ample $\mathbb{G}_m$-torsor $\overline{\mathfrak{L}_g(M)} \to \overline{\mathfrak{A}_g(M)}$ over $\overline{\mathbb{Q}}$.*

Proof. See [53, 10.5, 10.9, 10.10, 11.16] for the first four assertions. For (5) see [53, 6.25, 9.24, 12.4]. For (6) see [53, 8.6, 8.13, 9.13, 9.16, 12.4]. □

Denote by $\mathrm{GSp}_0 := \mathbb{G}_m$ and $P_0 := \mathbb{G}_a \rtimes \mathbb{G}_m$ with the standard action of $\mathbb{G}_m$ on $\mathbb{G}_a$. Pink proved the following lemma ([53, 2.26])

Lemma 1.1.35 (Reduction Lemma)**.** *Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum with generic Mumford-Tate group.*

1. *If V is trivial, then there exist a connected pure Shimura datum $(G_0, \mathcal{D}^+)$ and an embedding*

$$(P, \mathcal{X}^+) \hookrightarrow (G_0, \mathcal{D}^+) \times \prod_{i=1}^{r} (P_0, \mathcal{X}_0^+)$$

where $r = \dim(U)$ *(see [53, 2.8, 2.14] for definition of* $(P_0, \mathcal{X}_0^+)$*);*

2. *If* V *is not trivial, then there exist a connected pure Shimura datum* $(G_0, \mathcal{D}^+)$ *and Shimura morphisms*

$$(P', \mathcal{X}'^+) \twoheadrightarrow (P, \mathcal{X}^+)$$

$$\text{and } (P', \mathcal{X}'^+) \xhookrightarrow{\lambda} (G_0, \mathcal{D}^+) \times \prod_{i=1}^{r} (P_{2g}, \mathcal{X}_{2g}^+)$$

such that $\mathrm{Ker}(P' \to P)$ *is of dimension 1 and of weight -2. Moreover* $\lambda|_V : V \simeq V_{2g} \to \oplus_{i=1}^r V_{2g}$ *is the diagonal map,* $\lambda|_{U'} : U' \simeq \oplus_{i=1}^r U_{2g}$ *and* $G \xrightarrow{\lambda|_G} G_0 \times \prod_{i=1}^r \mathrm{GSp}_{2g} \to \mathrm{GSp}_{2g}$ *is non-trivial for each projection.*

Proof. The statement except the last claim of the "Moreover" part is [53, 2.26 statement & pp 45]. For the last part, call $p_i : G \to \mathrm{GSp}_{2g}$ the composite with the i-th projection. If p_i is trivial, then $p_i(P', \mathcal{X}'^+)$ is trivial since a connected mixed Shimura datum is trivial if its pure part is trivial. This contradicts the dimension of V. $\square$

1.1.4 A group theoretical proposition

Proposition 1.1.36. *Let* $1 \to N \to Q \xrightarrow{\varphi} Q' \to 1$ *be an exact sequence of algebraic groups over* $\mathbb{Q}$*. Then the following diagram with solid arrows is commutative and all the lines and columns are exact:*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & W_N := \mathcal{R}_u(N) & \longrightarrow & N & \underset{\pi_N}{\overset{s_N}{\rightleftarrows}} & G_N := N/W_N & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & W_Q := \mathcal{R}_u(Q) & \longrightarrow & Q & \underset{\pi_Q}{\overset{s_Q}{\rightleftarrows}} & G_Q := Q/W_Q & \longrightarrow & 1. \\
 & & \downarrow & & \downarrow \varphi & & \downarrow \overline{\varphi} & & \\
1 & \longrightarrow & W_{Q'} := \mathcal{R}_u(Q') & \longrightarrow & Q' & \underset{\pi_{Q'}}{\overset{s_{Q'}}{\rightleftarrows}} & G_{Q'} := Q'/W_{Q'} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Moreover, if we fix a morphism s_Q *which splits the middle line (such an* s_Q *exists by Levi decomposition), then we can deduce* s_N *and* $s_{Q'}$ *which split the other two lines. Note that in this case, the action of* G_N *on* $W_{Q'}$ *induced by* s_Q *is trivial.*

Proof. The two bottom lines are already exact. By group theory, we know $\varphi(W_Q(\overline{\mathbb{Q}})) = W_{Q'}(\overline{\mathbb{Q}})$ ([13, Corollary 14.11]), and since the set of closed points of W_Q (resp. $W_{Q'}$) is dense on W_Q (resp. $W_{Q'}$), we have $\varphi(W_Q) = W_{Q'}$. In consequence, we have the map $\overline{\varphi}$, which is surjective since φ is. Now we get the solid diagram with exact lines and columns but with W_N replaced by $N \cap W_Q$ and G_N replaced by $N/(N \cap W_Q)$. But $N/(N \cap W_Q)$, being normal in G_Q, is reductive ([13, 14.2 Corollary(b)]). Hence $N \cap W_Q = \mathcal{R}_u(N) = W_N$ and we get the desired solid diagram.

If we have an s_Q, then to get a desired $s_{Q'}$ (and s_N) is equivalent to prove that $\varphi \circ s_Q(G_N)$ is trivial, i.e. the intersection of this image with $W_{Q'}$ (in Q') is trivial and the projection of this image to $G_{Q'}$ (under $\pi_{Q'}$) is trivial. The projection is trivial by a simple diagram-chasing. The neutral component of the intersection is trivial since it is reductive and unipotent, and hence the intersection is trivial since $W_{Q'}$ is unipotent over $\mathbb{Q}$. Now the triviality of the action of G_N on $W_{Q'}$ induced by s_Q is automatic. □

Corollary 1.1.37. *Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum. Suppose $N \lhd P$. Then there are decompositions*

$$V = V_N \oplus V_N^{\perp} \qquad \textit{(resp. } U = U_N \oplus U_N^{\perp}\textit{)}$$

as G-modules, where $V_N := V \cap N$ (resp. $U_N := U \cap N$), such that the action of $G_N := N/V_N$ on $V_N^{\perp}$ (resp. $U_N^{\perp}$) is trivial.

Proof. To prove the decomposition of V, apply Proposition 1.1.36 to the exact sequence

$$1 \to V_N \rtimes G_N \to V \rtimes G \to (V/V_N) \rtimes (G/G_N) \to 1,$$

then since G is reductive, the vertical line on the left (in the diagram of the proposition) splits. The conjugation by P on V factors through G by reason of weights, and hence equals to the action of G on V induced by any Levi decomposition s_P. So the action of G_N on $V_N^{\perp}$ is trivial by the last assertion of Proposition 1.1.36.

To prove the decomposition of U, it suffices to apply Proposition 1.1.36 to the exact sequence

$$1 \to U_N \rtimes G_N \to U \rtimes G \to (U/U_N) \rtimes (G/G_N) \to 1.$$

□

In fact we have a better result if $(P, \mathcal{X}^+)$ is with generic Mumford-Tate group.

Proposition 1.1.38. *Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum such that $P = \mathrm{MT}(\mathcal{X}^+)$. Suppose $N \lhd P$ such that N possesses no non-trivial torus quotient. Then G_N acts trivially on U.*

Proof. By Reduction Lemma (Lemma 1.1.35), we may assume that $(P, \mathcal{X}^+) \hookrightarrow (G_0, \mathcal{D}^+) \times \prod_{i=1}^r (P_{2g}, \mathcal{X}_{2g}^+)$ $(g \geqslant 0)$. Since N possesses no non-trivial torus quotient, G_N is semi-simple (the last line of the proof of Proposition 1.2.4). So

$$G_N = G_N^{\mathrm{der}} < G^{\mathrm{der}} < (G_0 \times \prod_{i=1}^r \mathrm{GSp}_{2g})^{\mathrm{der}} = G_0^{\mathrm{der}} \times \prod_{i=1}^r \mathrm{Sp}_{2g}$$

where $\mathrm{Sp}_0 := 1$. Hence G_N acts trivially on U since $G_0^{\mathrm{der}} \times \prod_{i=1}^r \mathrm{Sp}_{2g}$ acts trivially on $\oplus_{i=1}^r U_{2g}$. □

1.2 Weakly special subvarieties

1.2.1 Definition and basic properties

Definition 1.2.1. *(Pink, [54, Definition 4.1(b)]) Let S be a connected mixed Shimura variety. Consider any Shimura morphisms $T' \xleftarrow{[\varphi]} T \xrightarrow{[i]} S$ and any point $t' \in T'$. Then any irreducible component of $[i]([\varphi]^{-1}(t'))$ is called a* ***weakly special subvariety*** *of S. We will prove later in Remark 1.2.5 that weakly special subvarieties of S are indeed closed subvarieties.*

Since any Shimura morphism is related to a Shimura morphism between Shimura data, we will try to rephrase this definition in the context of Shimura data:

Definition 1.2.2. *Given a connected mixed Shimura datum $(P, \mathcal{X}^+)$, a* ***weakly special subset*** *of $\mathcal{X}^+$ is a connected component of $i(\varphi^{-1}(y')) \subset \mathcal{X}^+$ for a point $y' \in \mathcal{Y}'^+$, where i, φ, $\mathcal{Y}'^+$ are in the following diagram of Shimura morphisms*

$$\begin{array}{ccc} & (Q, \mathcal{Y}^+) & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle i} \\ (Q', \mathcal{Y}'^+) & & (P, \mathcal{X}^+) \end{array}.$$

Remark 1.2.3. 1. *In the definition above, let $N := \mathrm{Ker}(Q \to Q')$ and let $U_N := U_Q \cap N$, then $i(\varphi^{-1}(y'))$ is a connected component of $N(\mathbb{R})U_N(\mathbb{C})y$ where $\varphi(y) = y'$. So $i(\varphi^{-1}(y'))$ is smooth as an analytic variety. In particular, its connected components and complex analytic irreducible components coincide. As a result, we can replace "a connected component" by "a complex analytic irreducible component" in Definition 1.2.2.*

2. *If furthermore N is connected, then $i(\varphi^{-1}(y'))$ itself is connected (hence also complex analytic irreducible). The proof is as follows: Consider the image of $\varphi^{-1}(y')$ under the projection $(Q, \mathcal{Y}^+) \xrightarrow{\pi} (G_Q, \mathcal{Y}_{G_Q}^+) := (Q, \mathcal{Y}^+)/W_Q$. By the decomposition ([39, 3.6])*

$$(G_Q^{\mathrm{ad}}, \mathcal{Y}_{G_Q}^+) = (G_N^{\mathrm{ad}}, \mathcal{Y}_1^+) \times (G_2, \mathcal{Y}_2^+)$$

where $G_N := N/W\cap N$*, we have* $\pi(\varphi^{-1}(y')) = \mathcal{Y}_1^+ \times \{y_2\}$*. So* $\pi(\varphi^{-1}(y')) = G_N(\mathbb{R})^+\pi(y)$*. But* $W_N(\mathbb{R})U_N(\mathbb{C})$ *(*$W_N := W \cap N$*) is connected, hence* $\varphi^{-1}(y') = N(\mathbb{R})^+U_N(\mathbb{C})y$*, which is connected. In consequence,* $i(\varphi^{-1}(y'))$ *also is connected.*

Proposition 1.2.4. *For any weakly special subvariety of* S *(resp. weakly special subset of* $\mathcal{X}^+$*), the Shimura morphisms in Definition 1.2.1 (resp. Definition 1.2.2) can be chosen such that*

- *the underlying homomorphism of algebraic groups* i *is injective, and hence* i *is an embedding in the sense of [53, 2.3];*
- *the underlying homomorphism of algebraic groups* φ *is surjective, and its kernel* N *is connected. Moreover,* N *possesses no non-trivial torus quotient (or equivalently,* $G_N := N/(W \cap N)$ *is semi-simple);*
- φ *is a quotient Shimura morphism.*

Proof. If $P = \mathrm{MT}(\mathcal{X}^+)$, then the first two points except the statement in the bracket are proved by [54, Proposition 4.4]. The general cases follow directly from Proposition 1.1.19(1). The third assertion can be proved by the universal property of quotient Shimura data given in Proposition 1.1.22. Now we are left to prove the statement in the bracket.

$G_N \lhd G$ since $G_N = N/(W \cap N) \hookrightarrow G = P/W$ and $N \lhd P$, and hence G_N is reductive ([13, 14.2, Corollary(b)]). By [13, 14.2 Proposition(2)], G_N is the almost-product of G_N^{der} and $Z(G_N)^\circ$, and $Z(G_N)^\circ$ equals the radical of G_N which is a torus. So N possesses no non-trivial torus quotient iff G_N possesses no non-trivial torus quotient iff G_N is semi-simple. □

Remark 1.2.5. *We can now prove that weakly special subvarieties of* S *are closed. By the proposition above, we can choose* i *to be injective. Then* $[i]$ *is finite by Proposition 1.1.27(1). Hence* $[i]([\varphi]^{-1}(t'))$ *is closed.*

Lemma 1.2.6. *Suppose that the Shimura morphisms* $T' \xleftarrow{[\varphi]} T \xrightarrow{[i]} S$ *are associated with the morphisms of mixed Shimura data*

$$(Q', \mathcal{Y}'^+) \xleftarrow{\varphi} (Q, \mathcal{Y}^+) \xrightarrow{i} (P, \mathcal{X}^+)$$

so that we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{Y}'^+ & \xleftarrow{\quad\varphi\quad} & \mathcal{Y}^+ & \xrightarrow{\quad i \quad} & \mathcal{X}^+ \\ \downarrow{\scriptstyle \mathrm{unif}_{\mathcal{Y}'^+}} & & \downarrow{\scriptstyle \mathrm{unif}_{\mathcal{Y}^+}} & & \downarrow{\scriptstyle \mathrm{unif}_{\mathcal{X}^+}} \\ T' = \Delta'\backslash\mathcal{Y}'^+ & \xleftarrow{[\varphi]} & T = \Delta\backslash\mathcal{Y}^+ & \xrightarrow{[i]} & S = \Gamma\backslash\mathcal{X}^+ \end{array},$$

then for any point $y' \in \mathcal{Y}'^+$*, any irreducible component of* $\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ *is also an irreducible component of* $[i]([\varphi]^{-1}(\mathrm{unif}_{\mathcal{Y}'^+}(y')))$*.*

Proof. Let $N := \mathrm{Ker}(\varphi)$ and let U_Q be the weight -2 part of Q, then we have

$$\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y'))) \subset [i]([\varphi]^{-1}(\mathrm{unif}_{\mathcal{Y}'^+}(y'))),$$

and both of them are of constant dimension d, where d is the dimension of any orbit of $N(\mathbb{R})^+(U_Q \cap N)(\mathbb{C})$. This allows us to conclude. □

The following Proposition tells us that the two definitions of weak specialness are compatible.

Proposition 1.2.7. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ be the uniformization. Then a subvariety Z of S is weakly special if and only if Z is the image of some weakly special subset of $\mathcal{X}^+$.*

Proof. The "if" part is immediate by Lemma 1.2.6. We prove the "only if" part. We assume that i, φ are as in Proposition 1.2.4. For any weakly special subvariety $Z \subset S$, suppose that we have a diagram as in Lemma 1.2.6 and that Z is an irreducible component of $[i]([\varphi]^{-1}(t'))$. Since

$$[i]([\varphi]^{-1}(t')) \subset \bigcup_{y' \in \mathrm{unif}_{\mathcal{Y}^+}^{-1}(t')} \mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y'))) = \mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(\mathrm{unif}_{\mathcal{Y}^+}^{-1}(t')))),$$

there exists a $y' \in \mathcal{Y}'^+$ lying over t' such that Z is an irreducible component of $\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ by Lemma 1.2.6. By Remark 1.2.3.2, $i(\varphi^{-1}(y'))$ is complex analytic irreducible, so $\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ is also complex analytic irreducible when S is regarded as an analytic variety. Hence $\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ is irreducible as an algebraic variety. So $Z = \mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$. □

Next we come to special subvarieties of connected mixed Shimura varieties.

Definition 1.2.8. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$.*

1. *A **special subvariety** of S is the image of any Shimura morphism $T \to S$ of connected mixed Shimura varieties;*
2. *A point $x \in \mathcal{X}^+$ and its image in S are called **special** if the homomorphism $x : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $T_{\mathbb{C}}$ for a torus $T \subset P$.*

Remark 1.2.9. *By definition, $x \in \mathcal{X}^+$ is special if and only if it is the image of a Shimura morphism $(T, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$. Hence a special point is just a special subvariety of dimension* 0*.*

The following result is easy to prove. It tells us that special subvarieties of S are precisely connected mixed Shimura subvarieties of S.

Lemma 1.2.10. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S$ be the uniformizing map, then a subvariety of S is special if and only if it is of the form $\mathrm{unif}(\mathcal{Y}^+)$ for some $(Q, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$.*

Proposition 1.2.11. *Every special subvariety of S contains a Zariski dense subset of special points.*

Proof. [54, Proposition 4.14]. □

The relation between special and weakly special subvarieties is:

Proposition 1.2.12. *A subvariety of S is special if and only if it is weakly special and contains a special point.*

Proof. [54, Proposition 4.2, Proposition 4.15]. □

We close this section by proving that this definition of weakly special subvarieties is compatible with the one (which is already known) for pure Shimura varieties.

Proposition 1.2.13. *A weakly special subvariety of a pure Shimura variety S is a subvariety of the same form as in [65, Definition 2.1].*

Proof. This is pointed out in [54, Remark 4.5]. We give a (relatively) detailed proof here. We prove the result for weakly special subsets. Assume that S is associated with the connected pure Shimura datum $(P,\mathcal{X}^+)$. For a subset of the same form as in [65, Definition 2.1], take $(Q,\mathcal{Y}^+)=(H,X_H^+)$ and $(Q',\mathcal{Y}'^+)=(H_1,X_1^+)$ (same notation as [65, Definition 2.1]). Then by definition such a subset is weakly special (as in Definition 1.2.2).

On the other hand, suppose that we have a weakly special subset $\widetilde{F}$ defined by a diagram as in Definition 1.2.2 satisfying Proposition 1.2.4. Let $N:=\mathrm{Ker}(\varphi)$, then the homogeneous spaces of the connected pure Shimura data $(Q',\mathcal{Y}'^+)=(Q,\mathcal{Y}^+)/N$ and $(Q,\mathcal{Y}^+)/Z(Q)N=(Q^{\mathrm{ad}},\mathcal{Y}^{\mathrm{ad}+})/N^{\mathrm{ad}}$ are canonically isomorphic to each other ([38, Proposition 5.7]). Hence we may replace $(Q',\mathcal{Y}'^+)$ by $(Q^{\mathrm{ad}},\mathcal{Y}^{\mathrm{ad}+})/N^{\mathrm{ad}}$. But by [39, 3.6, 3.7], $(Q^{\mathrm{ad}},\mathcal{Y}^{\mathrm{ad}+})=(N^{\mathrm{ad}},\mathcal{Y}_1^+)\times(Q_2,\mathcal{Y}_2^+)$. So $\widetilde{F}$ is of the same form as in [65, Definition 2.1]. □

1.2.2 Weakly special subvarieties in Kuga varieties

In this section, we consider only connected mixed Shimura varieties of Kuga type. Through the whole section, $S=\Gamma\backslash\mathcal{X}^+$ will be a connected mixed Shimura variety of Kuga type which is associated with the connected mixed Shimura datum $(P,\mathcal{X}^+)$ with $\Gamma=\Gamma_V\rtimes\Gamma_G$ neat. Then $W_{-2}(P)$ is trivial by definition. Denote by $V=\mathcal{R}_u(P)$ and

$$\begin{array}{ccc} (P,\mathcal{X}^+) & \xrightarrow{\pi} & (G,\mathcal{X}_G^+):=(P,\mathcal{X}^+)/V \\ {\scriptstyle \mathrm{unif}}\downarrow & & \downarrow{\scriptstyle \mathrm{unif}_{\mathcal{X}_G^+}} \\ S & \xrightarrow{[\pi]} & S_G \end{array}.$$

By Example 1.1.24, there is a natural bijection $V(\mathbb{R}) \times \mathcal{X}_G^+ \simeq \mathcal{X}^+$. By Proposition 1.1.26, $S \xrightarrow{[\pi]} S_G$ is a family of abelian varieties. Let $[\varepsilon]\colon S_G \to S$ be the zero-section of $[\pi]$. Then $[\varepsilon]$ corresponds to $\varepsilon\colon (G, \mathcal{X}_G^+) \hookrightarrow (P, \mathcal{X}^+)$. The Shimura morphism ε is a section of π and determines a Levi-decomposition of $P = V \rtimes^{\varepsilon} G$. A particular example is $\mathfrak{A}_g \to \mathcal{A}_g$, where ε is the natural inclusion $\mathrm{GSp}_{2g} = \{0\} \times \mathrm{GSp}_{2g} < V_{2g} \rtimes \mathrm{GSp}_{2g} = P_{2g,\mathrm{a}}$.

The goal of this section is to prove the following proposition:

Proposition 1.2.14. *Let B be an irreducible subvariety of S_G and $X := [\pi]^{-1}(B)$. Define $\mathcal{C}$ to be the isotrivial part of $X \to B$, i.e. the largest isotrivial abelian subscheme of X over B. Then*

$$\{\textit{translates of abelian subscheme of } X \to B \textit{ by a torsion section and then by a constant section of } \mathcal{C} \to B\} = \{X \cap E |\ E \textit{ weakly special in } S\}.$$

Let us define constant sections of $\mathcal{C} \to B$. By definition of isotriviality, there exists a finite cover $B' \to B$ such that $\mathcal{C} \times_B B' \simeq \mathcal{C}_{b_0} \times B'$ for any $b_0 \in B$. A **constant section of** $\mathcal{C} \to B$ is then defined to be the image of the graph of a constant morphism $B' \to \mathcal{C}_{b_0}$ in $\mathcal{C} \times_B B'$ under the projection $\mathcal{C} \times_B B' \to \mathcal{C}$.

Proposition 1.2.14 has the following corollary, which describes weakly special subvarieties of connected mixed Shimura varieties of Kuga type in geometric terms.

Corollary 1.2.15. *An irreducible subvariety Y of S is weakly special iff the followings hold:*

1. *$[\pi]Y$ is a totally geodesic subvariety of S_G;*
2. *Y is the translate of an abelian subscheme of $[\pi]^{-1}([\pi]Y)$ (over $[\pi]Y$) by a torsion section and then by a constant section of the isotrivial part of $[\pi]^{-1}[\pi]Y \to [\pi]Y$.*

Proof. This follows directly from [39, 4.3] and Proposition 1.2.14. □

We start from the following proposition which is not hard to prove using Levi decomposition [55, Theorem 2.3]. Another proof can be found in [33, Section 5.1].

Proposition 1.2.16. *To give a Shimura subdatum $(Q, \mathcal{Y}^+)$ of $(P, \mathcal{X}^+)$ is equivalent to give:*

- *a pure Shimura subdatum $(G_Q, \mathcal{Y}_{G_Q}^+)$ of $(G, \mathcal{X}_G^+)$;*
- *a G_Q-submodule V_Q of V (V is a G-module, and therefore a G_Q-module);*
- *an element $\overline{v}_0 \in (V/V_Q)(\mathbb{Q})$.*

Proof. We only give the constructions here.

1. Given $(Q, \mathcal{Y}^+) \subset (P, \mathcal{X}^+)$, we have $V_Q := \mathcal{R}_u(Q) < \mathcal{R}_u(P) = V$. Therefore the inclusion $(Q, \mathcal{Y}^+) \subset (P, \mathcal{X}^+)$ induces

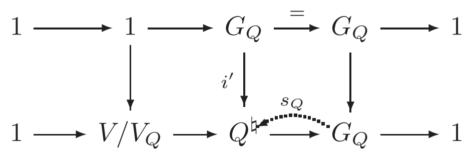
$$(G_Q, \mathcal{Y}^+_{G_Q}) := (Q, \mathcal{Y}^+)/V_Q \subset (G, \mathcal{X}^+_G) = (P, \mathcal{X}^+)/V.$$

The fact that V_Q is a G_Q-submodule of V is clear. Now it suffices to find $\overline{v}_0 \in (V/V_Q)(\mathbb{Q})$.

Consider the group $Q^\natural := (V/V_Q) \rtimes G_Q$, where the action is induced by the natural one of G_Q on V. By definition, $Q^\natural = \pi^{-1}(G_Q)/V_Q$. Now the inclusion $(Q, \mathcal{Y}^+) \subset (P, \mathcal{X}^+)$ induces another inclusion (which we call i')

$$G_Q = Q/V_Q \subset \pi^{-1}(G_Q)/V_Q = Q^\natural.$$

We have the following diagram, whose solide arrows commute:



where s_Q is the homomorphism $G_Q = \{0\} \rtimes G_Q < (V/V_Q) \rtimes G_Q = Q^\natural$. Now i' and s_Q are two Levi-decompositions for $Q^\natural$. By [55, Theorem 2.3], s_Q equals the conjugation of i' by an element $\overline{v}_0 \in (V/V_Q)(\mathbb{Q})$. Moreover, the choice of $\overline{v}_0$ is unique.

2. Conversely, given the three data as in the Proposition, the underlying group Q is the conjugate of $V_Q \rtimes G_Q < V \rtimes G$ (compatible Levi-decompositions) by $(v_0, 1)$ in P. The space

$$\mathcal{Y}^+ = \big(v_0 + V_Q(\mathbb{R})\big) \times \mathcal{Y}^+_{G_Q} \subset V(\mathbb{R}) \times \mathcal{X}^+_G \simeq \mathcal{X}^+$$

where v_0 is any lift of $\overline{v}_0$ to $V(\mathbb{Q})$.

□

Proposition 1.2.17. *A subvariety Y of S is weakly special iff there exist*

- *a pure Shimura subdatum $(G_Q, \mathcal{Y}^+_{G_Q})$ of $(G, \mathcal{X}^+_G)$;*
- *a point $v_0 \in V(\mathbb{Q})$;*
- *a normal semi-simple connected subgroup G_N of G_Q and a point $\widetilde{y}_G \in \mathcal{Y}^+_{G_Q}$;*
- *a G_Q-submodule V_N of V;*
- *a G_Q-submodule $V_N^\perp$ of V on which G_N acts trivially, and a point $v \in V_N^\perp(\mathbb{R})$*

such that

$$Y = \mathrm{unif}\Big(\big(v_0 + v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+\widetilde{y}_G\Big).$$

Here $\big(v_0 + v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+\widetilde{y}_G \subset V(\mathbb{R}) \times \mathcal{X}_G^+ \simeq \mathcal{X}^+$.

Proof. 1. Given a weakly special subvariety Y of S, let $(Q, \mathcal{Y}^+)$, N and $\widetilde{y}$ be as in Definition 1.2.2 and Proposition 1.2.4. By Proposition 1.2.16, $(Q, \mathcal{Y}^+)$ corresponds to a Shimura subdatum $(G_Q, \mathcal{Y}_{G_Q}^+)$ of $(G, \mathcal{X}_G^+)$, a G_Q-submodule V_Q of V and a point $\overline{v}_0 \in (V/V_Q)(\mathbb{Q})$. Let v_0 be any lift of $\overline{v}_0$ to $V(\mathbb{Q})$. Let $G_N := N/(V_Q \cap N)$, then G_N is a connected nomral subgroup of G_Q, and hence is reductive. Since N possesses no non-trivial torus quotient, G_N is semi-simple. Let $\widetilde{y}_G := \pi(\widetilde{y})$.

Let $V_N := V_Q \cap N$, then V_N is a G_Q-submodule of V_Q since N is normal in Q. By Corollary 1.1.37, there exists a G_Q-submodule $V_N^\perp$ of V_Q such that $V_Q = V_N \oplus V_N^\perp$ and G_N acts trivially on $V_N^\perp$. Write $\widetilde{y} = (\widetilde{y}_V, \widetilde{y}_G) \in (v_0 + V_Q(\mathbb{R})) \times \mathcal{Y}_{G_Q}^+ = \mathcal{Y}^+ \subset \mathcal{X}^+$ (here we use the second part of the proof of Proposition 1.2.16).

To simplify the computation below, we introduce a new Shimura subdatum $(Q', \mathcal{Y}')$ of $(P, \mathcal{X}^+)$: $(Q', \mathcal{Y}')$ is defined to be the conjugate of $(Q, \mathcal{Y}^+)$ by $(-v_0, 1)$. By the second part of the proof of Proposition 1.2.16, $(Q', \mathcal{Y}') = (V_Q \rtimes G_Q, V_Q(\mathbb{R}) \times \mathcal{Y}_{G_Q}^+) \subset (V \rtimes \mathrm{GSp}_{2g}, \mathcal{X}^+)$. Let $N' := V_N \rtimes G_N < V \rtimes \mathrm{GSp}_{2g}$, then N' is the conjugate of N by $(-v_0, 1)$. Let $\widetilde{y}' := (\widetilde{y}_V - v_0, \widetilde{y}_G) \in \mathcal{Y}'^+$.

Let v be the $V_N^\perp(\mathbb{R})$-factor of $\widetilde{y}_V - v_0$ under $V_Q = V_N \oplus V_N^\perp$. Then since G_N acts trivially on $V_N^\perp$, we have

$$N'(\mathbb{R})^+\widetilde{y}' = \big(v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+\widetilde{y}_G \subset \mathcal{Y}'^+.$$

Hence $N(\mathbb{R})^+\widetilde{y} = \big(v_0 + v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+\widetilde{y}_G$. Now the conclusion follows.

2. Conversely given all these data, let the Shimura subdatum $(Q, \mathcal{Y}^+)$ be the one obtained from $(G_Q, \mathcal{Y}_{G_Q}^+)$, $V_N \oplus V_N^\perp$ and v_0 by Proposition 1.2.16. Let N be the subgroup of Q which is defined to be $V_N \rtimes G_N$ conjugated by $(v_0, 1)$ in P. Then since G_N acts trivially on $V_N^\perp$, we have $N \lhd Q$. Let $\widetilde{y} := (v_0 + v, \widetilde{y}_G)$. Now we have

$$\big(v_0 + v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+\widetilde{y}_G = N(\mathbb{R})^+\widetilde{y}.$$

The group N is by definition connected and it possesses no non-trivial torus quotient since G_N is semi-simple. Hence Y is weakly special by definition.

□

Now we can prove Proposition 1.2.14:

Proof of Proposition 1.2.14. 1. Prove "$\supset$". For this it suffices to prove:

For any weakly special subvariety Y of S, Y is the translate of an abelian subscheme of $[\pi]^{-1}([\pi]Y)$ (over $[\pi]Y$) by a torsion section and then by a constant section of the isotrivial part of $[\pi]^{-1}[\pi]Y \to [\pi]Y$.

Let Y be a weakly special subvariety of S. Then associated to Y there are data as in Proposition 1.2.17 and

$$Y = \mathrm{unif}\left(\left(v_0 + v + V_N(\mathbb{R})\right) \times G_N(\mathbb{R})^+\widetilde{y}_G\right).$$

Let $B' := [\pi]Y$ and $X' := [\pi]^{-1}(B')$.

Now $X' \to B'$ is an abelian scheme. Since V_N is a G_Q-submodule of V, $\mathrm{unif}\left(V_N(\mathbb{R}) \times G_N(\mathbb{R})^+\widetilde{y}_G\right)$ is an abelian subscheme of X' over B'. Therefore,

$$\mathrm{unif}\left(\left(v_0 + V_N(\mathbb{R})\right) \times G_N(\mathbb{R})^+\widetilde{y}_G\right)$$

is the translate of B' by a torsion section of $X' \to B'$. But $v \in V_N^{\perp}(\mathbb{R})$ and G_N acts trivially on $V_N^{\perp}$, so $\mathrm{unif}\left(V_N^{\perp}(\mathbb{R}) \times G_N(\mathbb{R})^+\widetilde{y}_G\right)$ is an isotrivial abelian scheme over B'. Therefore Y is the translate of an abelian subscheme of $X' \to B'$ by a torsion section and then by a constant section of the isotrivial part of $X' \to B'$.

2. Prove "$\subset$". Let Y be a subvariety of X such that Y is the translate of an abelian subscheme of $X \to B$ translated by a torsion section and then by a section of $\mathcal{C} \to B$, where $\mathcal{C} \to B$ is the isotrivial part of $X \to B$. Let us find a weakly special subvariety E of S associated with the data in Proposition 1.2.17 such that $Y = E \cap X$.

   Let B' be the smallest weakly special subvariety of S_G containing B. Then by definition there exist a Shimura subdatum $(G_Q, \mathcal{Y}_{G_Q}^+)$, a connected semi-simple normal subgroup G_N of G_Q and a point $\widetilde{y}_G \in \mathcal{Y}_{G_Q}^+$ such that $B' = \mathrm{unif}_G\left(G_N(\mathbb{R})^+\widetilde{y}_G\right)$. Moreover by [39, 3.6, 3.7], G_N can be taken to be the connected algebraic monodromy group of $(B')^{\mathrm{sm}}$, i.e. the neutral component of the Zariski closure of $\Gamma_{B'^{\mathrm{sm}}}$:=the image of $\pi_1((B')^{\mathrm{sm}}) \to \pi_1(S_G) = \Gamma_G$.

   Let $X' := [\pi]^{-1}(B')$. Then the isotrivial part $\mathcal{C}'$ of $X' \to B'$ is

$$\mathrm{unif}\left(V'(\mathbb{R}) \times G_N(\mathbb{R})^+\widetilde{y}_G\right),$$

   where V' is the largest G_Q-submodule of V on which G_N acts trivially. This V' is the $V_N^{\perp}$ we want in Proposition 1.2.17.

   A key step is to prove that as subvarieties of S, we have

$$\mathcal{C} = \mathcal{C}' \cap X \tag{1.2.1}$$

   It is clear that $\mathcal{C}' \cap X \subset \mathcal{C}$. For the other inclusion, suppose that $\mathcal{C}$ is defined by the G_Q-submodule V'' of V (i.e. $\mathcal{C} = \mathrm{unif}(V''(\mathbb{R}) \times \widetilde{B})$ for

$\widetilde{B} := \mathrm{unif}_G^{-1}(B))$, then $\Gamma_{B'^{\mathrm{sm}}}$ acts trivially on V''. However the action of G on V is algebraic, therefore $\overline{\Gamma_{B'^{\mathrm{sm}}}}^{\mathrm{Zar}}$ acts trivially on V''. So G_N acts trivially on V''. By the maximality of V', $V'' \subset V'$. So $\mathcal{C} \subset \mathcal{C}'$. Now (1.2.1) follows.

Now since Y is the translate of an abelian subscheme by a torsion section and then by a constant section of $\mathcal{C} \to B$, there exists, by (1.2.1), a G_Q-submodule V_N of V such that

$$Y = \mathrm{unif}\left(\left(v_0 + v + V_N(\mathbb{R})\right) \times \widetilde{B}\right)$$

where $v_0 \in V(\mathbb{Q})$ corresponds to the torsion section and $v \in V'(\mathbb{R})$ corresponds to the constant section of $\mathcal{C} \to B$. In other words,

$$Y = E \cap X, \text{ where } E = \mathrm{unif}\left(\left(v_0 + v + V_N(\mathbb{R})\right) \times G_N(\mathbb{R})^+ \widetilde{y}_G\right)$$

and E is the weakly special subvariety of S we desire.

□

1.3 The bi-algebraic setting

1.3.1 Realization of the uniformizing space

Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum. We first define the dual $\mathcal{X}^\vee$ of $\mathcal{X}^+$ (see [53, 1.7(a)] or [37, Chapter VI, Proposition 1.3]):

Let M be a faithful representation of P and take any $x_0 \in \mathcal{X}^+$. The weight filtration on M is constant, so the Hodge filtration $x \mapsto \mathrm{Fil}^{\cdot}_x(M_{\mathbb{C}})$ gives an injective map $\mathcal{X}^+ \hookrightarrow \mathrm{Grass}(M)(\mathbb{C})$ to a certain flag variety. In fact, this injective map factors through

$$\mathcal{X}^+ = P(\mathbb{R})^+ U(\mathbb{C})/C(x_0) \hookrightarrow P(\mathbb{C})/F^0_{x_0}P(\mathbb{C}) \hookrightarrow \mathrm{Grass}(M)(\mathbb{C})$$

where $C(x_0)$ is the stabilizer of x_0 in $P(\mathbb{R})^+U(\mathbb{C})$. The first injection is an open immersion ([53, 1.7(a)] or [37, Chapter VI, (1.2.1)]). We define the dual $\mathcal{X}^\vee$ of $\mathcal{X}^+$ to be

$$\mathcal{X}^\vee := P(\mathbb{C})/F^0_{x_0}P(\mathbb{C}).$$

$\mathcal{X}^\vee$ is a connected smooth complex algebraic variety.

Proposition 1.3.1. *Under the open immersion $\mathcal{X}^+ \hookrightarrow \mathcal{X}^\vee$, $\mathcal{X}^+$ is realized as a semi-algebraic set which is also a complex manifold.*

Proof. $\mathcal{X}^+$ is smooth since it is a homogeneous space, and the open immersion endows it with a complex structure. For semi-algebraicity, consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{X}^+ & \hookrightarrow & \mathcal{X}^\vee \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi^\vee} \\ \mathcal{X}_G^+ & \hookrightarrow & \mathcal{X}_G^\vee \end{array} .$$

As $\pi^\vee$ is algebraic, the conclusion follows from [64, Lemme 2.1]. □

Remark 1.3.2. *It is not hard to see that $\mathcal{X}^\vee$ is a projective variety if and only if $(P, \mathcal{X}^+)$ is pure. The argument is as follows: $\mathcal{X}^\vee$ is a holomorphic vector bundle over $\mathcal{X}_G^\vee$ where the fibre is homeomorphism to $W(\mathbb{R})U(\mathbb{C})$. $\mathcal{X}_G^\vee$ is projective, so $\mathcal{X}^\vee$ is projective if and only if it is a trivial vector bundle over $\mathcal{X}_G^\vee$, i.e. if and only if W is trivial.*

Let us take a closer look at the semi-algebraic structrue of $\mathcal{X}^+$. By [71, pp 6], there exists a Shimura morphism $i\colon (G, \mathcal{X}_G^+) \to (P, \mathcal{X}^+)$ such that $\pi \circ i = \mathrm{id}$. The morphism i defines a Levi decomposition of $P = W \rtimes G$. By definition $\mathcal{X}^+ \subset \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_\mathbb{C})$. Define a bijective map

$$\begin{array}{ccc} W(\mathbb{R})U(\mathbb{C}) \times \mathcal{X}_G^+ & \longrightarrow & \mathcal{X}^+ \\ (w, x) & \mapsto & \mathrm{int}(w) \circ i(x) \end{array}.$$

Identify P with the 4-tuple (G, V, U, Ψ) as in §1.1.2.5. Since $W \simeq U \times V$ as $\mathbb{Q}$-varieties, we can define a bijection induced by the one above

$$\rho : U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+ \xrightarrow{\sim} \mathcal{X}^+ \tag{1.3.1}$$

$P(\mathbb{R})^+U(\mathbb{C})$ acts on $\mathcal{X}^+$ by definition. There is also a natural action of $P(\mathbb{R})^+U(\mathbb{C})$ on $U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$ which is defined as follows. Under the notation of §1.1.2.5, for any $(u, v, g) \in P(\mathbb{R})^+U(\mathbb{C})$ and $(u', v', x) \in U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$,

$$(u, v, g) \cdot (u', v', x) := (u + gu' + \frac{1}{2}\Psi(v, v'), v + gv', gx). \tag{1.3.2}$$

This action is algebraic since Ψ is a polynomial over $\mathbb{Q}$ (see §2.2). The map ρ is $P(\mathbb{R})^+U(\mathbb{C})$-equivariant by an easy calculation.

Proposition 1.3.3. *The map ρ is semi-algebraic.*

Proof. It is enough to prove that the graph of ρ is semi-algebraic. This is true since ρ is $P(\mathbb{R})^+U(\mathbb{C})$-equivariant and the actions of $P(\mathbb{R})^+U(\mathbb{C})$ on both sides are algebraic and transitive. Explicitly, fix a point $x_0 \in U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$, the graph of ρ

$$\begin{aligned} Gr(\rho) &= \{(gx_0, \rho(gx_0)) \in (U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+) \times \mathcal{X}^+ |\ g \in P(\mathbb{R})^+U(\mathbb{C})\} \text{ (transitivity)} \\ &= \{(gx_0, g\rho(x_0)) \in (U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+) \times \mathcal{X}^+ |\ g \in P(\mathbb{R})^+U(\mathbb{C})\} \text{ (equivariance)} \\ &= P(\mathbb{R})^+U(\mathbb{C}) \cdot (x_0, \rho(x_0)) \end{aligned}$$

is semi-algebraic since the action of $P(\mathbb{R})^+U(\mathbb{C})$ on $(U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+) \times \mathcal{X}^+$ is algebraic. □

Remark 1.3.4. *If U is trivial, then $\mathcal{X}^+ = V(\mathbb{R}) \rtimes \mathcal{X}_G^+$ under the notation of Example 1.1.24. In this case, the complex structure of $\mathcal{X}^+$ given via $\mathcal{X}^\vee$ is the same as the one given in Example 1.1.24 since for the projection $\mathcal{X}^+ \xrightarrow{\pi} \mathcal{X}_G^+$, the complex structure of any fibre $\mathcal{X}_{x_G}^+$ ($x_G \in \mathcal{X}_G^+$) given by $\mathcal{X}^\vee$ is the same as the one obtained from $\mathcal{X}_{x_G}^+ \simeq V(\mathbb{C})/F_{x_G}^0 V(\mathbb{C})$ (see [53, 3.13, 3.14]). In particular this holds for $\mathcal{X}_{2g,\mathrm{a}}^+$ (see §1.1.3 for notation). Therefore for any $\mathfrak{A}_g(M)$, the fundamental set $[0,N)^{2g} \times \mathcal{F}_G \subset V_{2g}(\mathbb{R}) \times \mathbb{H}_g^+ \simeq \mathcal{X}_{2g,\mathrm{a}}^+$ is the one considered in [47].*

1.3.2 Algebraicity in the uniformizing space

Definition 1.3.5. *Let $\widetilde{Y}$ be an analytic subset of $\mathcal{X}^+$, then*

1. *$\widetilde{Y}$ is called an* ***irreducible algebraic subset*** *of $\mathcal{X}^+$ if it is a complex analytic irreducible component of the intersection of its Zariski closure in $\mathcal{X}^\vee$ and $\mathcal{X}^+$;*

2. *$\widetilde{Y}$ is called* ***algebraic*** *if it is a finite union of irreducible algebraic subsets of $\mathcal{X}^+$.*

In view of Definition 1.3.5, we are in the following bi-algebraic situation: both $\mathcal{X}^+$ are S are algebraic, but $\mathrm{unif}\colon \mathcal{X}^+ \to S$ is transcendental. Hence a priori there is no relation between the algebraic structures on $\mathcal{X}^+$ and on S. Therefore a natural question arises: what are the bi-algebraic objects? This question will be answered in the following sections. We state the result here:

Theorem 1.3.6. *A subset $Y \subset S$ is weakly special iff $\widetilde{Y}$ (a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$) is algebraic in $\mathcal{X}^+$ and Y is an irreducible subvariety of S.*

Remark 1.3.7. *Recall the following result of Pila-Tsimerman [49, Lemma 4.1]: maximal connected irreducible semi-algebraic subsets of $\mathcal{X}^+$ which are contained in a complex analytic subset of $\mathcal{X}^+$ are all algebraic (see the paragraph before Theorem 3.1.2 for the definition of "connected irreducible semi-algebraic subsets"). Hence an equivalent way to restate Theorem 1.3.6 is to replace "$\widetilde{Y}$ is algebraic in $\mathcal{X}^+$" by "$\widetilde{Y}$ is a semi-algebraic subset of $\mathcal{X}^+$".*

A more refined version as well as the proof of this theorem will be given in Corollary 2.3.3. Here we only prove the easy part of the theorem, which is:

Lemma 1.3.8. *Any weakly special subset of $\mathcal{X}^+$ is irreducible algebraic.*

Proof. Suppose that $\widetilde{Z}$ is a weakly special subset of $\mathcal{X}^+$. Use the notation of Definition 1.2.2 and assume that i and φ satisfy the properties in Proposition 1.2.4. Let $N := \mathrm{Ker}(Q \to Q')$ and let y be a point of the weakly special subset, then $\widetilde{Z} = N(\mathbb{R})^+ U_N(\mathbb{C}) y$ is complex analytic irreducible by Remark 1.2.3.2. But $N(\mathbb{R})^+ U_N(\mathbb{C}) y = N(\mathbb{C}) y \cap \mathcal{X}^+$ and $N(\mathbb{C}) y$ is algebraic, so $\widetilde{Z}$ is irreducible algebraic by definition. ∎

We finish this section by the functoriality of algebraicity:

Lemma 1.3.9 (functoriality of algebraicity)**.** *Let* $f\colon (Q, \mathcal{Y}^+) \to (P, \mathcal{X}^+)$ *be a Shimura morphism. Then there exists a unique morphism* $f^\vee\colon \mathcal{Y}^\vee \to \mathcal{X}^\vee$ *of algebraic varieties such that the diagram commutes:*

$$\begin{array}{ccc} \mathcal{Y}^+ & \hookrightarrow & \mathcal{Y}^\vee \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^\vee} \\ \mathcal{X}^+ & \hookrightarrow & \mathcal{X}^\vee \end{array}.$$

Furthermore, for any irreducible algebraic subset $\widetilde{Z}$ *of* $\mathcal{Y}^+$*, the closure in the archimedean topology of* $f(\widetilde{Z})$ *is irreducible algebraic in* $\mathcal{X}^+$ *and* $f(\widetilde{Z})$ *contains a dense open subset of this closure.*

In particular, if f *is an embedding, then an irreducible algebraic subset of* $\mathcal{Y}^+$ *is an irreducible component of the intersection of an irreducible algebraic subset of* $\mathcal{X}^+$ *with* $\mathcal{Y}^+$*.*

Proof. Fix a point $x_0 \in \mathcal{Y}^+$, then we have

$$\begin{array}{ccc} \mathcal{Y}^+ = Q(\mathbb{R})^+ U_Q(\mathbb{C})/C(x_0) & \hookrightarrow & \mathcal{Y}^\vee = Q(\mathbb{C})/F^0_{x_0}Q(\mathbb{C}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^\vee} \\ \mathcal{X}^+ = P(\mathbb{R})^+ U_P(\mathbb{C})/C(f(x_0)) & \hookrightarrow & \mathcal{X}^\vee = P(\mathbb{C})/F^0_{f(x_0)}P(\mathbb{C}) \end{array},$$

where $C(x_0)$ (resp. $C(f(x_0))$) denotes the stabilizer of x_0 (resp. $f(x_0)$) in $Q(\mathbb{R})U_Q(\mathbb{C})$ (resp. $P(\mathbb{R})U_P(\mathbb{C})$). The map $f^\vee$ is unique since $Q(\mathbb{R})U_Q(\mathbb{C})/C(x_0)$ is dense in $\mathcal{Y}^\vee$.

To prove the second statement, it is enough to prove the result for $f^\vee(\overline{\widetilde{Z}}^{\mathrm{Zar}}) \subset \mathcal{X}^\vee$ where $\overline{\widetilde{Z}}^{\mathrm{Zar}}$ is the Zariski closure of $\widetilde{Z}$ in $\mathcal{Y}^\vee$. This is then an algebro-geometric result, which follows easily from Chevalley's Theorem ([22, Chapitre IV, 1.8.4]) and [41, I.10, Theorem 1]. ∎

Chapter 2

Ax's theorem of log type

2.1 Results for the unipotent part

Given a connected mixed Shimura variety S, let S_G be its pure part. We have a projection $S \xrightarrow{[\pi]} S_G$. For any point $b \in S_G$, denote by E the fiber S_b. Suppose that S is associated with the mixed Shimura datum $(P, \mathcal{X}^+)$, which can be further assumed to satisfy $P = \mathrm{MT}(\mathcal{X}^+)$ by Proposition 1.1.19. Let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ be the uniformization. Now $E = S_b \simeq \Gamma_W\backslash W(\mathbb{R})U(\mathbb{C})$ with the complex structure determined by $b \in S_G$ ($E = S_b = \Gamma_W\backslash W(\mathbb{C})/F_b^0W(\mathbb{C})$), where $\Gamma_W := \Gamma \cap W(\mathbb{Q})$. Write $T := \Gamma_U\backslash U(\mathbb{C})$ and $A := \Gamma_A\backslash V(\mathbb{C})/F_b^0V(\mathbb{C})$ where $\Gamma_U := \Gamma\cap U(\mathbb{Q})$ and $\Gamma_V := \Gamma_W/\Gamma_U$, then A is a complex abelian variety and E is an algebraic torus over A whose fibers are isomorphic to T.

Lemma 2.1.1. *If E admits a structure of algebraic group whose group law is compatible with the group law of W, then W (hence E) is commutative. In this case E is a semi-abelian variety.*

Proof. If E is an algebraic group, then T is a normal subgroup of E. Hence E acts on T by conjugation, and this action factors via A, and then it is trivial by [13, 8.10 Proposition]. Therefore T is in the center of E. Now consider the commutator pairing $E\times E \to E$. This factors through a morphism $A\times A \xrightarrow{f} T$. But as a morphism from an abelian variety to an algebraic torus over $\mathbb{C}$, f is then constant. So the commutator pairing $E \times E \to E$ is trivial, and hence E is commutative.

The commutator pairing $W\times W \to W$ induces an alternating form $\Psi\colon V \times V \to U$ (see §1.1.2.5) which induces the morphism f above. We have proved in the last paragraph that $\Psi(V(\mathbb{R}), V(\mathbb{R})) \subset \Gamma_U$ with $\Gamma_U := \Gamma \cap U(\mathbb{Q})$. But $\Psi(V(\mathbb{R}), v)$ is continuous for any $v \in V(\mathbb{R})$ and $\Psi(0, V(\mathbb{R})) = 0$, so $\Psi(V(\mathbb{R}), V(\mathbb{R})) = 0$. Hence the commutator pairing $W \times W \to W$ is trivial, and therefore W is commutative. ☐

2.1.1 Weakly special subvarieties of a complex semi-abelian variety

Proposition 2.1.2. *Use the notation as at the beginning of the section. Weakly special subvarieties of E are precisely the subsets of E of the form*

$$\mathrm{unif}(W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{z})$$

where W_0 *is a* $\mathrm{MT}(b)$*-subgroup of* W *(i.e. a subgroup of* W *normalized by* $\mathrm{MT}(b)$*),* $U_0 := W_0 \cap U$*,* $\mathrm{unif}(\widetilde{z}_G) = b$ *and* $\widetilde{z}_V \in (N_W(W_0)/U)(\mathbb{R})$ *(*$\widetilde{z} = (\widetilde{z}_U, \widetilde{z}_V, \widetilde{z}_G)$ *under* (1.3.1)*).*

In particular, if E *can be given the structure of an algebraic group whose group law is compatible with that of* W *(i.e.* W *is commutative), then the weakly special subvarieties of* E *are precisely the translates of subgroups of* E*.*

Proof. Let Z be a weakly special variety of E and let $\widetilde{Z}$ be a complex analytic irreducible component of $\mathrm{unif}^{-1}(Z)$, then there exists a diagram as in Definition 1.2.2 such that $\widetilde{z}\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $Q_{\mathbb{C}}$, $N \lhd Q$ and $\widetilde{Z} = N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{z}$ for some $\widetilde{z} \in \widetilde{Z}$. As is explained in [54, paragraph 2, pp 265], $G_N = 1$. We prove that $N = W_N$ satisfies the conditions which we require. Let $U_N := W_N \cap U$, then U_N is a $\mathrm{MT}(b)$-module by Proposition 1.1.19(2). Denote by $V_N := W_N/U_N$, $\pi_{P/U}\colon (P, \mathcal{X}^+) \to (P/U, \mathcal{X}^+_{P/U})$ and $[\pi_{P/U}]\colon S \to S_{P/U}$. Then $[\pi_{P/U}](Z)$ is a subvariety of A since Z is a subvariety of E. So $\pi_{P/U}(\widetilde{Z}) = V_N(\mathbb{R}) + \pi_{P/U}(\widetilde{z})$ is the translate of a complex subspace of $V(\mathbb{R}) = V(\mathbb{C})/F^0_bV(\mathbb{C})$, and therefore V_N is a $\mathrm{MT}(b)$-module. So W_N is stable under the action of $\mathrm{MT}(b)$. Now $\widetilde{z}_V \in (N_W(N)/U)(\mathbb{R})$ since $\widetilde{z}\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $N_P(N)_{\mathbb{C}}$.

Conversely let $\widetilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{z}$ with W_0, $\widetilde{z}$ as stated. Fix a Levi decomposition $P = W \rtimes G$. Let $G' := \mathrm{MT}(b)$, let $W' := N_W(W_0)$ and let $Q := W' \rtimes G'$. Then $W_0 \lhd Q$ and hence $\widetilde{z}\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $Q_{\mathbb{C}}$. Therefore $(Q, \mathcal{Y}^+)$, where $\mathcal{Y}^+ := Q(\mathbb{R})^+(U \cap Q)(\mathbb{C})\widetilde{z}$, is a connected mixed Shimura subdatum of $(P, \mathcal{X}^+)$ such that $b \in \mathrm{unif}(\mathcal{Y}^+)$. Now consider the morphisms of connected mixed Shimura data

$$(Q, \mathcal{Y}^+)/W_0 \xleftarrow{\varphi} (Q, \mathcal{Y}^+) \xrightarrow{i} (P, \mathcal{X}^+).$$

In the fibres above the point $b \in S_G$ these maps are simply

$$S_{Q,b}/Z \twoheadleftarrow S_{Q,b} \hookrightarrow E = S_b.$$

Hence Z is a weakly special subvariety by definition. □

Corollary 2.1.3. 1. *Weakly special subvarieties of a complex abelian variety are precisely the translates of its abelian subvarieties;*

2. *Weakly special subvarieties of an algebraic torus over* $\mathbb{C}$ *are precisely the translates of its subtori.*

Proof. This is a direct consequence of Proposition 2.1.2. □

2.1.2 Smallest weakly special subvariety containing a given subvariety of an abelian variety or an algebraic torus over $\mathbb{C}$

Proposition 2.1.4. *1. Let X be a complex abelian variety and let Z be an irreducible subvariety of X. Denote by*

$$\widetilde{X} = \pi_1(X, z) \otimes_{\mathbb{Z}} \mathbb{R} = H_1(X, \mathbb{R}) \simeq \mathbb{C}^n \xrightarrow{u} X$$

the universal cover of X ($z \in Z^{\mathrm{sm}}$), then the smallest weakly special subvariety of X containing Z is a translate of $u(\pi_1(Z^{\mathrm{sm}}, z) \otimes \mathbb{R})$.

2. Let X be an algebraic torus over $\mathbb{C}$ and let Z be an irreducible subvariety of X. Denote by

$$\widetilde{X} = \pi_1(X, z) \otimes_{\mathbb{Z}} \mathbb{C} = H_1(X, \mathbb{C}) \simeq \mathbb{C}^n \xrightarrow{u} X$$

the universal cover of X ($z \in Z^{\mathrm{sm}}$), then the smallest weakly special subvariety of X containing Z is a translate of $u(\pi_1(Z^{\mathrm{sm}}, z) \otimes \mathbb{C})$.

Proof. 1. If X is a complex abelian variety, then the result is due to Ullmo-Yafaev. Their proof of [65, Proposition 5.1] has in fact revealed this property. Here we restate the proof with more details.

Let $Z^{\mathrm{de}} \xrightarrow{s} Z$ be a desingularization of Z^{de} such that there exists a Zariski open subset Z_0^{de} of Z^{de} such that $Z_0^{\mathrm{de}} \xrightarrow[s]{\sim} Z^{\mathrm{sm}}$. By the commutative diagram

$$\begin{array}{ccccc}
\pi_1(Z_0^{\mathrm{de}}, z) & \xrightarrow{\sim} & \pi_1(Z^{\mathrm{sm}}, z) & & \\
\downarrow & & \downarrow & \searrow & \\
\pi_1(Z^{\mathrm{de}}, z) & \longrightarrow & \pi_1(Z, z) & \longrightarrow & \pi_1(X, z)
\end{array},$$

where $z \in Z^{\mathrm{sm}}$ (the surjectivity on the left is due to [31, 2.10.1]), we know that the image of $\pi_1(Z^{\mathrm{de}}, z)$ and the image of $\pi_1(Z^{\mathrm{sm}}, z)$ in $\pi_1(X, z)$ are the same.

Let $\mathrm{Alb}(Z^{\mathrm{de}})$ be the Albanese variety of Z^{de} normalized by z, then the map $\tau \colon Z^{\mathrm{de}} \to Z \to X$ factors uniquely through the Albanese morphism([70, Theorem 12.15]):

$$\begin{array}{ccccc}
Z^{\mathrm{de}} & \twoheadrightarrow & Z & \hookrightarrow & X \\
& \underset{\mathrm{alb}}{\searrow} & & \underset{\Gamma}{\nearrow} & \\
& & \mathrm{Alb}(Z^{\mathrm{de}}) & &
\end{array}$$

Let $A := \Gamma(\mathrm{Alb}(Z^{\mathrm{de}}))$, then it is the smallest weakly special subvariety (i.e. the translate of an abelian subvariety) of X containing Z since $\mathrm{alb}(Z^{\mathrm{de}})$ generates $\mathrm{Alb}(Z^{\mathrm{de}})$ ([70, Lemma 12.11]).

It suffices to prove that the image of $\pi_1(Z^{\mathrm{de}}, z)$ in $\pi_1(X, z) \simeq H_1(X, \mathbb{Z})$ is of finite index in $H_1(A, \mathbb{Z})$. This is true since the image of $\pi_1(Z^{\mathrm{de}}, z)$ in $H_1(X, \mathbb{Z})$ contains

$$(\Gamma \circ \mathrm{alb})_* H_1(Z^{\mathrm{de}}, \mathbb{Z}) \simeq \Gamma_* H_1(\mathrm{Alb}(Z^{\mathrm{de}}), \mathbb{Z}) \simeq \Gamma_* \pi_1(\mathrm{Alb}(Z^{\mathrm{de}}))$$

(the first isomorphism is given by the definition of Albanese varieties via Hodge theory, see e.g. the proof of [70, Lemma 12.11]), which is of finite index in $\pi_1(A, z) \simeq H_1(A, \mathbb{Z})$ by [31, 2.10.2].

2. If X is an algebraic torus over $\mathbb{C}$, then we can first of all translate Z by a point such that the translate contains the origin of X. Now we are done if we can prove that the smallest subtorus containing this translate of Z is $u(\pi_1(Z^{\mathrm{sm}}, z) \otimes_{\mathbb{Z}} \mathbb{C})$.

   Suppose $T \simeq (\mathbb{C}^*)^m$ is the smallest sub-torus of X containing Z with $j \colon Z^{\mathrm{sm}} \hookrightarrow T$ the inclusion. We are done if we can prove $[\pi_1(T, z) : j_* \pi_1(Z^{\mathrm{sm}}, z)] < \infty$. If not, then

$$j_* \pi_1(Z^{\mathrm{sm}}, z) \ \subset \ \mathrm{Ker}(\mathbb{Z}^m \overset{\rho}{\twoheadrightarrow} \mathbb{Z}) \tag{2.1.1}$$

   for some map ρ. Since the covariant functor $T \mapsto X_*(T)$ ($X_*(T)$ is the co-character group of T) is an equivalence between the category {algebraic tori over $\mathbb{C}$ and their morphisms as algebraic groups} and the category {free $\mathbb{Z}$-modules of finite rank}, the map ρ corresponds to a surjective map (with connected kernel) of tori $p \colon T \twoheadrightarrow T'$. The composition of the maps $Z^{\mathrm{sm}} \xrightarrow{j} T \xrightarrow{p} T' = \mathbb{G}_{m,\mathbb{C}}$ should be dominant by the choice of T. But then we have

$$[\pi_1(T', p(z)) : (p \circ j)_* \pi_1(Z^{\mathrm{sm}}, z)] < \infty$$

   ([31, 2.10.2]), which contradicts (2.1.1) by the following lemma.

Lemma 2.1.5. *For any $\mathbb{C}$-split torus $T \simeq (\mathbb{C}^*)^n$, we have a canonical isomorphism*

$$X_*(T) \xrightarrow[\sim]{\psi_T} \pi_1(T, 1).$$

Here "canonical" means that for any morphism (between algebraic groups) $f \colon T \to T'$ between two such $\mathbb{C}$-split tori, the following diagram commutes:

$$\begin{array}{ccc} X_*(T) & \xrightarrow[\sim]{\psi_T} & \pi_1(T, 1) \\ {\scriptstyle X_*(f)} \downarrow & & \downarrow {\scriptstyle f_*} \\ X_*(T') & \xrightarrow[\sim]{\psi_{T'}} & \pi_1(T', 1) \end{array}$$

Proof. Denote by $U_1 := \{z \in \mathbb{C} \mid |z| = 1\}$ and $i\colon U_1 \hookrightarrow \mathbb{C}^*$ the inclusion. Then the map ψ_T is defined by

$$\begin{array}{ccc} X_*(T) & \xrightarrow{\psi_T} & \pi_1(T,1) \\ \nu & \mapsto & [\nu \circ i] \end{array}.$$

This is a group homomorphism. It is surjective since a representative of the generators of $\pi_1(T,1)$ is given by the n coordinate embeddings $U_1 \hookrightarrow \mathbb{C}^* \hookrightarrow T = (\mathbb{C}^*)^n$. ψ_T is injective since $X_*(T) \simeq \pi_1(T,1) \simeq \mathbb{Z}^n$ is torsion-free. The rest of the lemma is immediate by the construction of ψ_T. □

□

2.2 Monodromy groups of admissible variations of mixed Hodge structures

2.2.1 Arbitrary variation of mixed $\mathbb{Z}$-Hodge structures

Let $(\mathbb{V}, W_\cdot, \mathcal{F}^\cdot)$ be a variation of mixed $\mathbb{Z}$-Hodge structures over a complex manifold S (see §1.1.1.4 for definition). Let $\pi\colon \widetilde{S} \to S$ be a universal covering and choose a trivialization $\pi^*\mathbb{V} \simeq \widetilde{S} \times V$. For $s \in S$, $\mathrm{MT}_s \subset \mathrm{GL}(\mathcal{V}_s)$ denote the Mumford-Tate group of its fibre. The choice of a point $\widetilde{s} \in \widetilde{S}$ with $\pi(\widetilde{s}) = s$ gives an identification $\mathcal{V}_s \simeq V$, whence an injective homomorphism $i_{\widetilde{s}}\colon \mathrm{MT}_s \hookrightarrow \mathrm{GL}(V)$. By [1, §4, Lemma 4], on $S^\circ := S \setminus \Sigma$ where Σ is a meager subset of S, $M := \mathrm{Im}(i_{\widetilde{s}}) \subset \mathrm{GL}(V)$ does not depend on s, nor on the choice of $\widetilde{s}$. We call S° the **Hodge-generic locus** and the group M the **generic Mumford-Tate group** of $(\mathbb{V}, W_\cdot, \mathcal{F}^\cdot)$.

On the other hand, if we choose a base-point $s \in S$ and a point $\widetilde{s} \in \widetilde{S}$ with $\pi(\widetilde{s}) = s$, then then local system $\mathbb{V}$ corresponds to a representation $\rho\colon \pi_1(S, s) \to \mathrm{GL}(V)$, called the monodromy representation. The algebraic monodromy group is defined as the smallest algebraic subgroup of $\mathrm{GL}(V)$ over $\mathbb{Q}$ which contains the image of ρ. We write H_s^{mon} for its connected component of the identity, called the **connected algebraic monodromy group**. Given the trivialization of $\pi^*\mathbb{V}$, the group $H_s^{\mathrm{mon}} \subset \mathrm{GL}(V)$ is independent of the choice of s and $\widetilde{s}$.

Suppose now that $(\mathbb{V}, W_\cdot, \mathcal{F}^\cdot)$ is graded-polarizable, then $H_s^{\mathrm{mon}} < M$ for any $s \in S^\circ$ by [1, §4, Lemma 4].

2.2.2 Admissible variations of $\mathbb{Z}$-mixed Hodge structures

We now recall the concept of "admissible" variations of mixed Hodge structures which was introduced by Steenbrick-Zucker and studied by Kashiwara and Hain-Zucker. We give the definition here, but instead of the exact definition, we shall only use the notion of "admissibility" and the fact that it can be defined using "curve test". We will use Δ (resp. Δ^*) to denote the unit disc (resp. punctured unit disc).

Definition 2.2.1. *(see [45, Definition 14.49])*

1. *A variation of mixed Hodge structures $(\mathbb{V}, W_., \mathcal{F}^\cdot)$ over the punctured unit disc Δ^* is called **admissible** if*

   - *it is graded-polarizable;*
   - *the monodromy T is unipotent and the weight filtration $M(N, W_.)$ of $N := \log T$ relative to $W_.$ exists;*
   - *the filtration $\mathcal{F}^\cdot$ extends to a filtration $\widetilde{\mathcal{F}}^\cdot$ of $\widetilde{\mathcal{V}}$ which induced ${}^k\widetilde{\mathcal{F}}$ on $\mathrm{Gr}_k^W \widetilde{\mathcal{V}}$ for each k.*

2. *Let S be a smooth connected complex algebraic variety and let $\overline{S}$ be a compactification of S such that $\overline{S} \setminus S$ is a normal crossing divisor. A graded-polarizable variation of mixed Hodge structures $(\mathbb{V}, W_., \mathcal{F}^\cdot)$ on S is called **admissible** if for every holomorphic map $i\colon \Delta \to \overline{S}$ which maps Δ^* to S and such that $i^*\mathbb{V}$ has unipotent monodromy, the variation $i^*(\mathbb{V}, W_., \mathcal{F}^\cdot)$ is admissible. (This definition is sometimes called the "curve test" version).*

Remark 2.2.2. *This definition is equivalent to the one in [25, 1.5]. See [61, Properties 3.13 and Appendix], [28, §1 and Theorem 4.5.2] and [25, 1.5] for details.*

The following lemma is an easy property of admissibility and is surely known by many people, but I cannot find any reference, so I give a proof here.

Lemma 2.2.3. *Let S be a smooth connected complex algebraic variety and let $(\mathbb{V}, W_., \mathcal{F}^\cdot)$ be an admissible variation of mixed Hodge structures on S. Then for any smooth connected (not necessarily closed) subvariety $j\colon Y \hookrightarrow S$, $j^*(\mathbb{V}, W_., \mathcal{F}^\cdot)$ is also admissible on Y.*

Proof. Take smooth compactifications $\overline{Y}$ of Y and $\overline{S}$ of S such that $\overline{Y} \setminus Y$ and $\overline{S} \setminus S$ are normal crossing divisors and such that $j\colon Y \hookrightarrow S$ extends to a morphism $\overline{j}\colon \overline{Y} \to \overline{S}$. This can be done by first choosing any compactifications of Y^{cp} of Y and S^{cp} of S with normal crossing divisors and then taking a suitable resolution of singularities of the closure of the graph of j in $Y^{\mathrm{cp}} \times S^{\mathrm{cp}}$. Now the conclusion follows from our "curve test" version of the definition. ☐

2.2.3 Consequences of admissibility

Y.André proved:

Theorem 2.2.4. *Let* $(\mathbb{V}, W_\cdot, \mathcal{F}^\cdot)$ *be an admissible variation of mixed Hodge structures over a smooth connected complex algebraic variety* S*, then for any* $s \in S$*, the connected monodromy group* H_s^{mon} *is a normal subgroup of the generic Mumford-Tate group* M *and also its derived group* M^{der}*.*

Proof. [1, §5, Theorem 1] states that $H_s^{\mathrm{mon}} \lhd M^{\mathrm{der}}$, and in the proof he first proved that $H_s^{\mathrm{mon}} \lhd M$. □

Now we state a theorem which roughly says that all the variations of mixed Hodge structure obtained from representations of the underlying group of a connected mixed Shimura datum are admissible. Explicitly, let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ be the uniformization. Suppose that Γ is neat. Consider any $\mathbb{Q}$-representation $\xi\colon P \to \mathrm{GL}(V)$. By [55, Proposition 4.2], there exists a Γ-invariant lattice $V_{\mathbb{Z}}$ of V. Now ξ and $V_{\mathbb{Z}}$ together give rise to a VMHS on S whose underlying local system is $\Gamma\backslash(\mathcal{X}^+ \times V_{\mathbb{Z}})$. This variation is (graded-)polarizable by [53, 1.18(d)]. Wildeshaus proved:

Theorem 2.2.5. *Let* S*,* $(P, \mathcal{X}^+)$*,* $\xi\colon P \to \mathrm{GL}(V)$ *and* $V_{\mathbb{Z}}$ *be as in the paragraph above, then the variation of mixed Hodge structures obtained as above is admissible.*

Proof. [71, Theorem 2.2] says that the corresponding $\mathbb{Q}$-variation is admissible, and Γ gives a $\mathbb{Z}$-structure as in the discussion above. □

Remark 2.2.6. *In this language, we can rephrase Definition 1.1.18 as:* P *is the generic Mumford-Tate group (of the variation in Theorem 2.2.5). For any Hodge generic point* $x \in \mathcal{X}^+$*, the only* $\mathbb{Q}$*-subgroup* N *of* P^{der} *such that* $N(\mathbb{R})^+U_N(\mathbb{C})$*, where* $U_N := U \cap N$*, stabilizes* x *is the trivial group.*

2.3 The smallest weakly special subvariety containing a given subvariety

In this section, our goal is to prove a theorem (Theorem 2.3.1) which (in some sense) generalizes [39, 3.6, 3.7]. In particular, we get a criterion of weak specialness as a corollary (Corollary 2.3.3) which generalizes [65, Theorem 4.1].

2.3.1 Connected algebraic monodromy group associated with a subvariety of a mixed Shimura variety

Before the proof, let us do some technical preparation at first.

Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ be the uniformization. We may assume $P = \mathrm{MT}(\mathcal{X}^+)$ by Proposition 1.1.19. There exists a $\Gamma' \leqslant \Gamma$ of finite index such that Γ' is neat. Let $S' := \Gamma'\backslash\mathcal{X}^+$ and let

$\mathrm{unif}' \colon \mathcal{X}^+ \to S'$ be its uniformization. Choose any faithful $\mathbb{Q}$-representation $\xi \colon P \to \mathrm{GL}(M)$ of P, then Theorem 2.2.5 claims that ξ (together with a choice of a Γ'-invariant lattice of M) gives rise to an admissible variation of mixed Hodge structure on S'. The generic Mumford-Tate group of this variation is P.

Suppose that Y is an irreducible subvariety of S. Let Y' be an irreducible component of $p^{-1}(Y)$ under $p \colon S' = \Gamma' \backslash \mathcal{X}^+ \to S = \Gamma \backslash \mathcal{X}^+$, then Y' is an irreducible subvariety of S' which maps surjectively to Y under p. The variation we constructed above can be restricted to Y'^{sm}, and this restriction is still admissible by Lemma 2.2.3. The **connected algebraic monodromy group associated with** Y^{sm} is defined to be the connected algebraic monodromy group of the restriction of the VMHS defined in the last paragraph to Y'^{sm}, i.e. the neutral component of the Zariski closure of the image of $\pi_1(Y'^{\mathrm{sm}}, y') \to \pi_1(S', y') \to P$.

Let us briefly prove that the connected algebraic monodromy group associated with Y^{sm} is well-defined. Suppose that we have another covering $S'' \xrightarrow{p'} S'$ with S'' smooth. Let Y'' be an irreducible component of $p'^{-1}(Y')$. Let $Y_0''^{\mathrm{sm}} := Y''^{\mathrm{sm}} \cap p'^{-1}(Y'^{\mathrm{sm}})$, then by the commutative diagram

$$\begin{array}{ccccc}
\pi_1(Y_0''^{\mathrm{sm}}, y'') = \pi_1(Y''^{\mathrm{sm}}, y'') & \longrightarrow & \pi_1(S'', y'') & \longrightarrow & P \\
\downarrow & & \downarrow & & \downarrow = \\
\pi_1(Y'^{\mathrm{sm}}, y') & \longrightarrow & \pi_1(S', y') & \longrightarrow & P
\end{array},$$

where the equality in the top-left cornor is given by [31, 2.10.1] and the morphism on the left is of finite index by [31, 2.10.2], the neutral components of the Zariski closures of the images of $\pi_1(Y''^{\mathrm{sm}}, y'')$ and $\pi_1(Y'^{\mathrm{sm}}, y')$ in P coincide.

2.3.2 Ax's theorem of log type

Theorem 2.3.1 (Ax of log type)**.** *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif} \colon \mathcal{X}^+ \to S = \Gamma \backslash \mathcal{X}^+$ be the uniformization. Let Y be an irreducible subvariety of S and*

- *let $\widetilde{Y}$ be a complex analytic irreducible component of* $\mathrm{unif}^{-1}(Y)$*;*
- *take $\widetilde{y}_0 \in \widetilde{Y}$;*
- *let N be the connected algebraic monodromy group associated with Y^{sm}.*

Then

1. *The set $\widetilde{F} := N(\mathbb{R})^+ U_N(\mathbb{C}) \widetilde{y}_0$, where $U_N := U \cap N$, is a weakly special subset of $\mathcal{X}^+$ (or equivalently, $F := \mathrm{unif}(\widetilde{F})$ is a weakly special subvariety of S). Moreover N is the largest subgroup of Q such that $N(\mathbb{R})^+ U_N(\mathbb{C})$ stabilizes $\widetilde{F}$, where $(Q, \mathcal{Y}^+)$ is the smallest connected mixed Shimura subdatum with $\widetilde{F} \subset \mathcal{Y}^+$;*

2. *The Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$ (which means the complex analytic irreducible component of the intersection of the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^\vee$ and $\mathcal{X}^+$ which contains $\widetilde{Y}$) is $\widetilde{F}$;*

3. *The smallest weakly special subset containing $\widetilde{Y}$ is $\widetilde{F}$ and F is the smallest weakly special subvariety of S containing Y.*

Proof. 1. Let S_Y be the smallest special subvariety containing Y. Such an S_Y exists since the irreducible components of intersections of special subvarieties are special (which can easily be shown by means of generic Mumford-Tate group). By definition of special subvarieties, there exists a connected mixed Shimura subdatum $(Q, \mathcal{Y}^+)$ such that S_Y is the image of $\Gamma_Q\backslash\mathcal{Y}^+$ in S where $\Gamma_Q := \Gamma \cap Q(\mathbb{Q})$. We may furthermore assume $(Q, \mathcal{Y}^+)$ to have generic Mumford-Tate group by Proposition 1.1.19.

Let N be the connected algebraic monodromy group associated with Y^{sm}, then $N \lhd Q$ (and also $N \lhd Q^{\mathrm{der}}$) by the discussion at the beginning of this section (which claims that the variation we use to define N is admissible), Remark 2.2.6 (which claims that the generic Mumford-Tate group of this variation is Q) and Theorem 2.2.4.

Then $\widetilde{F}$ is a weakly special subset of $\mathcal{Y}^+$ since it is the inverse image of the point $\varphi(\widetilde{y}_0)$ under the Shimura morphism $(Q, \mathcal{Y}^+) \xrightarrow{\varphi} (Q, \mathcal{Y}^+)/N$. Then $\widetilde{F}$ is also a weakly special subset of $\mathcal{X}^+$ by definition. By the choice of $(Q, \mathcal{Y}^+)$, $\widetilde{F}$ is Hodge generic in $\mathcal{Y}^+$, and hence $\varphi(\widetilde{F})$ is a Hodge generic point in $\mathcal{Y}'^+$. Now $\mathrm{Stab}_{Q^{\mathrm{der}}(\mathbb{Q})}(\widetilde{F})^\circ = N(\mathbb{Q})$ by Remark 2.2.6.

2. We prove that $\widetilde{F}$ is the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$. We first show that the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$ defined as in the statement of the theorem exists. To see this, denote by $\widetilde{Y}^\vee$ the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^\vee$. Recall that $\mathcal{X}^+$ is realized as a semi-algebraic open subset (w.r.t. the archimedean topology) of $\mathcal{X}^\vee$ as in §1.3.1. Hence $\widetilde{Y}^\vee \cap \mathcal{X}^+$ has only finitely many complex analytic irreducible components[1], which we call $I_1, ..., I_r$. If $\widetilde{Y}$ is contained in both I_i and I_j where I_i and I_j are distinct, then

$$\widetilde{Y} \subset I_i \cap I_j \subset (\widetilde{Y}^\vee \cap \mathcal{X}^+)^{\mathrm{sing}} \subset (\widetilde{Y}^\vee)^{\mathrm{sing}} \cap \mathcal{X}^+ \subsetneq \widetilde{Y}^\vee \cap \mathcal{X}^+$$

[1]This is true for any irreducible subvariety Z of $\mathcal{X}^\vee$ by induction on $\dim Z$: since the collection of all semi-algebraic sets forms an o-minimal theory, $(Z \cap \mathcal{X}^+)^{\mathrm{sm}}$ decomposes into finitely many connected components, each of which semi-algebraic (To better understand this, recall the theorem of Klingler-Ullmo-Yafaev [29, Appendix] which says that for $(P, \mathcal{X}^+)$ pure, a subset of $\mathcal{X}^+$ is irreducible algebraic iff it is semi-algebraic and complex analytic irreducible. Their argument can be generalized to the mixed case without much difficulty.). Remark that these connected components are also precisely the complex analytic irreducible components since the ambient subset of $\mathcal{X}^+$ is smooth. Now $(Z \cap \mathcal{X}^+)^{\mathrm{sing}} = Z^{\mathrm{sing}} \cap \mathcal{X}^+$ also has only finitely many complex analytic irreducible components by induction hypothesis. So we can conclude.

But $(\widetilde{Y}^\vee)^{\mathrm{sing}}$ is an algebraic subvariety of $\mathcal{X}^\vee$. So this contradicts the fact that $\widetilde{Y}^\vee$ is the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^\vee$. Hence $\widetilde{Y}$ is contained in a unique complex analytic irreducible component of $\widetilde{Y}^\vee \cap \mathcal{X}^+$. So the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$ defined as in the statement of the theorem exists.

Next we prove that it suffices to prove $\widetilde{Y} \subset \widetilde{F}$. Assume this. Let $\overline{\widetilde{Y}}$ be the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$, then $\overline{\widetilde{Y}} \subset \widetilde{F}$ since $\widetilde{Y} \subset \widetilde{F}$ and $\widetilde{F}$ is algebraic (Lemma 1.3.8). On the other hand, $\Gamma_{Y^{\mathrm{sm}}} := \mathrm{Im}(\pi_1(Y^{\mathrm{sm}}) \to \pi_1(S) \to P)$ stabilizes $\widetilde{Y}$, so $\Gamma_{Y^{\mathrm{sm}}}\widetilde{y}_0 \subset \widetilde{Y}$. The group $\Gamma_{Y^{\mathrm{sm}}}$ is Zariski dense in N, and hence Zariski dense in $N_{\mathbb{C}}$. But $\widetilde{F}$ is a complex analytic irreducible component of $N(\mathbb{C})\widetilde{y}_0 \cap \mathcal{X}^+$, so $\Gamma_{Y^{\mathrm{sm}}}\widetilde{y}_0$ is Zariski dense in $\widetilde{F}$. Hence we have $\widetilde{F} \subset \overline{\widetilde{Y}}$. As a result, $\widetilde{F} = \overline{\widetilde{Y}}$.

Now we prove that $\widetilde{Y} \subset \widetilde{F}$ (or equivalently, $Y \subset F$).

The fact that $\widetilde{Y} \subset \widetilde{F}$ has nothing to do with the level structure. Hence we may assume $\Gamma = \Gamma_W \rtimes \Gamma_G$ with $\Gamma_W \subset W(\mathbb{Z})$, $\Gamma_U := \Gamma_W \cap U \subset U(\mathbb{Z})$, $\Gamma_V := \Gamma_W/\Gamma_U \subset V(\mathbb{Z})$ and $\Gamma_G \subset G(\mathbb{Z})$ small enough such that they are all neat and such that $\Gamma \subset P^{\mathrm{der}}(\mathbb{Q})$ (Remark 1.1.13(2)). We write $\Gamma_{P/U} := \Gamma/\Gamma_U$.

We may replace $(P, \mathcal{X}^+)$ by $(Q, \mathcal{Y}^+)$ and S by S_Y (same notation as in (1)) since $\widetilde{Y}$, $\widetilde{F} \subset \mathcal{Y}^+$ and Y, $F \subset S_Y$. In other words, we may assume that Y is Hodge generic in S and $(P, \mathcal{X}^+)$ is irreducible.

Consider the following diagram:

$$\begin{array}{ccccc}
\mathcal{X}^+ & \xrightarrow{\pi_{P/U}} & \mathcal{X}^+_{P/U} & \xrightarrow{\pi_G} & \mathcal{X}^+_G \\
\downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}_{P/U}} & & \downarrow{\scriptstyle \mathrm{unif}_G} \\
S = \Gamma\backslash\mathcal{X}^+ & \xrightarrow{[\pi_{P/U}]} & S_{P/U} := \Gamma_{P/U}\backslash\mathcal{X}^+_{P/U} & \xrightarrow{[\pi_G]} & S_G := \Gamma_G\backslash\mathcal{X}^+_G
\end{array}$$

Denote by π and $[\pi]$ the composites of the maps in the two lines respectively. Denote by $\widetilde{Y}_G := \pi(\widetilde{Y})$, $Y_G := [\pi](Y)$ and $\widetilde{Y}_{P/U} := \pi_{P/U}(\widetilde{Y})$, $Y_{P/U} := [\pi_{P/U}](Y)$; $\widetilde{F}_G := \pi(\widetilde{F})$, $F_G := [\pi](F)$ and $\widetilde{F}_{P/U} := \pi_{P/U}(\widetilde{F})$, $F_{P/U} := [\pi_{P/U}](F)$. Denote by $\widetilde{y}_{0,P/U} := \pi_{P/U}(\widetilde{y}_0)$ and $\widetilde{y}_{0,G} := \pi(\widetilde{y}_0)$.

Now to make the proof more clear, we divide it into several steps.

Step I. Prove that $\widetilde{Y}_G \subset \widetilde{F}_G$.

We begin the proof with the following lemma:

Lemma 2.3.2. *In the context above, the connected algebraic monodromy group associated with* $\overline{Y_G}^{\mathrm{sm}}$ *(resp.* $\overline{Y_{P/U}}^{\mathrm{sm}}$*) is* G_N *(resp.* N/U_N *where* $U_N := U \cap N$*).*

Proof. We only prove the statement for $\overline{Y_G}^{\mathrm{sm}}$. The proof for $\overline{Y_{P/U}}^{\mathrm{sm}}$ is similar. Take $Y_0^{\mathrm{sm}} := Y^{\mathrm{sm}} \cap \pi^{-1}(Y_G^{\mathrm{sm}})$, then we have the commutative diagram below:

$$\begin{array}{ccccc}
\pi_1(Y_0^{\mathrm{sm}}, y) & \longrightarrow & \pi_1(Y_G^{\mathrm{sm}}, y_G) & \twoheadrightarrow & \pi_1(\overline{Y_G}^{\mathrm{sm}}, \zeta_G) \\
\downarrow & & & \searrow & \downarrow \\
\pi_1(Y^{\mathrm{sm}}, y) & \longrightarrow & \pi_1(S, y) & \longrightarrow & \pi_1(S_G, y_G) \\
 & & \downarrow & & \downarrow \\
 & & P & \longrightarrow & G
\end{array} .$$

Here, the morphism on the left and the right morphism on the top are surjective since $\mathrm{codim}_{Y^{\mathrm{sm}}}(Y^{\mathrm{sm}} - Y_0^{\mathrm{sm}}) \geqslant 1$ and $\mathrm{codim}_{\overline{Y_G}^{\mathrm{sm}}}(\overline{Y_G}^{\mathrm{sm}} - Y_G^{\mathrm{sm}}) \geqslant 1$ ([31, 2.10.1]). Now [31, 2.10.2] shows that the image of $\pi_1(Y_0^{\mathrm{sm}}, y)$ is of finite index in $\pi_1(Y_G^{\mathrm{sm}}, y_G)$, so the neutral components of the Zariski closures of $\pi_1(Y^{\mathrm{sm}}, y)$ and $\pi_1(\overline{Y_G}^{\mathrm{sm}}, y_G)$ in G coincide. Hence we are done. □

Let $\widetilde{Z}$ be the closure (w.r.t. archimedean topology) of $\widetilde{Y}_G$ in $\mathcal{X}_G^+$, then $\widetilde{Z}$ is a complex analytic irreducible component of $\mathrm{unif}_G^{-1}(\overline{Y_G})$. For the pure connected Shimura datum $(G^{\mathrm{ad}}, \mathcal{X}_G^+)$, we have a decomposition ([39, 3.6])

$$(G^{\mathrm{ad}}, \mathcal{X}_G^+) = (G_N^{\mathrm{ad}}, \mathcal{X}_{G,1}^+) \times (G_2, \mathcal{X}_{G,2}^+).$$

By [39, 3.6, 3.7] and Lemma 2.3.2, $\widetilde{Z} \subset \mathcal{X}_{G,1}^+ \times \{\widetilde{y_{G,2}}\}$, i.e. $\widetilde{Z} \subset G_N(\mathbb{R})^+\widetilde{x}_G$ for some $\widetilde{x}_G \in \mathcal{X}_G^+$. But $\widetilde{y}_{0,G} \in \widetilde{Y}_G \subset \widetilde{Z}$, so $\widetilde{F}_G = G_N(\mathbb{R})^+\widetilde{y}_{0,G} \subset G_N(\mathbb{R})^+\widetilde{x}_G$. This implies that $\widetilde{F}_G = G_N(\mathbb{R})^+\widetilde{x}_G$. As a result, $\widetilde{Y}_G \subset \widetilde{Z} \subset \widetilde{F}_G$.

Step II. Consider the Shimura morphism

$$(P, \mathcal{X}^+) \xrightarrow{\rho} (P', \mathcal{X}^{+\prime}) := (P, \mathcal{X}^+)/N.$$

Then $\widetilde{F} = \rho^{-1}(\rho(\widetilde{F}))$ by definition of ρ. So in order to prove $\widetilde{Y} \subset \widetilde{F}$, it is enough to show that $\rho(\widetilde{Y}) \subset \rho(\widetilde{F})$. Hence we may replace $(P, \mathcal{X}^+)$ by $(P', \mathcal{X}^{+\prime})$. In other words, we may assume $N = \mathbf{1}$.

In this case $\widetilde{F}$ is just a point $\widetilde{x} \in \mathcal{X}^+$. Call $\widetilde{x}_{P/U} := \pi_{P/U}(\widetilde{x})$, $\widetilde{x}_G := \pi(\widetilde{x})$ and $x := \mathrm{unif}(\widetilde{x})$, $x_{P/U} := \mathrm{unif}_{P/U}(\widetilde{x}_{P/U})$, $x_G := \mathrm{unif}_G(\widetilde{x}_G)$. Then since $Y_G \subset F_G$, we have $Y \subset E$ where E is the fibre of $S \xrightarrow{[\pi]} S_G$ over x_G. Denote by A the fibre of $S_{P/U} \xrightarrow{[\pi]_G} S_G$ over x_G and T the fibre of $S \xrightarrow{[\pi_{P/U}]} S_{P/U}$ over $x_{P/U}$, then by [53, 3.13, 3.14] A is an abelian variety and T is an algebraic torus.

Step III. Prove that $\widetilde{Y}_{P/U} \subset \widetilde{F}_{P/U}$, i.e. $\widetilde{Y}_{P/U} = \{\widetilde{x}_{P/U}\}$.

By *Step I*, $Y_{P/U} \subset A$. We have the following morphisms

$$\pi_1(Y_{P/U}^{\mathrm{sm}}) \to \pi_1(A) \to \pi_1(S_{P/U}) = \Gamma_{P/U} \to P/U = V \rtimes G.$$

The neutral component of the Zariski closure of $\pi_1(Y_{P/U}^{\mathrm{sm}})$ (resp. $\pi_1(A)$) in $P/U = V \rtimes G$ is $\mathbf{1}$ (resp. V), so the image of

$$\pi_1(Y_{P/U}^{\mathrm{sm}}) \to \pi_1(A)$$

is a finite group.

Now $Y_{P/U}$ is irreducible since Y is irreducible. So by Proposition 2.1.4, $Y_{P/U} \subset A$ is a point. Equivalently, $\widetilde{Y}_{P/U}$ is a point. So $\widetilde{Y}_{P/U} \subset \widetilde{F}_{P/U}$ since $\widetilde{Y}_{P/U} \cap \widetilde{F}_{P/U} \neq \emptyset$ (both of them contain $\widetilde{y}_{0,P/U}$).

Step IV. Prove that $\widetilde{Y} \subset \widetilde{F}$, i.e. $\widetilde{Y} = \{\widetilde{x}\}$.

By *Step I*, $Y \subset E$. By *Step III*, $Y_{P/U} = \{x_{P/U}\}$. So $Y \subset T$. We have the following morphisms

$$\pi_1(Y^{\mathrm{sm}}) \to \pi_1(T) \to \pi_1(S) = \Gamma \to P = W \rtimes G.$$

The neutral component of the Zariski closure of $\pi_1(Y^{\mathrm{sm}})$ (resp. $\pi_1(T)$) in $P = W \rtimes G$ is $\mathbf{1}$ (resp. U), so the image of

$$\pi_1(Y^{\mathrm{sm}}) \to \pi_1(T)$$

is a finite group.

Now since Y is irreducible, by Proposition 2.1.4, $Y \subset T$ is a point. Equivalently, $\widetilde{Y}$ is a point. So $\widetilde{Y} \subset \widetilde{F}$ since $\widetilde{Y} \cap \widetilde{F} \neq \emptyset$ (both of them contain $\widetilde{y}_0$).

3. Since every weakly special subset of $\mathcal{X}^+$ is algebraic by Lemma 1.3.8, $\widetilde{F}$ is also the smallest weakly special subset which contains $\widetilde{Y}$. Therefore F is the smallest weakly special subvariety of S which contains Y.

□

Corollary 2.3.3. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ *be the uniformization map. Let Y be an irreducible subvariety of S, then Y is weakly special if and only if one (equivalently any) irreducible component of* $\mathrm{unif}^{-1}(Y)$ *is algebraic.*

If Y is weakly special, then $Y = \mathrm{unif}(N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{y})$ where N is the connected algebraic monodromy group associated with Y^{sm}, $U_N := U \cap N$ and $\widetilde{y}$ is any point of $\mathrm{unif}^{-1}(Y)$.

Proof. The "only if" part is immediate by Lemma 1.3.8. Now we prove the "if" part.

We first of all quickly show that if one irreducible component of $\mathrm{unif}^{-1}(Y)$ is algebraic, so are the others. The proof is the same as [65, first paragraph of the proof of Theorem 4.1]. Suppose that $\widetilde{Y}$ is an irreducible component of $\mathrm{unif}^{-1}(Y)$ which is algebraic, i.e. $\widetilde{Y}$ is an irreducible component of $\mathcal{X}^+ \cap Z$ for some algebraic subvariety Z of $\mathcal{X}^\vee$. Then for any $\gamma \in \Gamma \subset P(\mathbb{R})U(\mathbb{C})$,

$$\gamma\widetilde{Y} = \gamma(\mathcal{X}^+ \cap Z) \subset \mathcal{X}^+ \cap \gamma Z = \gamma\gamma^{-1}(\mathcal{X}^+ \cap \gamma Z) \subset \gamma\widetilde{Y}.$$

Hence it follows that $\gamma\widetilde{Y} = \mathcal{X}^+ \cap \gamma Z$ is algebraic.

Next under the notation of Theorem 2.3.1, $\widetilde{Y} = \overline{\widetilde{Y}} = \widetilde{F}$ since $\widetilde{Y}$ is algebraic. Hence $\widetilde{Y}$ is weakly special, and so is Y.

Finally if Y is weakly special, then for any $\widetilde{y} \in \mathrm{unif}^{-1}(Y)$ and $\widetilde{Y}$ the irreducible component of $\mathrm{unif}^{-1}(Y)$ which contains $\widetilde{y}$, $\widetilde{Y} = \widetilde{F} = N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{y}$ by Theorem 2.3.1, and hence $Y = \mathrm{unif}(N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{y})$. □

Chapter 3

The mixed Ax-Lindemann theorem

Convention: In this chapter we always consider a connected mixed Shimura variety S and its uniformization $\mathcal{X}^+ \xrightarrow{\text{unif}} S$. Unless stated otherwise, all closures taken in S are assumed to be Zariski closures and all closures taken in $\mathcal{X}^+$ are assumed to be closures in the archimedean topology. It happens that they often coincide by Chevalley's theorem in the situations we will consider. But for simplicity I will not discuss this.

3.1 Statement of the theorem

3.1.1 Four equivalent statements for Ax-Lindemann

There are several equivalent forms for the Ax-Lindemann theorem. In this section we will give four different statements and explain their equivalence. The proof for this theorem, being the core of this chapter, will be executed in the following sections.

We start from the most usual form of the Ax-Lindemann theorem. It is also this statement that we will prove afterwards.

Theorem 3.1.1. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\text{unif}\colon \mathcal{X}^+ \to S$ *be the uniformization. Let Y be an irreducible algebraic subvariety of S and let $\widetilde{Z}$ be an irreducible algebraic subset of $\mathcal{X}^+$ contained in* $\text{unif}^{-1}(Y)$, *maximal for these properties. Then $\widetilde{Z}$ is weakly special.*

The next statement we give shall be called the *semi-algebraic form of Ax-Lindemann*. In fact this and its direct variant Theorem 3.1.4 are the forms which will be adopted in all the applications in this dissertation. Recall that a connected semi-algebraic subset of $\mathcal{X}^+$ is called **irreducible** if its $\mathbb{R}$-Zariski closure in $\mathcal{X}^\vee$ is an irreducible real algebraic variety. Note that any connected semi-algebraic subset of $\mathcal{X}^+$ has only finitely many irreducible components.

Theorem 3.1.2. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\text{unif}\colon \mathcal{X}^+ \to S$ *be the uniformization. Let Y be an irreducible algebraic subvariety of S and let $\widetilde{Z}$ be a connected irreducible semi-algebraic subset of $\mathcal{X}^+$ contained in* $\text{unif}^{-1}(Y)$, *maximal for these properties. Then $\widetilde{Z}$ is complex analytic and each complex analytic irreducible component of $\widetilde{Z}$ is weakly special.*

The equivalence of Theorem 3.1.1 and Theorem 3.1.2 follows easily from [49, Lemma 4.1], which claims that **maximal connected irreducible semi-algebraic subsets of $\mathcal{X}^+$ which are contained in** $\mathrm{unif}^{-1}(Y)$ **are all algebraic in the sense of Definition 1.3.5** (there is a typo in the proof of [49, Lemma 4.1]: $\mathbb{C}^{2n}$ should be $\mathbb{C}^n$).

The next two forms of Ax-Lindemann have more "equidistributional" taste. Their equivalence with the two statements above is not hard to check (Theorem 3.1.3 with Theorem 3.1.1, Theorem 3.1.4 with Theorem 3.1.2).

Theorem 3.1.3. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\mathrm{unif}\colon \mathcal{X}^+ \to S$ *be the uniformization. Let $\widetilde{Z}$ be any irreducible algebraic subset of $\mathcal{X}^+$. Then* $\overline{\mathrm{unif}(\widetilde{Z})}$ *is weakly special.*

Theorem 3.1.4. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\mathrm{unif}\colon \mathcal{X}^+ \to S$ *be the uniformization. Let $\widetilde{Z}$ be any semi-algebraic subset of $\mathcal{X}^+$. Then every irreducible component of* $\overline{\mathrm{unif}(\widetilde{Z})}$ *is weakly special.*

Let us explain now why Theorem 3.1.1 implies Theorem 3.1.3. Let S, $(P, \mathcal{X}^+)$ and $\widetilde{Z}$ be as in Theorem 3.1.3. Let $Y := \overline{\mathrm{unif}(\widetilde{Z})}$ and let $\widetilde{W}$ be an irreducible algebraic subset of $\mathcal{X}^+$ which contains $\widetilde{Z}$ and is contained in $\mathrm{unif}^{-1}(Y)$, maximal for these properties. Such a $\widetilde{W}$ exists by, for example, dimension reason. Then $Y = \overline{\mathrm{unif}(\widetilde{W})}$ and $\widetilde{W}$ is a maximal irreducible algebraic subset of $\mathcal{X}^+$ which is contained in $\mathrm{unif}^{-1}(Y)$. Theorem 3.1.1 then implies that $\widetilde{W}$ is weakly special. Hence $\mathrm{unif}(\widetilde{W})$ is an irreducible subvariety of S by Corollary 2.3.3. So $Y = \overline{\mathrm{unif}(\widetilde{W})} = \mathrm{unif}(\widetilde{W})$ is weakly special since $\widetilde{W}$ is weakly special in $\mathcal{X}^+$. Theorem 3.1.2 implies Theorem 3.1.4 by a similar argument because any semi-algebraic subset of $\mathcal{X}^+$ has only finitely many connected irreducible components.

Let us explain now why Theorem 3.1.3 implies Theorem 3.1.1. Let S, $(P, \mathcal{X}^+)$, Y and $\widetilde{Z}$ be as in Theorem 3.1.1. Then Theorem 3.1.3 tells us that $\overline{\mathrm{unif}(\widetilde{Z})}$ is a weakly special subvariety of S, which we shall call Y_0. By assumption of Y and $\widetilde{Z}$, Y_0 is a subvariety of Y. Let $\widetilde{Y_0}$ be the complex analytic irreducible component of $\mathrm{unif}^{-1}(Y_0)$ containing $\widetilde{Z}$. Then $\widetilde{Y_0}$ is irreducible algebraic by Corollary 2.3.3. But then the maximality assumption on $\widetilde{Z}$ tells us that $\widetilde{Z} = \widetilde{Y_0}$. Hence $\widetilde{Z}$ is weakly special. Theorem 3.1.4 implies Theorem 3.1.2 by a similar argument.

3.1.2 Ax-Lindemann for the unipotent part

In this subsection we state Ax-Lindemann for the unipotent part. There is nothing new in the statement, but it is better to state it here because we will prove it separately in §3.4.

Given a connected mixed Shimura variety S, let S_G be its pure part. We have a projection $S \xrightarrow{[\pi]} S_G$. For any point $b \in S_G$, denote by E the fiber S_b. Suppose that S is associated with the mixed Shimura datum $(P, \mathcal{X}^+)$, which can be further assumed to satisfy $P = \mathrm{MT}(\mathcal{X}^+)$ by Proposition 1.1.19. Let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma \backslash \mathcal{X}^+$ be the uniformization. Now $E = S_b \simeq \Gamma_W \backslash W(\mathbb{R})U(\mathbb{C})$ with the complex structure determined by $b \in S_G$ ($E = S_b = \Gamma_W \backslash W(\mathbb{C})/F_b^0 W(\mathbb{C})$), where $\Gamma_W := \Gamma \cap W(\mathbb{Q})$.

By abuse of notation we denote by $\mathrm{unif}\colon W(\mathbb{R})U(\mathbb{C}) = W(\mathbb{C})/F_b^0 W(\mathbb{C}) \to E$ for the uniformization of E. It is then the restriction of $\mathrm{unif}\colon \mathcal{X}^+ \to S$.

Theorem 3.1.5. *Let Y be an irreducible subvariety of E and let $\widetilde{Z}$ be a maximal irreducible algebraic subvariety which is contained in* $\mathrm{unif}^{-1}(Y)$. *Then $\widetilde{Z}$ is weakly special.*

Proof. If E is an algebraic torus over $\mathbb{C}$, this is a consequence of the Ax-Schanuel theorem [42, Corollary 3.6]. If E is an abelian variety, this is Pila-Zannier [51, pp9, Remark 1]. A proof using volume calculation and points counting method for these two cases can be found in the Appendix of this chapter. The general case will be proved in §3.4. ☐

3.2 Ax-Lindemann Part 1: Outline of the proof

In these three sections, we are going to prove Theorem 3.1.1. The organization of the proof is as follows: the outline of the proof is given in this section. After some preparation, the key proposition (Proposition 3.2.6) leading to the theorem will be stated and exploited (together with Theorem 3.1.5) to finish the proof in Theorem 3.2.8. We prove this key proposition in the next section using Pila-Wilkie's counting theorem and Theorem 3.1.5 will be proved in §3.4.

Now let us fix some notation which will be used through the whole proof:

Notation 3.2.1. *Consider the following diagram:*

$$\begin{array}{ccc} \mathcal{X}^+ & \xrightarrow{\quad\pi\quad} & \mathcal{X}_G^+ \\ \Big\downarrow \mathrm{unif} & & \Big\downarrow \mathrm{unif}_G \\ S = \Gamma\backslash\mathcal{X}^+ & \xrightarrow{[\pi]} & S_G := \Gamma_G\backslash\mathcal{X}_G^+ \end{array}$$

Now we begin the proof of Theorem 3.1.1. Let us first of all do some reduction:

- Since every point of $\mathcal{X}^+$ is weakly special, we may assume $\dim(\widetilde{Z}) > 0$.
- Let $(Q, \mathcal{Y}^+)$ be the smallest mixed Shimura subdatum of $(P, \mathcal{X}^+)$ s.t $\widetilde{Z} \subset \mathcal{Y}^+$ and let S_Q be the corresponding special subvariety of S. Then $Q = \mathrm{MT}(\mathcal{Y}^+)$ by Proposition 1.1.19(1). If we replace $(P, \mathcal{X}^+)$ by $(Q, \mathcal{Y}^+)$, S by S_Q, $\mathrm{unif}\colon \mathcal{X}^+ \to S$ by $\mathrm{unif}_Q\colon \mathcal{Y}^+ \to S_Q$ and Y by an irreducible

component Y_0 of $Y \cap S_Q$, then $\widetilde{Z}$ is again a maximal irreducible algebraic subset of $\mathrm{unif}_Q^{-1}(Y_0)$. By definition, $\widetilde{Z}$ is weakly special in $\mathcal{X}^+$ iff it is weakly special in $\mathcal{Y}^+$. So we may assume $P = \mathrm{MT}(\mathcal{X}^+)$ and that $\widetilde{Z}$ is Hodge generic.

- Furthermore, let Y_0 by the minimal irreducible subvariety of S such that $\widetilde{Z} \subset \mathrm{unif}^{-1}(Y_0)$, then $\widetilde{Z}$ is still maximal irreducible algebraic in $\mathrm{unif}^{-1}(Y_0)$. Hence we may assume that $Y = Y_0$. In fact it is not hard to see that after this reduction, $Y = \overline{\mathrm{unif}(\widetilde{Z})}$ and $\widetilde{Z}$ is weakly special iff Y is weakly special.
- By the previous reduction, there is a unique complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$ which contains $\widetilde{Z}$. Denote it by $\widetilde{Y}$. Denote by $\widetilde{Y}_G := \pi(\widetilde{Y})$, $Y_G := [\pi](Y)$ and $\widetilde{Z}_G := \pi(\widetilde{Z})$. Remark that by Lemma 1.3.9, $\overline{\widetilde{Z}_G}$ is an algebraic subset of $\mathcal{X}_G^+$.
- Replacing Γ by a subgroup of finite index does not matter for this problem, so we may assume that Γ is neat and $\Gamma \subset P^{\mathrm{der}}(\mathbb{Q})$ (Remark 1.1.13(2)).

Let $\widetilde{F}$ be the smallest weakly special subset containing $\widetilde{Y}$. By Theorem 2.3.1, $\widetilde{F} = N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{z}$ some $\widetilde{z} \in \widetilde{Z} \subset \widetilde{Y}$, where N is the connected algebraic monodromy group associated with Y^{sm} and $U_N := U \cap N$. The set $\widetilde{F}$ is Hodge generic in $(P, \mathcal{X}^+)$ since $\widetilde{Z}$ is, so $N \lhd P$ and $N \lhd P^{\mathrm{der}}$ by Theorem 2.2.4.

Define

$$\Gamma_{\widetilde{Z}} := \{\gamma \in \Gamma | \gamma \cdot \widetilde{Z} = \widetilde{Z}\} \qquad (\text{resp. } \Gamma_{G,\overline{\widetilde{Z}_G}} := \{\gamma_G \in \Gamma_G | \gamma_G \cdot \overline{\widetilde{Z}_G} = \overline{\widetilde{Z}_G}\})$$

and

$$H_{\widetilde{Z}} := (\overline{\Gamma_{\widetilde{Z}}}^{\mathrm{Zar}})^{\circ} \qquad (\text{resp. } H_{\overline{\widetilde{Z}_G}} := (\overline{\Gamma_{G,\overline{\widetilde{Z}_G}}}^{\mathrm{Zar}})^{\circ}).$$

Define $U_{H_{\widetilde{Z}}} := U \cap H_{\widetilde{Z}}$ and $W_{H_{\widetilde{Z}}} := W \cap H_{\widetilde{Z}}$. Both of them are normal in $H_{\widetilde{Z}}$. Then $H_{\widetilde{Z}}$ (resp. $H_{\overline{\widetilde{Z}_G}}$) is the largest connected subgroup of P^{der} (resp. G^{der}) such that $H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})$ (resp. $H_{\overline{\widetilde{Z}_G}}(\mathbb{R})^+$) stabilizes $\widetilde{Z}$ (resp. $\overline{\widetilde{Z}_G}$).

Define $V_{H_{\widetilde{Z}}} := W_{H_{\widetilde{Z}}}/U_{H_{\widetilde{Z}}}$ and $G_{H_{\widetilde{Z}}} := H_{\widetilde{Z}}/W_{H_{\widetilde{Z}}} \hookrightarrow P/W = G$.

The following two lemmas were proved for the pure case in [50] and [29].

Lemma 3.2.2. *The set $\widetilde{Y}$ is stable under $H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})$.*

Proof. Every fiber of $\mathcal{X}^+ \to \mathcal{X}_{P/U}^+$ can be canonically identified with $U(\mathbb{C})$. So it is enough to prove that $\widetilde{Y}$ is stable under $H_{\widetilde{Z}}(\mathbb{R})^+$: If $U_{H_{\widetilde{Z}}}(\mathbb{R})\widetilde{y} \subset \widetilde{Y}$ for $\widetilde{y} \in \widetilde{Y}$, then $U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{y} \subset \widetilde{Y}$ because $\widetilde{Y}$ is complex analytic and $U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{y}$ is the smallest complex analytic subset of $\mathcal{X}^+$ containing $U_{H_{\widetilde{Z}}}(\mathbb{R})\widetilde{y}$.

If not, then since $H_{\widetilde{Z}}(\mathbb{Q})$ is dense (w.r.t. the archimedean topology) in $H_{\widetilde{Z}}(\mathbb{R})^+$, there exists $h \in H_{\widetilde{Z}}(\mathbb{Q})$ such that $h\widetilde{Y} \neq \widetilde{Y}$. The set $\widetilde{Z}$ is contained in $\widetilde{Y} \cap h\widetilde{Y}$ by definition of $H_{\widetilde{Z}}$, and hence contained in a complex analytic irreducible component $\widetilde{Y}'$ of it.

Consider the Hecke operator T_h. Then $T_h(Y) = \mathrm{unif}(h \cdot \mathrm{unif}^{-1}(Y))$. Hence

$$Y \cap T_h(Y) = \mathrm{unif}(\mathrm{unif}^{-1}(Y) \cap (h \cdot \mathrm{unif}^{-1}(Y))).$$

On the other hand, $T_h(Y)$ is equidimensional of the same dimension as Y by definition, hence by reason of dimension, $h\widetilde{Y}$ is an irreducible component of $\mathrm{unif}^{-1}(T_h(Y)) = \Gamma h \Gamma \widetilde{Y}$. So $\mathrm{unif}(h\widetilde{Y})$ is an irreducible component of $T_h(Y)$.

Since $\widetilde{Y}'$ is a complex analytic irreducible component of $\widetilde{Y} \cap h\widetilde{Y}$, it is also a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y) \cap (h\widetilde{Y}) = \Gamma\widetilde{Y} \cap h\widetilde{Y}$. So $Y' := \mathrm{unif}(\widetilde{Y}')$ is a complex analytic irreducible component of $Y \cap \mathrm{unif}(h\widetilde{Y})$. So Y' is a complex analytic irreducible component of $Y \cap T_h(Y)$, and hence is algebraic since $Y \cap T_h(Y)$ is.

Since $h\widetilde{Y} \neq \widetilde{Y}$ and Y is irreducible, $\dim(Y') < \dim(Y)$. But $\widetilde{Z} \subset \widetilde{Y} \cap h\widetilde{Y} \subset \mathrm{unif}^{-1}(Y')$. This contradicts the minimality of Y. □

Lemma 3.2.3. $H_{\widetilde{Z}} \lhd N$.

Proof. We have $\widetilde{Z} \subset \widetilde{F} = N(\mathbb{R})^+ U_N(\mathbb{C})\widetilde{z}$ for some $\widetilde{z} \in \widetilde{Z}$, so the image of $\widetilde{Z}$ under the morphism

$$(P, \mathcal{X}^+) \to (P, \mathcal{X}^+)/N$$

is a point. But $H_{\widetilde{Z}}/(H_{\widetilde{Z}} \cap N)$ stabilizes this point which is Hodge generic (since $\widetilde{F}$ is Hodge generic in $\mathcal{X}^+$), and therefore is trivial by Remark 2.2.6. So $H_{\widetilde{Z}} < N$.

Let H' be the algebraic group generated by $\gamma^{-1} H_{\widetilde{Z}} \gamma$ for all $\gamma \in \Gamma_{Y^{\mathrm{sm}}}$, where $\Gamma_{Y^{\mathrm{sm}}}$ is the monodromy group of Y^{sm}. Since H' is invariant under conjugation by $\Gamma_{Y^{\mathrm{sm}}}$, it is invariant under $\overline{\Gamma_{Y^{\mathrm{sm}}}}^{\mathrm{Zar}}$, therefore invariant under conjugation by N.

By Lemma 3.2.2, $\widetilde{Y}$ is invariant under $H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})$. On the other hand, $\widetilde{Y}$ is also invariant under $\Gamma_{Y^{\mathrm{sm}}}$ by definition. So $\widetilde{Y}$ is invariant under the action of $H'(\mathbb{R})^+ U_{H'}(\mathbb{C})$ where $U_{H'} := U \cap H'$. Since $H'(\mathbb{R})^+ U_{H'}(\mathbb{C})\widetilde{Z}$ is semi-algebraic, there exists an irreducible algebraic subset of $\mathcal{X}^+$, say $\widetilde{E}$, which contains $H'(\mathbb{R})^+ U_{H'}(\mathbb{C})\widetilde{Z}$ and is contained in $\widetilde{Y}$ by [49, Lemma 4.1]. Now $\widetilde{Z} \subset \widetilde{E} \subset \widetilde{Y}$, so $\widetilde{Z} = \widetilde{E} = H'(\mathbb{R})^+ U_{H'}(\mathbb{C})\widetilde{Z}$ by maximality of $\widetilde{Z}$, and therefore $H' = H_{\widetilde{Z}}$ by definition of $H_{\widetilde{Z}}$. So $H_{\widetilde{Z}}$ is invariant under conjugation by N. Since $H_{\widetilde{Z}} < N$, $H_{\widetilde{Z}}$ is normal in N. □

Corollary 3.2.4.

$$G_{H_{\widetilde{Z}}},\ H_{\overline{\widetilde{Z}_G}} \lhd G^{\mathrm{der}} \ \textit{and}\ G_{H_{\widetilde{Z}}} \lhd H_{\overline{\widetilde{Z}_G}}.$$

Proof. We have $G_{H_{\widetilde{Z}}} \lhd G_N \lhd G^{\mathrm{der}}$, and so $G_{H_{\widetilde{Z}}} \lhd G^{\mathrm{der}}$ since all the three groups are reductive.

Working with $((G, \mathcal{X}_G^+),\ \overline{Y_G},\ \overline{\widetilde{Z}_G})$ instead of $((P, \mathcal{X}^+),\ Y,\ \widetilde{Z})$, we can prove (similar to Lemma 3.2.3) that $H_{\overline{\widetilde{Z}_G}} \lhd G_N$. Hence $H_{\overline{\widetilde{Z}_G}} \lhd G^{\mathrm{der}}$ by the same reason for $G_{H_{\widetilde{Z}}}$.

By definition $G_{H_{\widetilde{Z}}} < H_{\overline{\widetilde{Z}_G}}$. So $G_{H_{\widetilde{Z}}} \lhd H_{\overline{\widetilde{Z}_G}}$ since $G_{H_{\widetilde{Z}}} \lhd G^{\mathrm{der}}$. $\square$

So far the proof looks similar to the pure case. From now on it will be quite different. For the readers' convenience, we list here some differences between the proof of Ax-Lindemann for mixed Shimura varieties and for the pure case:

- We shall prove that $\widetilde{Z}$ is an $H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})$-orbit. To prove this, it suffices to prove $\dim H_{\widetilde{Z}} > 0$ when S is a pure Shimura variety. However this is far from enough for the mixed case, since this does not exclude the naive counterexample when $\dim \widetilde{Z}_G > 0$ but $H_{\widetilde{Z}}$ is unipotent. To overcome it, we should at least prove $\dim G_{H_{\widetilde{Z}}} > 0$. In fact we shall directly prove $G_{H_{\widetilde{Z}}} = H_{\overline{\widetilde{Z}_G}}$ (Proposition 3.2.6). This equality is not obvious because, as appears in the proof of Lemma 3.2.5, there is no reason a priori why $\overline{\widetilde{Z}_G}$, which is obviously algebraic in $\mathrm{unif}^{-1}(Y_G)$, should be maximal for this property. If one could prove direcly this is the case, then Klingler-Ullmo-Yafaev [29, Theorem 1.3] would give directly the result.

- As mentioned in the Introduction, we shall make essential use of the "family" version of Pila-Wilkie's theorem (Remark 3.3.4);

- If $P = G$ is reductive, then $H_{\widetilde{Z}} \lhd N \lhd P$ implies directly $H_{\widetilde{Z}} \lhd P$. This is obviously false when P is not reductive.

- For a general mixed Shimura variety S, the fiber of $S \xrightarrow{[\pi]} S_G$ is not necessarily an algebraic group (Lemma 2.1.1), hence not a semi-abelian variety. We do not have Ax-Lindemann for the fiber for this case. Thus we should execute a proof of Ax-Lindemann for the fiber. As the readers will see in §3.4, the proof of this case calls for much more careful study of $\widetilde{Z}$. First of all, when doing the estimate and using the family version of Pila-Wilkie for the fiber (*Step I*), we should introduce a seemingly strange subgroup which serves as G_N in the section. The reason for this will be explained in Remark 3.4.1. Secondly, to prove that $W_{H_{\widetilde{Z}}}$ is normal in W is not trivial, and the key to the solution (*Step IV*) is a well-known fact: any holomorphic morphism from a complex abelian variety to an algebraic torus over $\mathbb{C}$ is trivial.

Before proceeding, we prove the following lemma:

Lemma 3.2.5. *1. $\overline{\widetilde{Y}_G}$ is weakly special. Hence $\overline{\widetilde{Y}_G} = G_N(\mathbb{R})^+ \widetilde{z}_G$ for any point $\widetilde{z}_G \in \widetilde{Z}_G$;*

2. $\overline{\mathrm{unif}_G(\widetilde{Z}_G)} = \overline{Y_G}$.

Proof. 1. Let $\widetilde{Z}'$ be an irreducible algebraic subset of $\mathcal{X}_G^+$ which contains $\overline{\widetilde{Z}_G}$ and is contained in $\mathrm{unif}^{-1}(\overline{Y_G})$, maximal for these properties. By [29, Theorem 1.3], $Z' := \mathrm{unif}_G(\widetilde{Z}')$ is weakly special, and therefore Zariski closed by definition. Now $\widetilde{Z} \subset \pi^{-1}(\widetilde{Z}') \cap \mathrm{unif}^{-1}(Y)$. However,

$$\mathrm{unif}(\pi^{-1}(\widetilde{Z}') \cap \mathrm{unif}^{-1}(Y)) = \mathrm{unif}(\pi^{-1}(\widetilde{Z}')) \cap Y = [\pi]^{-1}(Z') \cap Y.$$

Then we must have $Y \subset [\pi]^{-1}(Z')$ since Y is the minimal irreducible closed subvariety of S such that $\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$. Therefore $\overline{Y_G} \subset Z'$. But $Z' \subset \overline{Y_G}$ by definition of Z', so $Z' = \overline{Y_G}$. This means that $\overline{Y_G}$ is weakly special.

2. Let $Y' := \overline{\mathrm{unif}_G(\widetilde{Z}_G)}$, then $\overline{\widetilde{Z}_G} \subset \mathrm{unif}_G^{-1}(Y')$. Then $\widetilde{Z} \subset \pi^{-1}(\mathrm{unif}_G^{-1}(Y')) = \mathrm{unif}^{-1}([\pi]^{-1}(Y'))$, and so

$$\widetilde{Z} \subset \mathrm{unif}^{-1}([\pi]^{-1}(Y')) \cap \mathrm{unif}^{-1}(Y) = \mathrm{unif}^{-1}([\pi]^{-1}(Y') \cap Y).$$

Hence there exists an irreducible component Y'' of $[\pi]^{-1}(Y') \cap Y$ such that $\widetilde{Z} \subset \mathrm{unif}^{-1}(Y'')$. But

$$[\pi](Y'') \subset [\pi]([\pi]^{-1}(Y') \cap Y) = Y' \cap Y_G,$$

so $\dim([\pi](Y'')) \leqslant \dim(Y' \cap Y_G)$. If $Y' \neq \overline{Y_G}$, then $\dim(Y' \cap Y_G) < \dim(Y_G)$ and therefore $\dim(Y'') < \dim(Y)$, which contradicts the minimality of Y. So $Y' = \overline{Y_G}$.

□

Proposition 3.2.6 (key proposition)**.** *The set $\overline{\widetilde{Z}_G}$ is weakly special and $G_{H_{\widetilde{Z}}} = H_{\overline{\widetilde{Z}_G}}$. In other words,*

$$\overline{\widetilde{Z}_G} = G_{H_{\widetilde{Z}}}(\mathbb{R})^+ \widetilde{z}_G$$

for any point $\widetilde{z}_G \in \widetilde{Z}_G$.

Now let us show how this proposition together with Theorem 3.1.5 implies Theorem 3.1.1. Before proceeding to the final argument, we shall prove the following group theoretical lemma:

Lemma 3.2.7. *Fixing a Levi decomposition $H_{\widetilde{Z}} = W_{H_{\widetilde{Z}}} \rtimes G_{H_{\widetilde{Z}}}$, there exists a compatible Levi decomposition $P = W \rtimes G$.*

Proof. Suppose that the fixed Levi decomposition of $H_{\widetilde{Z}}$ is given by $s_1 \colon G_{H_{\widetilde{Z}}} \to H_{\widetilde{Z}}$. Define $P_* := \pi^{-1}(G_{H_{\widetilde{Z}}})$, then $H_{\widetilde{Z}} < P_*$. Now choose any Levi decomposition $P = W \rtimes G$ defined by $s_2 \colon G \to P$. Then $G_{H_{\widetilde{Z}}}$, being a subgroup of G, is realized as a subgroup of P via s_2. Hence s_2 induces a Levi-decomposition $P_* = W \rtimes^{s_2} G_{H_{\widetilde{Z}}}$. We have thus a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W_{H_{\widetilde{Z}}} & \longrightarrow & H_{\widetilde{Z}} & \overset{s_1}{\rightleftarrows} & G_{H_{\widetilde{Z}}} & \longrightarrow & 1 \\ & & \cap\downarrow & & \downarrow & & =\downarrow & & \\ 1 & \longrightarrow & W & \longrightarrow & P_* & \overset{s_1}{\rightleftarrows} & G_{H_{\widetilde{Z}}} & \longrightarrow & 1 \end{array},$$

where the morphism s_1 in the second line is induced by the one in the first line. Now s_1, s_2 define two Levi decompositions of P_*. They differ by the conjugation by an element w_0 of $W(\mathbb{Q})$ by [55, Theorem 2.3]. So replacing s_2 by its conjugation by w_0 we can find a Levi decomposition of P which is compatible with the fixed $H_{\widetilde{Z}} = W_{H_{\widetilde{Z}}} \rtimes G_{H_{\widetilde{Z}}}$. □

Theorem 3.2.8. *1. $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z}$ for any $\widetilde{z} \in \widetilde{Z}$;*

2. $H_{\widetilde{Z}} \lhd P$.

Hence $\widetilde{Z}$ is weakly special by definition.

Proof. 1. Consider a fibre of $\widetilde{Z}$ over a Hodge-generic point $\widetilde{z}_G \in \widetilde{Z}_G$ such that $\pi|_{\widetilde{Z}}$ is flat at $\widetilde{z}_G$ (such a point exists by [1, §4, Lemma 1.4] and generic flatness). Suppose that $\widetilde{W}$ is an irreducible algebraic component of $\widetilde{Z}_{\widetilde{z}_G}$ such that $\dim(\widetilde{Z}_{\widetilde{z}_G}) = \dim(\widetilde{W})$, then since $\pi|_{\widetilde{Z}}$ is flat at $\widetilde{z}_G$,

$$\dim(\widetilde{Z}) = \dim(\widetilde{Z}_G) + \dim(\widetilde{Z}_{\widetilde{z}_G}) = \dim(\widetilde{Z}_G) + \dim(\widetilde{W}).$$

Consider the set $\widetilde{E} := H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{W}$. It is semi-algebraic (since $\widetilde{W}$ is algebraic and the action of $P(\mathbb{R})^+U(\mathbb{C})$ on $\mathcal{X}^+$ is algebraic). The fact $\widetilde{W} \subset \widetilde{Z}$ implies that $\widetilde{E} \subset \widetilde{Z}$. By [49, Lemma 4.1], there exists an irreducible algebraic subset of $\mathcal{X}^+$, say $\widetilde{E}_{\mathrm{alg}}$, which contains $\widetilde{E}$ and is contained in $\widetilde{Z}$. Now we have by Proposition 3.2.6

$$\pi(\widetilde{E}) = G_{H_{\widetilde{Z}}}(\mathbb{R})^+\widetilde{z}_G = H_{\overline{\widetilde{Z}_G}}(\mathbb{R})^+\widetilde{z}_G = \widetilde{Z}_G$$

and that the $\mathbb{R}$-dimension of every fiber of $\pi|_{\widetilde{E}}$ is at least $\dim_{\mathbb{R}}(\widetilde{W})$. So

$$\dim(\widetilde{E}_{\mathrm{alg}}) \geqslant \dim(\pi(\widetilde{E})) + \dim(\widetilde{W}) = \dim(\widetilde{Z}_G) + \dim(\widetilde{W}) = \dim(\widetilde{Z}).$$

So $\widetilde{E} = \widetilde{Z}$ since $\widetilde{Z}$ is irreducible.

Next let $\widetilde{W}'$ be an irreducible algebraic subset which contains $\widetilde{Z}_{\widetilde{z}_G}$ and is contained in $\mathrm{unif}^{-1}(Y)_{\widetilde{z}_G}$, maximal for these properties. Then $\widetilde{W}'$ is weakly special by Theorem 3.1.5. We have $\widetilde{W}' \subset \widetilde{Y}$ since $\widetilde{Y}$ is an irreducible component of $\pi^{-1}(Y)$. Consider $\widetilde{E}' := H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{W}'$. Then $\widetilde{E}' \subset \widetilde{Y}$ by Lemma 3.2.2. But $\widetilde{E}'$ is semi-algebraic, so by [49, Lemma 4.1], there exists an irreducible algebraic subset of $\mathcal{X}^+$, say $\widetilde{E}'_{\mathrm{alg}}$ which contains $\widetilde{E}'$ and is contained in $\widetilde{Y}$. So $\widetilde{Z} = \widetilde{E} \subset \widetilde{E}'_{\mathrm{alg}} \subset \widetilde{Y}$, and hence $\widetilde{Z} = \widetilde{E}'_{\mathrm{alg}} = \widetilde{E}'$ by the maximality of $\widetilde{Z}$. So $\widetilde{Z}_{\widetilde{z}_G} = \widetilde{W}'$ is weakly special.

Write $\widetilde{Z}_{\widetilde{z}_G} = W'(\mathbb{R})U'(\mathbb{C})\widetilde{z}$ with $W' < W$, $U' = W' \cap U$ and $\widetilde{z} \in \widetilde{Z}_{\widetilde{z}_G}$. Then $W_{H_{\widetilde{Z}}} < W'$. The complex structure of $\pi^{-1}(\widetilde{z}_G)$ comes from $W(\mathbb{R})U(\mathbb{C}) \simeq W(\mathbb{C})/F^0_{\widetilde{z}_G}W(\mathbb{C})$, where $F^0_{\widetilde{z}_G}W(\mathbb{C}) = \exp(F^0_{\widetilde{z}_G}\,\mathrm{Lie}\,W_{\mathbb{C}})$. So the fact that $\widetilde{Z}_{\widetilde{z}_G}$ is a complex subspace of $\pi^{-1}(\widetilde{z}_G)$ implies that W'/U' is a $\mathrm{MT}(\widetilde{z}_G) = G$-module. Hence W' is a G-group.

Define $P' := W'H_{\widetilde{Z}}$, then P' is a subgroup of P since $W' > W_{H_{\widetilde{Z}}}$ and $G_{H_{\widetilde{Z}}}W' = W'$. Now we have

$$\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{Z}_{\widetilde{z}_G} = H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})W'(\mathbb{R})U'(\mathbb{C})\widetilde{z} = P'(\mathbb{R})^+U'(\mathbb{C})\widetilde{z}.$$

So $H_{\widetilde{Z}} = P'$ because $H_{\widetilde{Z}}$ is the largest subgroup of P^{der} such that $H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})$ stabilizes $\widetilde{Z}$. So we have $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z}$.

2. First of all, $U_{H_{\widetilde{Z}}} \lhd P$ by Proposition 1.1.19(2).

   Next consider the complex structure of $\pi^{-1}(\widetilde{z}_G)$. It comes from $W(\mathbb{R})U(\mathbb{C}) \simeq W(\mathbb{C})/F^0_{\widetilde{z}_G}W(\mathbb{C})$. So the fact that $\widetilde{Z}_{\widetilde{z}_G}$ is a complex subspace of $\pi^{-1}(\widetilde{z}_G)$ implies that $V_{H_{\widetilde{Z}}}$ is a $\mathrm{MT}(\widetilde{z}_G) = G$-module. Hence $W_{H_{\widetilde{Z}}}$ is a G-group. Besides, $G_{H_{\widetilde{Z}}} \lhd G$ by Proposition 3.2.6. In particular, $G_{H_{\widetilde{Z}}}$ is reductive.

   Then let us prove $W_{H_{\widetilde{Z}}} \lhd P$. It suffices to prove $W_{H_{\widetilde{Z}}} \lhd W$. For any $\widetilde{z} \in \widetilde{Z}$, we have proved in (1) that $\widetilde{Z}_{\widetilde{z}_G} = W_{H_{\widetilde{Z}}}(\mathbb{R})U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z}$ is weakly special. Hence by Proposition 1.2.4, there is a connected mixed Shimura subdatum $(Q, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$ such that $\widetilde{z} \in \mathcal{Y}^+$ and $W_{H_{\widetilde{Z}}} \lhd Q$. Define W^* to be the G-subgroup (of W) generated by $W_Q := \mathcal{R}_u(Q)$, then $W_{H_{\widetilde{Z}}} \lhd W^*$ since $W_{H_{\widetilde{Z}}}$ is a G-group.

   Fix a Levi decomposition $H_{\widetilde{Z}} = W_{H_{\widetilde{Z}}} \rtimes G_{H_{\widetilde{Z}}}$ and choose a compatible Levi decomposition $P = W \rtimes G$ (as is shown in Lemma 3.2.7). Let P^* be the group generated by GQ, then $\mathcal{R}_u(P^*) = W^*$ and $P^*/W^* = G$. The group P^* defines a connected mixed Shimura datum $(P^*, \mathcal{X}^{*+})$ with $\mathcal{X}^{*+} = P^*(\mathbb{R})^+U^*(\mathbb{C})\widetilde{z}$. Now $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z} \subset \mathcal{X}^{*+}$. But $\widetilde{Z}$ is Hodge generic in $\mathcal{X}^+$ by assumption, hence $P = P^*$ and $W = W^*$. So $W_{H_{\widetilde{Z}}} \lhd W$ and hence $W_{H_{\widetilde{Z}}} \lhd P$.

   Use the notation in §1.1.2.5. We are done if we can prove:

$$\forall u \in U,\ \forall v \in V,\ \textit{and}\ \forall g \in G_{H_{\widetilde{Z}}},\ (u, v, 1)(0, 0, g)(-u, -v, 1) \in H_{\widetilde{Z}}.$$

   By Corollary 1.1.37, there exist decompositions

$$U = U_N \oplus U_N^{\perp} \qquad V = V_N \oplus V_N^{\perp}$$

as G-modules such that G_N acts trivially on $U_N^{\perp}$ and $V_N^{\perp}$. Now

$$\begin{aligned}
&(u, v, 1)(0, 0, g)(-u, -v, 1)\\
&= (u, v, g)(-u, -v, 1)\\
&= (u - g \cdot u, v - g \cdot v, g)\\
&= ((u_N + u_N^{\perp}) - g \cdot (u_N + u_N^{\perp}), (v_N + v_N^{\perp}) - g \cdot (v_N + v_N^{\perp}), g)\\
&= (u_N - g \cdot u_N, v_N - g \cdot v_N, g)\\
&= (u_N, v_N, 1)(0, 0, g)(-u_N, -v_N, 1) \in H_{\widetilde{Z}},
\end{aligned}$$

where the last inclusion follows from Lemma 3.2.3.

□

3.3 Ax-Lindemann Part 2: Estimate

This section is devoted to prove Proposition 3.2.6. The proof uses essentially the "block family" version of Pila-Wilkie's counting theorem [48, Theorem 3.6].

Keep notation and assumptions as in the last section and denote by $\pi\colon (P, \mathcal{X}^+) \to (G, \mathcal{X}_G^+)$. The group $G = Z(G)^{\circ} H_1 ... H_r$ is an almost direct product, where H_i's are non-trivial simple groups and are normal in G. We have a decomposition

$$(G^{\mathrm{ad}}, \mathcal{X}_G^+) \simeq \prod_{i=1}^{r} (H_i^{\mathrm{ad}}, \mathcal{X}_{H,i}^+)$$

by [39, 3.6]. Let $S_G^{\mathrm{ad}} := \Gamma_G^{\mathrm{ad}} \backslash \mathcal{X}_G^+$. Shrinking Γ_G^{ad} if necessary, we may assume $S_G^{\mathrm{ad}} \simeq \prod_{i=1}^{r} S_{H,i}$, where $S_{H,i}$ is a connected pure Shimura variety associated with $(H_i^{\mathrm{ad}}, \mathcal{X}_{H,i}^+)$.

Without loss of generality we may assume $G_N = H_1 ... H_l$. It suffices to prove $H_i < G_{H_{\widetilde{Z}}}$ for each $i = 1, ..., l$. The case $l = 0$ is trivial, so we assume that $l \geqslant 1$. Define $Q_i := \pi^{-1}(H_i)$.

3.3.1 Fundamental set and definability

The goal of this subsection is to prove that there exists $\mathcal{F} \subset \mathcal{X}^+$ a fundamental set for the action of Γ on $\mathcal{X}^+$ such that $\mathrm{unif}|_{\mathcal{F}}$ is definable.

First of all, by the Reduction Lemma (Lemma 1.1.35), it suffices to prove the existence of such a fundamental set for $(P, \mathcal{X}^+)$ pure and $(P, \mathcal{X}^+) = (P_{2g}, \mathcal{X}_{2g}^+)$ (see §3.5.1 for more details). The case where $(P, \mathcal{X}^+)$ is pure is guaranteed by Klingler-Ullmo-Yafaev [29, Theorem 4.1]. Now we prove the case $(P, \mathcal{X}^+) = (P_{2g}, \mathcal{X}_{2g}^+)$.

We draw the following diagram to make the notation more clear:

$$\begin{array}{ccc}
\mathcal{X}_{2g}^+ & \xrightarrow{\pi_{P/U}} & \mathcal{X}_{2g,\mathrm{a}}^+ \\
\Big\downarrow{\scriptstyle \mathrm{unif}} & & \Big\downarrow{\scriptstyle \mathrm{unif}_{P/U}} \\
S & \xrightarrow{[\pi_{P/U}]} & S_{P/U}
\end{array} .$$

In this case, $[\pi_{P/U}]\colon S \to S_{P/U}$ is an algebraic $\mathbb{G}_m$-torsor. By Peterzil-Starchenko [47, Theorem 1.3], there exists a fundamental set $\mathcal{F}_{P/U}$ for the action of Γ/Γ_U on $\mathcal{X}^+_{2g,\mathrm{a}}$ such that $\mathrm{unif}_{P/U}|_{\mathcal{F}_{P/U}}$ is definable (recall that if $g=0$, then $\mathcal{X}^+_{2g}=\mathbb{C}$, $S=\mathbb{C}^*$, $\mathrm{unif}=\exp$ and $S_{P/U}$ is a point). Let us now construct a fundamental set for the action of Γ on $\mathcal{X}^+_{2g}$ such that $\mathrm{unif}|_{\mathcal{F}}$ is definable and $\pi_{P/U}(\mathcal{F})=\mathcal{F}_{P/U}$.

Since any variety over a field is quasi-compact in the Zariski topology, there exists a finite Zariski open covering $\{V_\alpha\}_{\alpha\in\Lambda}$ of $S_{P/U}$ such that $S|_{V_\alpha}\simeq\mathbb{C}^*\times V_\alpha$ and these isomorphisms are algebraic. Define $U_\alpha := S|_{V_\alpha}=[\pi_{P/U}]^{-1}(V_\alpha)$ for every $\alpha\in\Lambda$. Then we have

$$\mathrm{unif}|_{\mathrm{unif}^{-1}(U_\alpha)}\colon\ \mathrm{unif}^{-1}(U_\alpha)\xrightarrow[\varphi]{\sim}U_{2g}(\mathbb{C})\times\mathrm{unif}^{-1}_{P/U}(V_\alpha)\to(\mathbb{C}^*)\times V_\alpha\simeq U_\alpha,$$

where φ is semi-algebraic (Proposition 1.3.3), the last isomorphism is algebraic and the middle morphism is $(\exp,\mathrm{unif}_{P/U}|_{\mathrm{unif}^{-1}_{P/U}}(V_\alpha))$. Let $\mathcal{F}_U := \{s\in\mathbb{C}|\ -1<\mathfrak{Re}(s)<1\}$ and let $\mathcal{F}_\alpha := \varphi^{-1}(\mathcal{F}_U\times\mathcal{F}_{P/U,\alpha})$. Then $\mathrm{unif}|_{\mathcal{F}_\alpha}$ is definable. Now $\mathcal{F}:=\cup\mathcal{F}_\alpha$ (remember that this is a finite union) satisfies the conditions we want.

Now we return to arbitrary $(P,\mathcal{X}^+)$. We have proved the existence of an $\mathcal{F}$ as stated at the beginning of this subsection. Let us choose such an $\mathcal{F}$ more carefully. First of all replace $\mathcal{F}$ by $\gamma\mathcal{F}$ if necessary to make sure $\mathcal{F}\cap\widetilde{Z}\neq\emptyset$. Next define $\mathcal{F}_G:=\pi(\mathcal{F})\subset\mathcal{X}^+_G\simeq\prod_{i=1}^r\mathcal{X}^+_{H,i}$. Denote by q_i the i-th projection and $\mathcal{F}_{H,i}:=q_i(\mathcal{F}_G)$. There exist some $\gamma_1=1,...,\gamma_s\in\Gamma_G<\Gamma$ such that $\prod_{i=1}^r\mathcal{F}_{H,i}\subset\cup_{j=1}^s\gamma_j\mathcal{F}_G$. Consider

$$\mathcal{F}' := (\bigcup_{j=1}^s\gamma_j\mathcal{F})\cap\pi^{-1}(\prod_{i=1}^r\mathcal{F}_{H,i}),$$

then $\mathcal{F}'$ is a fundamental set for the action of Γ on $\mathcal{X}^+$ and $\mathrm{unif}|_{\mathcal{F}'}$ is definable. Furthermore, $\pi(\mathcal{F}')=\prod_{i=1}^r\mathcal{F}_{H,i}$ and $\mathcal{F}_{H,i}=q_i\pi(\mathcal{F}')$. We still have $\mathcal{F}'\cap\widetilde{Z}\neq\emptyset$ since $\mathcal{F}\subset\mathcal{F}'$. Now replace $\mathcal{F}$ by $\mathcal{F}'$.

3.3.2 Counting points and conclusion

We shall work from now on with an $\mathcal{F}$ satisfying the conditions in the last paragraph of the previous subsection. By Lemma 3.2.5, $\overline{\widetilde{Y}_G}=\prod_{i=1}^l H_i(\mathbb{R})^+\widetilde{z}_G$. Fix a point $\widetilde{z}\in\mathcal{F}\cap\widetilde{Z}$. Define the following Shimura morphisms for each $i=1,...,l$

$$\begin{array}{ccc} (G,\mathcal{X}^+_G) & \xrightarrow{p_i} & (G_i,\mathcal{X}^+_{G,i}) := (G^{\mathrm{ad}},\mathcal{X}^+_G)/\prod_{j\neq i}H_j^{\mathrm{ad}} \\ \downarrow{\scriptstyle \mathrm{unif}_G} & & \downarrow{\scriptstyle \mathrm{unif}_{G,i}} \\ S_G & \xrightarrow{[p_i]} & S_{G,i} \end{array}.$$

Fix $i \in \{1, ..., l\}$. Define $\widetilde{Y}_{G,i} := p_i(\widetilde{Y}_G) = H_i^{\mathrm{ad}}(\mathbb{R})^+\pi_i(\widetilde{z}_G)$, $\widetilde{Z}_{G,i} := p_i(\widetilde{Z}_G)$ and $Y_{G,i} := [p_i](Y_G)$, then $\mathrm{unif}_{G,i}(\widetilde{Z}_{G,i})$ is Zariski dense in $\overline{Y_{G,i}}$ by Lemma 3.2.5. If $\dim(\widetilde{Z}_{G,i}) = 0$, then $\widetilde{Z}_{G,i}$ is a finite set of points since it is algebraic. But then $\mathrm{unif}_{G,i}(\widetilde{Z}_{G,i})$, and hence $\overline{Y_{G,i}} = \overline{\mathrm{unif}_{G,i}(\widetilde{Z}_{G,i})}$ is also a finite set of points. So $\dim(Y_{G,i}) = 0$, which contradicts $\widetilde{Y}_{G,i} = H_i^{\mathrm{ad}}(\mathbb{R})^+\pi_i(\widetilde{z}_G)$. To sum it up, $\dim(\widetilde{Z}_{G,i}) > 0$. For further convenience, we will denote by $\pi_i := p_i \circ \pi$.

Take an algebraic curve $C_{G,i} \subset \widetilde{Z}_{G,i}$ passing through $\pi_i(\widetilde{z})$. Now $\pi_i(\widetilde{Z} \cap \pi_i^{-1}(C_{G,i})) = \widetilde{Z}_{G,i} \cap C_{G,i} = C_{G,i}$, and hence there exists an algebraic curve $C \subset \widetilde{Z} \cap \pi_i^{-1}(C_{G,i})$ passing through $\widetilde{z}$ such that $\dim(\pi_i(C)) = 1$.

Let $\mathcal{F}_{G,i} := p_i(\mathcal{F}_G)$, then it is a fundamental set of $\mathrm{unif}_{G,i}$ and $\mathrm{unif}_{G,i}|_{\mathcal{F}_{G,i}}$ is definable. We define for any irreducible semi-algebraic subvariety A (resp. $A_{G,i}$) of $\mathrm{unif}^{-1}(Y)$ (resp. $\mathrm{unif}_{G,i}^{-1}(\overline{Y_{G,i}})$) the following sets: define

$$\Sigma^{(i)}(A) := \{g \in Q_i(\mathbb{R}) | \dim(gA \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F}) = \dim(A)\}$$
$$(\text{resp. } \Sigma_G^{(i)}(A_{G,i}) := \{g \in H_i^{\mathrm{ad}}(\mathbb{R}) | \dim(gA_{G,i} \cap \mathrm{unif}_{G,i}^{-1}(\overline{Y_{G,i}}) \cap \mathcal{F}_{G,i}) = \dim(A_{G,i})\})$$

and

$$\Sigma'^{(i)}(A) := \{g \in Q_i(\mathbb{R}) | g^{-1}\mathcal{F} \cap A \neq \emptyset\}$$
$$(\text{resp. } \Sigma_G'^{(i)}(A_{G,i}) := \{g \in H_i^{\mathrm{ad}}(\mathbb{R}) | g^{-1}\mathcal{F}_{G,i} \cap A_{G,i} \neq \emptyset\}).$$

Then $\Sigma^{(i)}(A)$ and $\Sigma_G^{(i)}(A_{G,i})$ are by definition definable. Let $\Gamma_{G,i}^{\mathrm{ad}} := p_i(\Gamma_G^{\mathrm{ad}})$.

Lemma 3.3.1. *$\Sigma'^{(i)}(A) \cap \Gamma = \Sigma^{(i)}(A) \cap \Gamma$ (resp. $\Sigma_G'^{(i)}(A_{G,i}) \cap \Gamma_{G,i}^{\mathrm{ad}} = \Sigma_G^{(i)}(A_{G,i}) \cap \Gamma_{G,i}^{\mathrm{ad}}$).*

Proof. The proof, which we include for completeness, is the same as [67, Lemma 5.2]. First of all $\Sigma^{(i)}(A) \cap \Gamma \subset \Sigma'^{(i)}(A) \cap \Gamma$ by definition. Conversely for any $\gamma \in \Sigma'^{(i)}(A) \cap \Gamma$, $\gamma^{-1}\mathcal{F} \cap A$ contains an open subspace of A since $\mathcal{F}$ is by choice open in $\mathcal{X}^+$. Hence $\gamma A \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F} = \gamma A \cap \mathcal{F}$ contains an open subspace of γA which must be of dimension $\dim(A)$. Hence $\gamma \in \Sigma^{(i)}(A) \cap \Gamma$. The proof for $A_{G,i}$ is the same. $\square$

This lemma implies

$$\Sigma^{(i)}(C) \cap \Gamma = \Sigma'^{(i)}(C) \cap \Gamma \subset \Sigma'^{(i)}(\widetilde{Z}) \cap \Gamma = \Sigma^{(i)}(\widetilde{Z}) \cap \Gamma$$
$$(\text{resp. } \Sigma_G^{(i)}(C_{G,i}) \cap \Gamma_{G,i}^{\mathrm{ad}} = \Sigma_G'^{(i)}(C_{G_i}) \cap \Gamma_{G,i}^{\mathrm{ad}} \subset \Sigma_G'^{(i)}(\overline{\widetilde{Z}_{G,i}}) \cap \Gamma_{G,i}^{\mathrm{ad}} = \Sigma^{(i)}(\overline{\widetilde{Z}_{G,i}}) \cap \Gamma_{G,i}^{\mathrm{ad}}) \quad (3.3.1)$$

Lemma 3.3.2. $\pi_i(\Gamma \cap \Sigma^{(i)}(C)) = \Gamma_{G,i}^{\mathrm{ad}} \cap \Sigma_G^{(i)}(C_{G,i})$.

Proof. By Lemma 3.3.1, it suffices to prove $\pi_i(\Gamma \cap \Sigma'^{(i)}(C)) = \Gamma_{G,i}^{\mathrm{ad}} \cap \Sigma_G'^{(i)}(C_{G,i})$. The inclusion $\subset$ is clear by definition. For the other inclusion, $\forall \gamma_{G,i} \in \Gamma_{G,i}^{\mathrm{ad}} \cap \Sigma_G'^{(i)}(C_{G,i})$, $\exists c_{G,i} \in C_{G,i}$ such that $\gamma_{G,i} \cdot c_{G,i} \in \mathcal{F}_{G,i}$.

Take a point $c \in C$ such that $\pi_i(c) = c_{G,i}$ and define $c_G := \pi(c) \in \mathcal{X}_G^+$. Suppose that under the decomposition

$$(G^{\mathrm{ad}}, \mathcal{X}_G^+) \simeq \prod_{i=1}^{r} (H_i^{\mathrm{ad}}, \mathcal{X}_{H,i}^+)$$

of [39, 3.6], $c_G = (c_{G,1}, ..., c_{G,r})$. Then by choice of $\mathcal{F}_G$, there exists $\gamma'_G \in \Gamma_G^{\mathrm{ad}}$ whose i-th coordinate is precisely the $\gamma_{G,i}$ in the last paragraph such that $\gamma'_G \cdot c_G \in \mathcal{F}_G$.

Let $\gamma_G \in \Gamma_G$ be such that its image under $\Gamma_G \to \Gamma_G^{\mathrm{ad}}$ is γ'_G, then $\gamma_G \cdot c \in \pi^{-1}(\mathcal{F}_G)$. Therefore there exist $\gamma_V \in \Gamma_V$, $\gamma_U \in \Gamma_U$ such that $(\gamma_U, \gamma_V, \gamma_G)c \in \mathcal{F}$. Denote by $\gamma = (\gamma_U, \gamma_V, \gamma_G)$, then $\gamma \in \Gamma \cap \Sigma'^{(i)}(C)$ and $\pi_i(\gamma) = \gamma_{G,i}$. □

For $T > 0$, define

$$\Theta_G^{(i)}(C_{G,i}, T) := \{\gamma_G \in \Gamma_{G,i}^{\mathrm{ad}} \cap \Sigma_G^{(i)}(C_{G,i}) | H(\gamma_G) \leqslant T\}.$$

Proposition 3.3.3. *There exists a constant $\delta > 0$ s.t. for all $T \gg 0$, $|\Theta_G^{(i)}(C_{G,i}, T)| \geqslant T^\delta$.*

Proof. This follows directly from [29, Theorem 1.3] applied to $((G_i, \mathcal{X}_{G,i}^+), S_{G,i}, \overline{\widetilde{Z}_{G,i}})$. □

Let us prove how these facts imply $H_i < G_{H_{\widetilde{Z}}}$.

Take a faithful representation $G^{\mathrm{ad}} \hookrightarrow \mathrm{GL}_n$ which sends Γ_G^{ad} to $\mathrm{GL}_n(\mathbb{Z})$. Consider the definable set $\Sigma_G^{(i)}(C_{G,i})$. By the theorem of Pila-Wilkie ([48, Theorem 3.6]), there exist $J = J(\delta)$ definable block families

$$B^j \subset \Sigma_G^{(i)}(C_{G,i}) \times \mathbb{R}^l, \qquad j = 1, ..., J$$

and $c = c(\delta) > 0$ such that for all $T \gg 0$, $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$ is contained in the union of at most $cT^{\delta/4n}$ definable blocks of the form B_y^j $(y \in \mathbb{R}^l)$. By Proposition 3.3.3, there exist a $j \in \{1, ..., J\}$ and a block $B_{G,i} := B_{y_0}^j$ of $\Sigma_G^{(i)}(C_{G,i})$ containing at least $T^{\delta/4n}$ elements of $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$.

Let $\Sigma^{(i)} := \Sigma^{(i)}(C) \cap \Sigma^{(i)}(\widetilde{Z})$, which is by definition a definable set. Consider $X^j := (\pi_i \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma^{(i)} \times \mathbb{R}^l)$, which is a definable family since π_i is algebraic.

By [69, Ch. 3, 3.6], there exists a number $n_0 > 0$ such that each fibre X_y^j has at most n_0 connected components. So the definable set $\pi_i^{-1}(B_{G,i}) \cap \Sigma^{(i)}$ has at most n_0 connected components. Now

$$\pi_i(\pi_i^{-1}(B_{G,i}) \cap \Sigma^{(i)} \cap \Gamma) = B_{G,i} \cap \pi_i(\Sigma^{(i)}(C) \cap \Gamma) = B_{G,i} \cap \Sigma_G^{(i)}(C_{G,i}) \cap \Gamma_{G,i}^{\mathrm{ad}} = B_{G,i} \cap \Gamma_{G,i}^{\mathrm{ad}}$$

by (3.3.1) and Lemma 3.3.2. So there exists a connected component B of $\pi_i^{-1}(B_{G,i}) \cap \Sigma^{(i)}$ such that $\pi_i(B \cap \Gamma)$ contains at least $T^{\delta/4n}/n_0$ elements of $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$.

We have $B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ since $\Sigma^{(i)}(\widetilde{Z})\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ by analytic continuation, and $\widetilde{Z} \subset \sigma^{-1}B\widetilde{Z}$ for any $\sigma \in B \cap \Gamma$. But B is connected, and therefore $\sigma^{-1}B\widetilde{Z} = \widetilde{Z}$ by maximality of $\widetilde{Z}$ and [49, Lemma 4.1]. So $\forall \sigma \in B \cap \Gamma$,

$$B \subset \sigma \, \mathrm{Stab}_{Q_i(\mathbb{R})}(\widetilde{Z}).$$

Fix a $\gamma_0 \in B \cap \Gamma$ such that $\pi_i(\gamma_0) \in \Theta_G^{(i)}(C_{G,i}, T^{1/2n})$. We have already shown that $\pi_i(B\cap\Gamma)$ contains at least $T^{\delta/4n}/n_0$ elements of $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$. For any $\gamma'_{G,i} \in \pi_i(B\cap\Gamma)\cap\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$, let γ' be one of its pre-images in $B\cap\Gamma$. Then $\gamma := \gamma'^{-1}\gamma_0$ is an element of $\Gamma\cap\mathrm{Stab}_{Q_i(\mathbb{R})}(\widetilde{Z}) = \Gamma_{\widetilde{Z}}\cap Q_i(\mathbb{R})$ such that $H(\pi_i(\gamma)) \ll T^{1/2}$. Therefore for $T \gg 0$, $\pi_i(\Gamma_{\widetilde{Z}})\cap H_i^{\mathrm{ad}}(\mathbb{R})$ contains at least $T^{\delta/4n}/n_0$ elements $\gamma_{G,i}$ such that $H(\gamma_{G,i}) \leqslant T$. Hence $\dim(\pi_i(H_{\widetilde{Z}})\cap H_i^{\mathrm{ad}}) > 0$ since $\pi_i(H_{\widetilde{Z}}) \cap H_i^{\mathrm{ad}}$ contains infinitely many rational points. But $\pi_i(H_{\widetilde{Z}}) = p_i\pi(H_{\widetilde{Z}}) = p_i(G_{H_{\widetilde{Z}}})$ by definition. So $H_i^{\mathrm{ad}} < p_i(G_{H_{\widetilde{Z}}})$ since H_i^{ad} is simple and $p_i(G_{H_{\widetilde{Z}}}) \cap H_i^{\mathrm{ad}} \lhd H_i^{\mathrm{ad}}$ by Corollary 3.2.4.

As a normal subgroup of G_N, $G_{H_{\widetilde{Z}}}$ is the almost direct product of some H_j's ($j = 1, ..., l$). So $H_i^{\mathrm{ad}} < p_i(G_{H_{\widetilde{Z}}})$ implies $H_i < G_{H_{\widetilde{Z}}}$. Now we are done.

Remark 3.3.4. *In the proof of the pure case by Klingler-Ullmo-Yafaev [29], it suffices to use a non-family version of Pila-Wilkie ([29, Theorem 6.1]). However this is not enough for our proof, since otherwise the n_0 would depend on T. Hence it is important to use a family version of Pila-Wilkie ([48, Theorem 3.6]).*

3.4 Ax-Lindemann Part 3: The unipotent part

We prove in this section Theorem 3.1.5.

We use the same notation as the first paragraph of §2.1 as well as the first paragraph of §3.1.2. Assume $\dim_{\mathbb{C}} T = m$ and $\dim_{\mathbb{C}} A = n$.

Proof of Theorem 3.1.5. First of all we may assume that $\widetilde{Z}$ is of positive dimension since every point is a weakly special subvariety of dimension 0. For any fundamental set $\mathcal{F}$ of the action of Γ_W on $W(\mathbb{R})U(\mathbb{C})$, define

$$\Sigma(\widetilde{Z}) := \{g \in W(\mathbb{R}) |\dim(g\widetilde{Z} \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F}) = \dim(\widetilde{Z})\}$$

and

$$\Sigma'(\widetilde{Z}) := \{g \in W(\mathbb{R}) | g^{-1}\mathcal{F} \cap \widetilde{Z} \neq \emptyset\},$$

then by Lemma 3.3.1,

$$\Sigma(\widetilde{Z}) \cap \Gamma_W = \Sigma'(\widetilde{Z}) \cap \Gamma_W \qquad (3.4.1)$$

Let $\Gamma_U := \Gamma \cap U(\mathbb{Q})$ and let $\Gamma_V := \Gamma_W/\Gamma_U$.

$\boxed{Case\ i:\ E{=}A.}$ This is [51, Theorem 2.1 and pp9 Remark 1]. A proof can be found in Appendix. In this case, $W = V$ and $\Gamma_V = \oplus_{i=1}^{2n}\mathbb{Z}e_i \subset \mathrm{Lie}(A) = \mathbb{C}^n = \mathbb{R}^{2n}$ is a lattice. Denote by unif: $\mathrm{Lie}(A) \to A$. Let $\mathcal{F}_V := \Sigma_{i=1}^{2n}(-1, 1)e_i$, then $\mathcal{F}_V$ is a fundamental set for the action of Γ_V on $\mathrm{Lie}(A)$ such that $\mathrm{unif}|_{\mathcal{F}_V}$ is definable.

$\boxed{Case\ ii:\ E{=}T.}$ This is a consequence of Ax's theorem [5] [42, Corollary 3.6]. A proof of this can be found in Appendix. In this case, $W = U$. Let

$\mathcal{F}_U := \{s \in \mathbb{C} | -1 < \Re e(s) < 1\}^m$, then $\mathcal{F}_U$ is a fundamental set for the action of Γ_U on $U(\mathbb{C})$ such that $\mathrm{unif}|_{\mathcal{F}_U}$ is definable.

Case iii : general E. Unlike the rest of the paper, the symbol π in this section denotes the map

$$\begin{array}{ccc} W(\mathbb{R})U(\mathbb{C}) & \xrightarrow{\pi} & V(\mathbb{R}) \\ \downarrow \mathrm{unif} & & \downarrow \mathrm{unif}_V \\ E & \xrightarrow{[\pi]} & A \end{array} . \qquad (3.4.2)$$

Take $\mathcal{F}_V \subset V(\mathbb{R})$ any fundamental set for the action of Γ_V on $V(\mathbb{R})$ such that $\mathrm{unif}_V|_{\mathcal{F}_V}$ is definable.We claim that:

$$\begin{gathered}\text{There exists a fundamental set } \mathcal{F} \text{ for the action of } \Gamma_W \text{ on } W(\mathbb{R})U(\mathbb{C}) \\ \text{such that } \mathrm{unif}|_{\mathcal{F}} \text{ is definable and } \pi(\mathcal{F}) = \mathcal{F}_V.\end{gathered} \qquad (3.4.3)$$

By Reduction Lemma (Lemma 1.1.35), it suffices to prove this for $E = E_1 \times_A ... \times_A E_m$ where E_i's are $\mathbb{G}_m$-torsors over A. But then it suffices to prove for the case $m = 1$. For this case, the proof is similar to §3.3.1.

Let Y_0 be the minimal closed irreducible subvariety of E such that $\widetilde{Z} \subset \mathrm{unif}^{-1}(Y_0)$, then $\widetilde{Z}$ is maximal irreducible algebraic in $\mathrm{unif}^{-1}(Y_0)$. Hence we may assume that $Y = Y_0$. Let N be the connected algebraic monodromy group of Y^{sm} and let $V_N := (N \cap W)/(N \cap U)$. Let $\widetilde{Y}$ be the complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$ which contains $\widetilde{Z}$. For further convenience, we will denote by $\widetilde{Z}_V := \pi(\widetilde{Z})$, $\widetilde{Y}_V := \pi(\widetilde{Y})$ and $Y_V := [\pi](Y)$.

Repeating the proof of Lemma 3.2.5 (but using the conclusion of *Case i* instead of [29, Theorem 1.1]), we get that $\overline{\widetilde{Y}_V} = V_N(\mathbb{R}) + \widetilde{z}_V$ for some $\widetilde{z}_V \in \widetilde{Z}_V$ is weakly special, and $\overline{\mathrm{unif}_V(\widetilde{Z}_V)} = \overline{Y_V}$. Remark that by GAGA, these closures could be taken in the complex analytic topology (i.e. the topology whose closed sets are complex analytic sets) or the Zariski topology. If V_N is trivial, then we are actually in the situation of *Case ii*, and therefore $\widetilde{Z}$ is weakly special. From now on, suppose that $\dim(V_N) > 0$. Replace S by its smallest special subvariety containing Y_0, then $N \lhd P$ by Theorem 2.2.5. Hence V_N is a $G = \mathrm{MT}(b)$-submodule of V.

Define $W_0 := (\overline{\Gamma_W \cap \mathrm{Stab}_{W(\mathbb{R})U(\mathbb{C})}(\widetilde{Z})}^{\mathrm{Zar}})^\circ$, $U_0 := W_0 \cap U$ and $V_0 := \pi(W_0) = W_0/U_0$. The proof is somehow technical, so we will divide it into several steps.

Step I. Let $V^\dagger$ be the smallest subgroup of V_N such that $\widetilde{Z}_V \subset V^\dagger(\mathbb{R}) + \widetilde{z}_V$. In Step I, we will prove $V^\dagger < V_0$.

Step I(i). We know that $A = \Gamma_V \backslash V(\mathbb{R})$ and $V(\mathbb{Q}) \simeq \Gamma_V \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider any $\mathbb{Q}$-quotient group V' of V of dimension 1

$$p' \colon V \to V'$$

such that $\dim(p'(V^\dagger)) = 1$. By abuse of notation, we shall denote its induced map $V(\mathbb{R}) \to V'(\mathbb{R})$ also by p'. Now let $\Gamma_{V'} := p'(\Gamma_V)$, then $\Gamma_{V'} \simeq \mathbb{Z}$ since p' is defined over $\mathbb{Q}$. Write $\Gamma_{V'} = \mathbb{Z}e'$, and let $\mathcal{F}_{V'} := (-1,1)e'$. Then $\mathcal{F}_{V'}$ is a fundamental set for the action of $\Gamma_{V'}$ on $V'(\mathbb{R})$. Define $A' = \Gamma_{V'} \backslash V'(\mathbb{R}) \simeq \mathbb{Z}\backslash\mathbb{R}$, $\mathrm{unif}_{V'} \colon V'(\mathbb{R}) \to A'$ the uniformization and $[p'] \colon A \to A'$ the map induced by p'. Then $\mathrm{unif}_{V'}|_{\mathcal{F}_{V'}}$ is definable (even in $\mathbb{R}_{an}$). Define $Y_{V'} := [p'](Y_V)$ and $\widetilde{Y}_{V'} := p'(\widetilde{Y}_V)$.

Let $V'' := \mathrm{Ker}(p')$. The exact sequence of free $\mathbb{Z}$-modules

$$1 \to \Gamma_{V''} := \Gamma_V \cap V''(\mathbb{Q}) \simeq \mathbb{Z}^{2n-1} \to \Gamma_V \simeq \mathbb{Z}^{2n} \to \Gamma_{V'} \simeq \mathbb{Z} \to 1$$

splits, and hence $\Gamma_V \simeq \Gamma_{V''} \oplus \Gamma_{V'}$. This induces $V \simeq V'' \oplus V'$. Write $\Gamma_{V''} = \sum_{i=2}^{2n} \mathbb{Z}e''_i$ and take $\mathcal{F}_{V''} := \sum_{i=2}^{n}(-1,1)e''_i$. Define $\mathcal{F}_V := \mathcal{F}_{V''} \oplus \mathcal{F}_{V'}$. Then $\mathcal{F}_V$ is a fundamental set for the action of Γ_V on $V(\mathbb{R})$ such that $\mathrm{unif}_V|_{\mathcal{F}_V}$ is definable (even in $\mathbb{R}_{an}$). Define $\mathcal{F}$ as in (3.4.3).

Since $p(V^\dagger) = V'$ by choice of V', $\dim_{\mathbb{R}} p'(\widetilde{Z}_V) > 0$ by minimality of $V^\dagger$. Hence $p'(\widetilde{Z}_V) = V'(\mathbb{R})$ since $p'(\widetilde{Z}_V)$ is connected.

Remark 3.4.1. *If we only request (V', p') to satisfy $p'(V_N) = 1$, then we do not know whether $\dim_{\mathbb{R}}(p'(\widetilde{Z}_V)) > 0$. This is because we are considering the real analytic topology (i.e. the topology whose closed sets are real analytic sets) on A' and the complex analytic topology (i.e. the topology whose closed sets are complex analytic sets) on A, and hence $\overline{\mathrm{unif}_V(\widetilde{Z}_V)} = \overline{Y_V}$ does NOT imply $\overline{\mathrm{unif}_{V'}(\widetilde{Z}_{V'})} = \overline{Y_{V'}}$. To overcome this problem, we introduce the seemingly strange subgroup $V^\dagger$ of V_N. We will prove (Step II) that V_0 is a* $\mathrm{MT}(b)$*-module with the help of $V^\dagger$. Then we prove the comparable result of Theorem 3.2.8(1) for the unipotent part in Step III.*

Let C be an $\mathbb{R}$-algebraic subvariety of $\widetilde{Z}$ of $\mathbb{R}$-dimension 1 such that $p'\pi(C) = V'(\mathbb{R})$. Define furthermore

$$\Sigma(C) := \{g \in W(\mathbb{R}) | \dim_{\mathbb{R}}(gC \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F}) = 1\}$$

and

$$\Sigma'(C) := \{g \in W(\mathbb{R}) | g^{-1}\mathcal{F} \cap C \neq \emptyset\}.$$

The set $\Sigma(C)$ is by definition definable. By Lemma 3.3.1,

$$\Sigma'(C) \cap \Gamma_W = \Sigma(C) \cap \Gamma_W \tag{3.4.4}$$

For $M > 0$, define

$$\Theta_{V'}(V'(\mathbb{R}), M) = \{\gamma_{V'} \in \Gamma_{V'} | H(\gamma_{V'}) \leqslant M\}.$$

Then

$$|\Theta_{V'}(V'(\mathbb{R}), M)| \gg M. \tag{3.4.5}$$

Step I(ii) is quite similar to the end of §3.3. Consider the definable set $V'(\mathbb{R})$. By the theorem of Pila-Wilkie ([48, Theorem 3.6]), there exist J definable block families

$$B^j \subset V'(\mathbb{R}) \times \mathbb{R}^l, \qquad j = 1, ..., J$$

and $c > 0$ such that for all $M \gg 0$, $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$ is contained in the union of at most $cM^{\delta/8}$ definable blocks of the form B^j_y ($y \in \mathbb{R}^l$). By (3.4.5), there exist a $j \in \{1, ..., J\}$ and a block $B_{V'} := B^j_{y_0}$ of $V'(\mathbb{R})$ containing at least $M^{\delta/8}$ elements of $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$.

Let $\Sigma := \Sigma(C) \cap \Sigma(\widetilde{Z})$, which is by definition a definable set. Consider $X^j := ((p'\pi) \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma \times \mathbb{R}^l)$, which is a definable family since $p'\pi$ is $\mathbb{R}$-algebraic.

By [69, Ch. 3, 3.6], there exists a number $n_0 > 0$ such that each fibre X^j_y has at most n_0 connected components. So the definable set $\pi^{-1}(B_{V'}) \cap \Sigma$ has at most n_0 connected components. Now

$$p'\pi((p'\pi)^{-1}(B_{V'}) \cap \Sigma \cap \Gamma_W) = B_{V'} \cap p'\pi(\Sigma(C) \cap \Gamma_W) = B_{V'} \cap (V'(\mathbb{R}) \cap \Gamma_{V'}) = B_{V'} \cap \Gamma_{V'}$$

by (3.4.1), (3.4.4) and the choice of $\mathcal{F}$ (remember that $\Gamma_V = \Gamma_{V''} \oplus \Gamma_{V'}$ and $\mathcal{F}_V = \mathcal{F}_{V''} \oplus \mathcal{F}_{V'}$). So there exists a connected component B of $(p'\pi)^{-1}(B_{V'}) \cap \Sigma$ such that $p'\pi(B \cap \Gamma_W)$ contains at least $M^{\delta/8}/n_0$ elements of $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$.

We have $B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ since $B \subset \Sigma(\widetilde{Z})$ by (complex) analytic continuation, and $\widetilde{Z} \subset \sigma_W^{-1} B\widetilde{Z}$ for any $\sigma_W \in B \cap \Gamma_W$. But B is connected, and therefore $\sigma_W^{-1} B\widetilde{Z} = \widetilde{Z}$ by maximality of $\widetilde{Z}$ and [49, Lemma 4.1]. So

$$B \subset \sigma_W \, \mathrm{Stab}_{W(\mathbb{R})}(\widetilde{Z}).$$

Fix a $\sigma_W \in B \cap \Gamma_W$ such that $p'\pi(\sigma_W) \in \Theta_{V'}(V'(\mathbb{R}), M^{1/4})$. We have shown that $p'\pi(B \cap \Gamma_W)$ contains at least $M^{\delta/8}/n_0$ elements of $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$. For any $\sigma_{V'} \in p'\pi(B \cap \Gamma) \cap \Theta_{V'}(V'(\mathbb{R}), M^{1/4})$, let σ'_W be one of its preimages in $B \cap \Gamma_W$. Then $\gamma_W := \sigma_W^{-1}\sigma'_W$ is an element of $\Gamma_W \cap \mathrm{Stab}_{W(\mathbb{R})}(\widetilde{Z})$ and $H(p'\pi(\gamma_W)) \ll M^{1/2}$. Therefore for $M \gg 0$, $p'\pi(\Gamma_W \cap \mathrm{Stab}_{W(\mathbb{R})}(\widetilde{Z}))$ contains at least $M^{\delta/8}/n_0$ elements $\gamma_{V'}$ such that $H(\gamma_{V'}) \leqslant M$. Therefore $\dim(p'\pi(W_0)) > 0$ since it is an infinite set. So $p'\pi(W_0) = V'$ since $\dim(V') = 1$. But V' is an arbitrary 1-dimensional quotient of V such that $p'(V^\dagger) = V'$. Therefore $V^\dagger < \pi(W_0) = V_0$.

Step II. We prove in this step that V_0 is a $\mathrm{MT}(b)$-module. This implies that W_0 is a $\mathrm{MT}(b)$-subgroup of W by Proposition 1.1.19(2).

By definition of $V^\dagger$, $\widetilde{Z}_V \subset V^\dagger(\mathbb{R}) + \widetilde{z}_V$. By definition of V_0, $V_0(\mathbb{R}) + \widetilde{z}_V \subset \widetilde{Z}_V$. Now the conclusion of *Step I* implies $V_0 = V^\dagger$ and $\widetilde{Z}_V = V_0(\mathbb{R}) + \widetilde{z}_V$. However $\widetilde{Z}_V$ is complex, so $V_0(\mathbb{R})$ is a complex subspace of $V(\mathbb{R})$. Therefore by considering the complex structure of $V(\mathbb{R})$, we get that $V_0(\mathbb{R})$ is a $\mathrm{MT}(b)(\mathbb{R})$-module. So V_0 is a $\mathrm{MT}(b)$-module.

Step III. can be seen as an analogue to the proof of Theorem 3.2.8(1). Consider a fibre of $\widetilde{Z}$ over a point $v \in \pi(\widetilde{Z})$ such that $\pi\colon W(\mathbb{C})/F_b^0W(\mathbb{C}) \to \mathrm{Lie}(A)$ is flat at v (such a point exists by generic flatness). Let $\widetilde{W}$ be an irreducible algebraic component of $\widetilde{Z}_v$ such that $\dim(\widetilde{Z}_v) = \dim(\widetilde{W})$, then since π is flat at v,

$$\dim(\widetilde{Z}) = \dim(\pi(\widetilde{Z})) + \dim(\widetilde{Z}_v) = \dim(\pi(\widetilde{Z})) + \dim(\widetilde{W}).$$

Consider the set $\widetilde{F} := W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{W}$. It is semi-algebraic. The fact $\widetilde{W} \subset \widetilde{Z}$ implies that $\widetilde{F} \subset \widetilde{Z}$. So by [49, Lemma 4.1], there exists an irreducible algebraic subvariety of $W(\mathbb{C})/F_b^0W(\mathbb{C})$, say $\widetilde{F}_{\mathrm{alg}}$, which contains $\widetilde{F}$ and is contained in $\widetilde{Z}$. Since

$$\pi(\widetilde{F}) = \pi(W_0)(\mathbb{R}) + v = \overline{\pi(\widetilde{Z})}$$

and every fiber of $\pi|_{\widetilde{F}_{\mathrm{alg}}}$ has $\mathbb{R}$-dimension at least $\dim_{\mathbb{R}}(\widetilde{W})$, we have

$$\dim(\widetilde{F}_{\mathrm{alg}}) \geqslant \dim(\pi(\widetilde{F})) + \dim(\widetilde{W}) = \dim(\pi(\widetilde{Z})) + \dim(\widetilde{W}) = \dim(\widetilde{Z}).$$

So $\widetilde{F} = \widetilde{Z}$ since $\widetilde{Z}$ is irreducible. In other words, $\widetilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{Z}_v$ and $\widetilde{Z}_v$ is irreducible for any $v \in \pi(\widetilde{Z})$.

Next for any $v \in \pi(\widetilde{Z})$, let $\widetilde{W}'$ be an irreducible algebraic subvariety which contains $\widetilde{Z}_v$ and is contained in $\mathrm{unif}^{-1}(Y)_v$, maximal for these properties. Then $\widetilde{W}'$ is weakly special by *Case ii*. Consider $\widetilde{F}' := W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{W}'$. Let $\widetilde{Y}$ be the irreducible component of $\mathrm{unif}^{-1}(Y)$ which contains $\widetilde{Z}$, then $\widetilde{W}' \subset \widetilde{Y}$ and so $\widetilde{F}' \subset \widetilde{Y}$ by Lemma 3.2.2. But $\widetilde{F}'$ is semi-algebraic, and hence by [49, Lemma 4.1] there exists an irreducible algebraic subvariety of $W(\mathbb{C})/F_b^0W(\mathbb{C})$, say $\widetilde{F}'_{\mathrm{alg}}$, which contains $\widetilde{F}'$ and is contained in $\widetilde{Y}$. So $\widetilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{Z}_v \subset \widetilde{F}'_{\mathrm{alg}} \subset \mathrm{unif}^{-1}(Y)$, and hence $\widetilde{Z} = \widetilde{F}'_{\mathrm{alg}} = \widetilde{F}'$ by the maximality of $\widetilde{Z}$. So $\widetilde{Z}_v = \widetilde{W}'$, i.e.

$$\text{For any } v \in \pi(\widetilde{Z}),\ \widetilde{Z}_v \text{ is a maximal irreducible algebraic subvariety of } W(\mathbb{C})/F^0W(\mathbb{C}) \text{ contained in } \mathrm{unif}^{-1}(Y)_v. \tag{3.4.6}$$

Now that $\widetilde{Z}_v = \widetilde{W}'$ is weakly special, we can write $\widetilde{Z}_v = U'(\mathbb{C}) + \widetilde{z}$ with $U' < U$ and $\widetilde{z} \in \widetilde{Z}_v$. Then $U_0 < U'$. The product $W' := W_0U'$ is a subgroup of W, and hence

$$\widetilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{Z}_v = W_0(\mathbb{R})U'(\mathbb{C})\widetilde{z} = W'(\mathbb{R})U'(\mathbb{C})\widetilde{z}.$$

So $W_0 = W'$ and $U_0 = U'$. In other words,

$$\widetilde{Z} = \widetilde{E} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{z} \tag{3.4.7}$$

for some point $\widetilde{z} \in \widetilde{Z}_v$.

Step IV. Let us now conclude that $\widetilde{Z}$ is weakly special.

First of all, $U_0 \lhd P$ by Proposition 1.1.19(2). Consider $(P, \mathcal{X}^+) \xrightarrow{\rho} (P, \mathcal{X}^+)/U_0$, then by definition $\widetilde{Z}$ is weakly special iff $\rho(\widetilde{Z})$ is. Replace $(P, \mathcal{X}^+)$ (resp. W, $\widetilde{Z}$, W_0, $\widetilde{z}$) by $(P, \mathcal{X}^+)/U_0$ (resp. W/U_0, $\rho(\widetilde{Z})$, $W_0/U_0 = V_0$, $\rho(\widetilde{z})$), then V_0 is a subgroup of W and $\widetilde{Z} = V_0(\mathbb{R})\widetilde{z}$. Use the notation of §1.1.2.5 and §1.3 and suppose $\widetilde{z} = (\widetilde{z}_U, \widetilde{z}_V)$. By Proposition 2.1.2, $\widetilde{Z}$ is weakly special iff $\widetilde{z}_V \in (N_W(V_0)/U)(\mathbb{R})$ iff $\Psi(V_0(\mathbb{R}), \widetilde{z}_V) = 0$. We shall prove the last claim.

Define $Z := \mathrm{unif}(\widetilde{Z})$, $z = \mathrm{unif}(\widetilde{z})$ and $z_V = [\pi](z) \in A$, then $\pi(\widetilde{Z}) = V_0(\mathbb{R}) + \widetilde{z}_V$ and $[\pi](Z) = A_0 + z_V$ where $A_0 = \Gamma_{V_0}\backslash V_0(\mathbb{R})$ is an abelian subvariety of A. We can compute the fiber

$$Z_{z_V} = \Big(\mathrm{unif}(\Gamma_W \widetilde{Z})\Big)_{z_V} = \widetilde{z}_U + \frac{1}{2}\Psi(\Gamma_V, \widetilde{z}_V) + \Gamma_U \mod \Gamma_U. \tag{3.4.8}$$

We have $\Psi(V(\mathbb{R}), V(\mathbb{R})) \subset U(\mathbb{R})$ since Ψ is defined over $\mathbb{Q}$. Let us prove $\Psi(\Gamma_V, \widetilde{z}_V) \subset U(\mathbb{Q})$. Fix an isomorphism $\Gamma_U \simeq \mathbb{Z}^m$, which induces an isomorphism $U(\mathbb{Q}) \simeq \mathbb{Q}^m$. Suppose that there exists a $u \in \Psi(\Gamma_V, \widetilde{z}_V) \setminus U(\mathbb{Q})$, then at least one of the coordinates of u is irrational. Without loss of generality we may assume that its first coordinate $u_1 \in \mathbb{R}\setminus\mathbb{Q}$. Denote by U_1 the $\mathbb{Q}$-subgroup of U corresponding to the first factor of $U(\mathbb{Q}) \simeq \mathbb{Q}^m$, then

$$\mathrm{unif}\left(\widetilde{z}_U + U_1(\mathbb{R})\right) \subset \overline{Z_{z_V}}$$

since $\{lu_1 \bmod \mathbb{Z} | l \in \mathbb{Z}\}$ is dense in $[0, 1)$. So $\overline{Z_{z_V}}$ contains

$$\mathrm{unif}\left(\widetilde{z}_U + U_1(\mathbb{C})\right),$$

and so does Y_{z_V} since $\overline{Z} \subset Y$. Let $v := v_0 + \widetilde{z}_V \in V(\mathbb{R})$, then $\widetilde{z}_U + U_1(\mathbb{C}) \subset \mathrm{unif}^{-1}(Y)_v$. However $\widetilde{Z}_{\widetilde{z}_v} = \widetilde{z}_U$ by (3.4.7) (recall that we have reduced to $W_0 = V_0$ and $U_0 = 0$), which contradicts (3.4.6). Hence $\Psi(\Gamma_V, \widetilde{z}_V) \subset U(\mathbb{Q})$, and therefore $(1/2)\Psi(N\Gamma_V, \widetilde{z}_V) \subset \Gamma_U$ for some $N \gg 0$ (since $\mathrm{rank}\,\Gamma_V < \infty$). Now we can construct a new lattice Γ'_W with $N\Gamma_V$ and Γ_U. Γ'_W is of finite index in Γ_W. Replacing Γ_W by Γ'_W does not change the assumption or the conclusion of Ax-Lindemann, so we may assume $(1/2)\Psi(\Gamma_V, \widetilde{z}_V) \subset \Gamma_U$. Now we can define C^∞-morphisms

$$\begin{aligned} f\colon A_0 + z_V &\longrightarrow T \\ a_0 + z_V &\mapsto \widetilde{z}_U + (1/2)\Psi(v_0, \widetilde{z}_V) \bmod \Gamma_U \end{aligned}$$

and

$$\begin{aligned} s\colon A_0 + z_V &\longrightarrow E|_{A_0+z_V} \\ a_0 + z_V &\mapsto (\widetilde{z}_U + (1/2)\Psi(v_0, \widetilde{z}_V), a_0 + z_V) \bmod \Gamma_W \end{aligned}$$

where v_0 is any point of $V_0(\mathbb{R})$ such that $\mathrm{unif}_V(v_0) = a_0$. But Z_a is a single point for all $a \in A_0 + z_V$ by (3.4.8), so s is the inverse of $[\pi]|_Z$, and therefore s is a holomorphic section of $E|_{A_0+z_V} \to A_0 + z_V$. Locally on $U_i \subset A_0 + z_V$,

s is represented by a holomorphic morphism $U_i \to T$, which must equal to $f|_{U_i}$ by definition. Hence f is holomorphic since being holomorphic is a local condition. So f is constant.

But $\Psi(0, \widetilde{z}_V) = 0$, and therefore $(1/2)\Psi(V_0(\mathbb{R}), \widetilde{z}_V) \subset \Gamma_U$. But $\Psi(V_0(\mathbb{R}), \widetilde{z}_V)$ is continuous and $\Psi(0, \widetilde{z}_V) = 0$, so $\Psi(V_0(\mathbb{R}), \widetilde{z}_V) = 0$. Hence we are done. □

3.5 Appendix

3.5.1 About the definability

This subsection is devoted to explain more details for the definability in §3.3.1. For any connected mixed Shimura variety $S = \Gamma\backslash\mathcal{X}^+$ associated with $(P, \mathcal{X}^+)$ whose uniformization is denoted by unif: $\mathcal{X}^+ \to S$, we have the following diagram by the reduction lemma (Lemma 1.1.35):

$$\begin{array}{ccc} (P', \mathcal{X}'^+) & \overset{i}{\hookrightarrow} & (G_0, \mathcal{D}^+) \times \prod_{j=1}^{r} (\mathrm{GSp}_{2g}, \mathcal{X}_{2g}^+) \\ {\scriptstyle p}\downarrow & & \\ (P, \mathcal{X}^+) & & \end{array}$$

where $\mathrm{Ker}(p\colon P' \to P) \subset U'$ is a $\mathbb{Q}$-vector group of dimension 1 or 0. Hence there exists a congruence group $\Gamma' \subset P'(\mathbb{Q})_+$ such that $p(\Gamma') = \Gamma$. Now in order to find a fundamental subset $\mathcal{F}$ for the action of Γ on $\mathcal{X}^+$ such that $\mathrm{unif}|_{\mathcal{F}}$ is definable, it suffices to find a fundamental subset $\mathcal{F}'$ for the action of Γ' on $\mathcal{X}'^+$ such that $\mathrm{unif}'|_{\mathcal{F}'}$ is definable (here unif: $\mathcal{X}'^+ \to S' := \Gamma'\backslash\mathcal{X}'^+$).

By [53, 3.8], there exists a congruence subgroup $\Gamma^\dagger \subset (G_0 \times \prod_{j=1}^r \mathrm{GSp}_{2g})(\mathbb{Q})_+$ such that $\Gamma' = \Gamma^\dagger \cap P'(\mathbb{Q})_+$ and $S' \xrightarrow{[i]} S^\dagger := \Gamma^\dagger\backslash(\mathcal{D}^+ \times \prod \mathcal{X}_{2g}^+)$ is a closed immersion. Applying Lemma 3.5.1 to

$$((P', \mathcal{X}'^+), \Gamma') \hookrightarrow \left((G_0, \mathcal{D}^+) \times \prod_{j=1}^{r} (\mathrm{GSp}_{2g}, \mathcal{X}_{2g}^+), \Gamma^\dagger \right),$$

it suffices to find a fundamental subset $\mathcal{F}^\dagger$ for the action of $\Gamma^\dagger$ on $\mathcal{D}^+ \times \prod \mathcal{X}_{2g}^+$ such that $\mathrm{unif}^\dagger|_{\mathcal{F}^\dagger}$ is definable. Replacing $\Gamma^\dagger$ by a smaller congruence subgroup does not change the conclusion, hence we may furthermore assume $\Gamma^\dagger = \Gamma_0 \times \prod_{j=1}^r \Gamma_i$ such that Γ_0 is a congruence subgroup of $G_0(\mathbb{Q})_+$ and Γ_j is a congruence subgroup of the j-th $\mathrm{GSp}_{2g}(\mathbb{Q})_+$-factor. Hence we are reduced to the situation as stated in §3.3.1.

Lemma 3.5.1. *Consider the diagram*

$$\begin{array}{ccc} (P_1, \mathcal{X}_1^+) & \overset{i}{\hookrightarrow} & (P_2, \mathcal{X}_2^+) \\ {\scriptstyle \mathrm{unif}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{unif}_2} \\ S_1 := \Gamma_1\backslash\mathcal{X}_1^+ & \overset{[i]}{\hookrightarrow} & S_2 := \Gamma_2\backslash\mathcal{X}_2^+ \end{array}$$

where $\Gamma_1 = \Gamma_2 \cap P_1(\mathbb{Q})_+$. *If there exists a fundamental set* $\mathcal{F}_2$ *for the action of* Γ_2 *on* $\mathcal{X}_2^+$ *such that* $\mathrm{unif}_2|_{\mathcal{F}_2}$ *is definable, then there exists a fundamental set* $\mathcal{F}_1$ *for the action of* Γ_1 *on* $\mathcal{X}_1^+$ *such that* $\mathrm{unif}_1|_{\mathcal{F}_1}$ *is definable.*

Proof. One possible way to prove this lemma is to repeat the proof of Ullmo [64, Proposition 2.4] (remark that Théorème 2.6 of *loc.cit.* holds for arbitrary linear algebraic groups over $\mathbb{Q}$). The proof we present here, which uses the o-minimal theory, is due to Pila-Tsimerman [50, Section 4.2].

First of all, note that $\mathrm{unif}_2^{-1}(S_1)$ is the (not disjoint) union over $\gamma \in \Gamma_2$ of $\gamma\mathcal{X}_1^+$. Secondly consider $\mathrm{unif}_2^{-1}(S_1) \cap \mathcal{F}_2$. Since $\mathrm{unif}_2|_{\mathcal{F}_2}$ is definable, this intersection has only finitely many connected components. Therefore there are finitely many elements $\gamma_j \in \Gamma_2$ $(1 \leqslant j \leqslant m)$ such that

$$\mathrm{unif}_2\left(\bigcup_{j=1}^{m} \gamma_j^{-1}\mathcal{X}_1^+ \cap \mathcal{F}_2\right) = S_1$$

and thus

$$\mathrm{unif}_2\left(\bigcup_{j=1}^{m} \mathcal{X}_1^+ \cap \gamma_j\mathcal{F}_2\right) = S_1.$$

Define Γ_2^1 to be the subgroup of Γ_2 which stabilizes $\mathcal{X}_1^+$. Then $\Gamma_1 \subset \Gamma_2^1$.

Now for any $x \in \mathcal{X}_1^+$, there exists a $\gamma \in \Gamma_2$ such that $\gamma x \in \mathcal{F}_2$ because $\mathcal{F}_2$ is a fundamental set for the action of Γ_2 on $\mathcal{X}_2^+$. As above this means that there exists a j with $1 \leqslant j \leqslant m$ such that $\gamma x \in \gamma_j^{-1}\mathcal{X}_1^+$ and $\gamma\mathcal{X}_1^+ = \gamma_j^{-1}\mathcal{X}_1^+$. Therefore there exists a $\gamma' \in \Gamma_2^1$ with $\gamma = \gamma_j^{-1}\gamma'$. Therefore $\gamma_j^{-1}\gamma' x \in \mathcal{F}_2 \cap \gamma_j^{-1}\mathcal{X}_1^+$ and so $\gamma' x \in \gamma_j\mathcal{F}_2 \cap \mathcal{X}_1^+$. To sum it up, $\mathcal{X}_1^e := \bigcup_{j=1}^m(\mathcal{X}_1^+ \cap \gamma_j\mathcal{F}_2)$ contains a fundamental set for the action of Γ_2^1 on $\mathcal{X}_1^+$. Now by picking coset representatives for Γ_1 in Γ_2^1, we can find a finite union of elements $\alpha_l \in \Gamma_2$ such that $\bigcup_l(\alpha_l\mathcal{X}_1^e \cap \mathcal{X}_1^+)$ contains a fundamental set, which we call $\mathcal{F}_1$, for the action of Γ_1 on $\mathcal{X}_1^+$. Then $\mathcal{F}_1$ is what we desire. □

3.5.2 A simplified proof of flat Ax-Lindemann

We prove here Theorem 3.1.5 when $E = T$ is an algebraic torus over $\mathbb{C}$ (which corresponds to the case $W = U$) and when $E = A$ is a complex abelian variety (which corresponds to the case $W = V$). The proof is a rearrangement of existing proofs (combining the point counting of Pila-Zannier [51] and volume calculation of Ullmo-Yafaev [67]). We use the notation of §3.4.

Case i : E=A. In this case, $W = V$ and $\Gamma_V = \oplus_{i=1}^{2n}\mathbb{Z}e_i \subset \mathrm{Lie}(A) = \mathbb{C}^n = \mathbb{R}^{2n}$ is a lattice. Denote by $\mathrm{unif}\colon \mathrm{Lie}(A) \to A$. Let $\mathcal{F}_V := \Sigma_{i=1}^{2n}(-1,1)e_i$, then $\mathcal{F}_V$ is a fundamental set for the action of Γ_V on $\mathrm{Lie}(A)$ such that $\mathrm{unif}|_{\mathcal{F}_V}$ is definable. Define the norm of $z = (x_1, y_1, ..., x_n, y_n) \in \mathrm{Lie}(A) = \mathbb{R}^{2n}$ to be

$$\parallel z \parallel := \mathrm{Max}(|x_1|, |y_1|, ..., |x_n|, |y_n|).$$

It is clear that $\forall z \in \mathrm{Lie}(A)$ and $\forall \gamma_V \in \Gamma_V$ such that $\gamma_V z \in \mathcal{F}_V$,

$$H(\gamma_V) \ll \| x_V \| . \tag{3.5.1}$$

Let $\omega_V := dz_1 \wedge d\overline{z}_1 + ... + dz_n \wedge d\overline{z}_n$ be the canonical $(1,1)$-form of $\mathrm{Lie}(A) = \mathbb{C}^n$. Let p_i $(i = 1, ..., n)$ be the n natural projections of $\mathrm{Lie}(A) = \mathbb{C}^n$ to $\mathbb{C}$. Let C be an algebraic curve of $\widetilde{Z}$ and define $C_M := \{z \in C | \| z \| \leqslant M\}$. We have

$$\begin{aligned} \int_{C\cap\mathcal{F}_V} \omega_V &\leqslant d \sum_{i=1}^{n} \int_{p_i(C\cap\mathcal{F}_V)} dz_i \wedge d\overline{z}_i \\ &\leqslant d \sum_{i=1}^{n} \int_{p_i(\mathcal{F}_V)} dz_i \wedge d\overline{z}_i = d \cdot O(1) \end{aligned} \tag{3.5.2}$$

and

$$\int_{C_M} \omega_V \geqslant O(M^2) \tag{3.5.3}$$

with $d = \deg(C)$ by [27, Theorem 0.1].

By (3.5.1)

$$C_M \subset \bigcup_{\gamma_V \in \Theta(\widetilde{Z}, M)} (C \cap \gamma^{-1} \mathcal{F}).$$

Integrating both sides w.r.t. $\omega_V$we have

$$M^2 \ll \#\Theta(\widetilde{Z}, M)$$

by (3.5.2) and (3.5.3).

Let $\mathrm{Stab}_V(\widetilde{Z}) := \overline{\Gamma_V \cap \mathrm{Stab}_{V(\mathbb{R})}(\widetilde{Z})}^{\mathrm{Zar}}$. Now by Pila-Wilkie [67, Theorem 3.4], there exists an semi-algebraic block $B \subset \Sigma(\widetilde{Z})$ of positive dimension containing arbtrarily many points $\gamma_V \in \Gamma_V$. We have $B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ since $\Sigma(\widetilde{Z})\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ by definition. Hence for any $\gamma_V \in \Gamma_V \cap B$, $\widetilde{Z} \subset \gamma_V^{-1} B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$, and therefore $\widetilde{Z} = \gamma_V^{-1} B\widetilde{Z}$ by maximality of $\widetilde{Z}$. So $\gamma_V^{-1}(B \cap \Gamma_V) \subset \mathrm{Stab}_V(\widetilde{Z})(\mathbb{Q})$, and hence $\dim(\mathrm{Stab}_V(\widetilde{Z})) > 0$. For any point $\widetilde{z} \in \widetilde{Z}$, $\mathrm{Stab}_V(\widetilde{Z})(\mathbb{R}) + \widetilde{z} \subset \widetilde{Z}$. By [51, Lemma 2.3], $\mathrm{Stab}_V(\widetilde{Z})(\mathbb{R})$ is full and complex. Define $V' := V/\mathrm{Stab}_V(\widetilde{Z})$ and $\Gamma_{V'} := \Gamma_V/(\Gamma_V \cap \mathrm{Stab}_V(\widetilde{Z})(\mathbb{Q}))$, and then $A' := V'(\mathbb{R})/\Gamma_{V'}$ is a quotient abelian variety of A. Let Y' (resp. $\widetilde{Z}'$) be the Zariski closure of the projection of Y (resp. $\widetilde{Z}$) in A' (resp. $V'(\mathbb{R})$). We prove that the image of $\widetilde{Z}'$ is a point. If not, then proceeding as before for the triple $(A', Y', \widetilde{Z}')$ can we prove $\dim(\mathrm{Stab}_{V'}(\widetilde{Z}')) > 0$. This contradicts the definition (maximality) of $\mathrm{Stab}_V(\widetilde{Z})$. Hence $\widetilde{Z}$ is a translate of $\mathrm{Stab}_V(\widetilde{Z})(\mathbb{R})$. So $\widetilde{Z}$ is weakly special.

Case ii : E=T. Define the norm of $x_U = (x_{U,1}, x_{U,2}, ..., x_{U,m}) \in U(\mathbb{C})$ to be

$$\| x_U \| := \mathrm{Max}(\| x_{U,1} \|, \| x_{U,2} \|, ..., \| x_{U,m} \|).$$

It is clear that for all $x_U \in U(\mathbb{C})$ and for all $\gamma_U \in \Gamma_U$ such that $\gamma_U x_U \in \mathcal{F}_U$,

$$H(\gamma_U) \ll \| x_U \| . \tag{3.5.4}$$

Let $\omega|_T = dz_1 \wedge d\overline{z}_1 + ... + dz_m \wedge d\overline{z}_m$ be the canonical $(1,1)$-form of $U(\mathbb{C}) \simeq \mathbb{C}^m$. Let p_i $(i = 1, ..., m)$ be the m natural projections of $U(\mathbb{C}) \simeq \mathbb{C}^m$ to $\mathbb{C}$. Let C be an algebraic curve of $\widetilde{Z}$ and define $C_M := \{x \in C | \| x \| \leqslant M\}$. We have

$$\begin{aligned} \int_{C_M \cap \mathcal{F}_U} \omega|_T &\leqslant d \sum_{i=1}^{m} \int_{p_i(C_M \cap \mathcal{F}_U)} dz_i \wedge d\overline{z}_i \\ &\leqslant d \sum_{i=1}^{m} \int_{\{s \in \mathbb{C} | -1 < \Re e(s) < 1, \|s\| \leqslant M\}} dz_i \wedge d\overline{z}_i = d \cdot O(M) \end{aligned} \tag{3.5.5}$$

where $d := \deg(C)$. On the other hand by [27, Theorem 0.1],

$$\int_{C_M} \omega|_T \geqslant O(M^2). \tag{3.5.6}$$

By (3.5.4)

$$C_M \subset \bigcup_{\gamma \in \Theta(\widetilde{Z}, M)} (C_M \cap \gamma^{-1}\mathcal{F}).$$

Integrating both side w.r.t. $\omega|_T$ and taking into account that

$$\gamma \cdot C_M \subset (\gamma C)_{2M} \qquad \text{if } H(\gamma) \leqslant M,$$

we have

$$M^2 \ll \#\Theta(\widetilde{Z}, M) \cdot M$$

by (3.5.5) and (3.5.6). Hence $\#\Theta(\widetilde{Z}, M) \gg M$.

Let $\mathrm{Stab}_U(\widetilde{Z}) := \overline{\Gamma_U \cap \mathrm{Stab}_{U(\mathbb{C})}(\widetilde{Z})}^{\mathrm{Zar}}$. Now by Pila-Wilkie [48, Theorem 3.6], there exists an semi-algebraic subset $B \subset \Sigma(\widetilde{Z})$ of positive dimension containing arbtrarily many points $\gamma_U \in \Gamma_U$. We have $B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ since $\Sigma(\widetilde{Z})\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ by definition. Hence for any $\gamma_U \in \Gamma_U \cap B$, $\widetilde{Z} \subset \gamma_U^{-1} B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$, and therefore $\widetilde{Z} = \gamma_U^{-1} B\widetilde{Z}$ by maximality of $\widetilde{Z}$. So $\gamma_U^{-1}(B \cap \Gamma_U) \subset \mathrm{Stab}_U(\widetilde{Z})(\mathbb{Q})$, and hence $\dim(\mathrm{Stab}_U(\widetilde{Z})) > 0$. Let $U' := U/\mathrm{Stab}_U(\widetilde{Z})$, $\Gamma_{U'} := \Gamma_U/(\Gamma_U \cap \mathrm{Stab}_U(\widetilde{Z})(\mathbb{Q}))$ and $T' := U'(\mathbb{C})/\Gamma_{U'}$. T' is an algebraic torus over $\mathbb{C}$. Let Y' (resp. $\widetilde{Z}'$) be the Zariski closure of the projection of Y (resp. $\widetilde{Z}$) in T' (resp. $U'(\mathbb{C})$). We prove that $\widetilde{Z}'$ is a point. If not, then proceeding as before for the triple $(T', Y', \widetilde{Z}')$ we can prove $\dim(\mathrm{Stab}_{U'}(\widetilde{Z}')) > 0$. This contradicts the definition (maximality) of $\mathrm{Stab}_U(\widetilde{Z})$. Hence $\widetilde{Z}$ is a translate of $\mathrm{Stab}_U(\widetilde{Z})(\mathbb{C})$. So $\widetilde{Z}$ is weakly special.

Chapter 4

From Ax-Lindemann to André-Oort

4.1 Distribution of positive-dimensional weakly special subvarieties

4.1.1 Weakly special subvarieties defined by a fixed $\mathbb{Q}$-subgroup

Let $S = \Gamma\backslash\mathcal{X}^+$ be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S$ be the uniformization. Suppose that N is a connected subgroup of P such that $N/(W \cap N) \hookrightarrow G$ is semi-simple. A subvariety of S is said to be **weakly special defined by** N if it is of the form $\mathrm{unif}(i(\varphi^{-1}(y')))$ under the notation of Definition 1.2.2 such that $N = \mathrm{Ker}(\varphi)$. Let $\mathfrak{F}(N)$ be the set of all weakly special subvarieties of S defined by N. The goal of this subsection is to prove:

Proposition 4.1.1. *If $\mathfrak{F}(N) \neq \emptyset$ and $N \ntriangleleft P$, then $\cup_{Z\in\mathfrak{F}(N)} Z$ is a finite union of proper special subvarieties of S.*

Proof. Take any $F \in \mathfrak{F}(N)$. Let $\mathcal{F}$ be a fundamental domain for the action Γ on $\mathcal{X}^+$. Suppose that $x' \in \mathcal{F}$ is such that $F = \mathrm{unif}(N(\mathbb{R})^+U_N(\mathbb{C})x')$. Consider $Q' := N_P(N)$, the normalizer of N in P. By definition of weakly special subvarieties, there exists $(R', \mathcal{Z}^+) \hookrightarrow (P, \mathcal{X}^+)$ such that $h_{x'} : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $R'_{\mathbb{C}}$ and $N \lhd R'$. Hence $R' < Q'$. Define $G_{Q'} := Q'/(W \cap Q')$. Then $G_{Q'}/(Z(G) \cap G_{Q'})$ is reductive by [15, Lemma 4.3] or [63, Proposition 3.28], and hence $G_{Q'}$ is reductive. Write

$$1 \to W \cap Q' \to Q' \xrightarrow{\pi_{Q'}} G_{Q'} \to 1.$$

The group $G_{Q'} = Z(G_{Q'})^\circ G^{\mathrm{nc}}_{Q'} G^{\mathrm{c}}_{Q'}$ is an almost-direct product, where $G^{\mathrm{nc}}_{Q'}$ (resp. $G^{\mathrm{c}}_{Q'}$) is the product of the $\mathbb{Q}$-simple factors whose set of $\mathbb{R}$-points is non-compact (resp. compact). Let $G_Q := Z(G_{Q'})^\circ G^{\mathrm{nc}}_{Q'}$ and then define $Q := \pi_{Q'}^{-1}(G_Q)$, then $h_{x'}$ factors through $Q_{\mathbb{C}}$ and $R' < Q$ by Definition 1.1.12(4). So $N \lhd Q$ and $(Q, \mathcal{Y}^+)$, where $\mathcal{Y}^+ := Q(\mathbb{R})^+U_Q(\mathbb{C})x'$, is a connected mixed Shimura subdatum of $(P, \mathcal{X}^+)$. But then $F \subset \mathrm{unif}(\mathcal{Y}^+) \subset \cup_{Z\in\mathfrak{F}(N)} Z$.

Define $\mathfrak{Y}_Q := \{x \in \mathcal{X}^+ | h_x \text{ factors through } Q_{\mathbb{C}}\}$, then $Q(\mathbb{R})^+U_Q(\mathbb{C})\mathfrak{Y}_Q = \mathfrak{Y}_Q$. The discussion of last paragraph tells us that $F \subset \mathrm{unif}(\mathfrak{Y}_Q)$ for any $F \in \mathfrak{F}(N)$. On the other hand, for any $x \in \mathfrak{Y}_Q$, $(Q, \mathcal{Y}^+)$, where $\mathcal{Y}^+ := Q(\mathbb{R})^+U_Q(\mathbb{C})x$, is a connected mixed Shimura subdatum of $(P, \mathcal{X}^+)$ and hence $\mathrm{unif}(N(\mathbb{R})^+U_N(\mathbb{C})x) \in \mathfrak{F}(N)$. Therefore $\mathrm{unif}(\mathfrak{Y}_Q) \subset \cup_{Z\in\mathfrak{F}(N)} Z$. To sum it up, $\cup_{Z\in\mathfrak{F}(N)} Z = \mathrm{unif}(\mathfrak{Y}_Q)$.

Now we are done if we can prove

Claim 4.1.2. *The set $\mathfrak{Y}_Q$ is a finite union of $Q(\mathbb{R})^+U_Q(\mathbb{C})$-conjugacy classes. In other words, $\mathfrak{Y}_Q$ is a finite union of connected mixed Shimura subdata of $(P,\mathcal{X}^+)$.*

Fix a special point x of $\mathcal{X}^+$ contained in $\mathfrak{Y}_Q$. There exists by definition a torus $T_x \subset Q$ such that $h_x : \mathbb{S}_\mathbb{C} \to Q_\mathbb{C}$ factors through $T_{x,\mathbb{C}}$. Furthermore, we may and do assume that $T_{x,\mathbb{C}}$ is a maximal torus of $Q_\mathbb{C}$. Let T be a maximal torus of $P_\mathbb{C}$ defined over $\mathbb{Q}$ such that $T > T_x$. Take a Levi decomposition $P = W \rtimes G$ such that $T < G < P$. Then the composite $\mathbb{S}_\mathbb{C} \xrightarrow{h_x} T_{x,\mathbb{C}} < P_\mathbb{C} \xrightarrow{\pi} G_\mathbb{C} < P_\mathbb{C}$ equals h_x and is defined over $\mathbb{R}$ by Definition 1.1.12(1).

For any other special point y of $\mathcal{X}^+$ contained in $\mathfrak{Y}_Q$, there exists $g \in Q(\mathbb{C})$ such that $gT_{x,\mathbb{C}}g^{-1} = T_{y,\mathbb{C}}$. The number of the $Q(\mathbb{R})$-conjugacy classes of maximal tori of $Q_\mathbb{R}$ defined over $\mathbb{R}$ is at most

$$\#(\mathrm{Ker}(H^1\left(\mathbb{R}, N_{Q(\mathbb{R})}(T_{x,\mathbb{R}})\right) \to H^1(\mathbb{R},Q))) < \infty,$$

where $N_{Q(\mathbb{R})}(T_{x,\mathbb{R}})$ is the normalizer of $T_{x,\mathbb{R}}$ in $Q(\mathbb{R})$. So it is equivalent to prove the finiteness of the $Q(\mathbb{R})^+U_Q(\mathbb{C})$-conjugay classes in $\mathfrak{Y}_Q$ and to prove the finiteness of the $Q(\mathbb{R})^+$-conjugacy classes of the morphisms $\mathbb{S} \to T_{x,\mathbb{R}}$. But $T_x < T < G$, so the $Q(\mathbb{R})^+$-conjugacy classes of the morphisms $\mathbb{S} \to T_{x,\mathbb{R}}$ equals the $G_Q(\mathbb{R})^+$-conjugacy classes of the morphisms $\mathbb{S} \to T_{x,\mathbb{R}}$. In otherwords, it suffices to prove the claim for $(G,\mathcal{X}_G^+)$. Now the result follows from [15, Lemma 4.4(ii)] (or [39, 2.4] or [66, Lemma 3.7]).

□

4.1.2 The distribution theorem

Now we use the result of the previous subsection to prove the following theorem about the distribution of positive-dimensional weakly special subvarieties. This is a direct generalization of the comparative result of Ullmo for pure Shimura varieties [64, Théorème 4.1].

Theorem 4.1.3. *Let $S = \Gamma\backslash\mathcal{X}^+$ be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P,\mathcal{X}^+)$. Let Y be a Hodge generic irreducible subvariety of S. Then there exists an $N \lhd P$ such that for the diagram*

$$\begin{array}{ccc} (P,\mathcal{X}^+) & \xrightarrow{\rho} & (P',\mathcal{X}'^+) := (P,\mathcal{X}^+)/N \\ \downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}'} \\ S & \xrightarrow{[\rho]} & S' \end{array}, \qquad (4.1.1)$$

- *the union of positive-dimensional weakly special subvarieties which are contained in $Y' := \overline{[\rho](Y)}$ is NOT Zariski dense in Y';*
- *$Y = [\rho]^{-1}(Y')$.*

Proof. Without any loss of generality, we assume that the union of positive-dimensional weakly special subvarieties which are contained in Y is Zariski dense in Y.

Take a fundamental domain $\mathcal{F}$ for the action of Γ on $\mathcal{X}^+$ such that unif $|_{\mathcal{F}}$ is definable. Such an $\mathcal{F}$ exists by §3.3.1.

By Reduction Lemma (Lemma 1.1.35), we may assume

$$(P, \mathcal{X}^+) \overset{\lambda}{\hookrightarrow} (G_0, \mathcal{D}^+) \times \prod_{i=1}^{r} (P_{2g}, \mathcal{X}^+_{2g}),$$

i.e. replace $(P, \mathcal{X}^+)$ by $(P', \mathcal{X}'^+)$ in the reduction lemma if necessary. Identify $(P, \mathcal{X}^+)$ with its image under λ.

Let $\mathcal{T}$ be the set of the triples (U', V', G') consisting of an $\mathbb{R}$-subgroup of $U_{\mathbb{R}}$, an $\mathbb{R}$-sub-Hodge structure of $V_{\mathbb{R}}$ and a connected $\mathbb{R}$-subgroup of $G_{\mathbb{R}}$ which is semi-simple and has no compact factors. Let

$$\mathcal{G} := \mathbb{G}_m(\mathbb{R})^r \times \mathrm{GSp}_{2g}(\mathbb{R}) \times G(\mathbb{R}),$$

then $\mathcal{G}$ acts on $\mathcal{T}$ by $(g_U, g_V, g) \cdot (U', V', G') := (g_U U', g_V V', gG'g^{-1})$. Also we define the action of a triple $(U'(\mathbb{R}), V'(\mathbb{R}), G'(\mathbb{R}))$ on $\mathcal{X}^+ \simeq U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}^+_G$ as (1.3.2). This action is algebraic.

Lemma 4.1.4. *Up to the action of $\mathcal{G}$ on $\mathcal{T}$, there exist only finitely many such triples.*

Proof. First of all by root system theory and Galois cohomology, there exist only finitely many semi-simple subgroups of $G_{\mathbb{R}}$ up to conjugation by $G(\mathbb{R})$.

Secondly, V' is by definition a symplectic subspace of $V_{\mathbb{R}}$. Hence a symplectic base of V' extends to a symplectic base of $V_{\mathbb{R}} = V_{2g,\mathbb{R}}$. But $\mathrm{GSp}_{2g}(\mathbb{R})$ acts transitively on the set of symplectic bases of $V_{2g,\mathbb{R}}$, so there are only finitely many choices for V' up to the action of $\mathrm{GSp}_{2g}(\mathbb{R})$.

Finally, observe that for all $(\lambda_1, ..., \lambda_r) \in \mathbb{G}_m(\mathbb{R})^r$ and $(u_1, ..., u_r) \in U \simeq \oplus_{i=1}^r U_{2g}^{(i)}$,

$$(\lambda_1, ..., \lambda_r) \cdot (u_1, ..., u_r) = (\lambda_1 u_1, ..., \lambda_r u_r)$$

Now $(u_1, ..., u_r)$ and $(u_1', ..., u_r')$ are under the same orbit of the action of $\mathbb{G}_m(\mathbb{R})^r$ if and only if $u_i u_i' \geqslant 0$ with $u_i u_i' = 0 \Rightarrow u_i = u_i' = 0$ for all $i = 1, ..., r$. Hence up to the action of $\mathbb{G}_m(\mathbb{R})^r$, there are only finitely many U''s. □

Let $\mathfrak{W}(Y)$ (resp. $\mathfrak{W}_l(Y)$) be the union of weakly special subvarieties of positive dimension (resp. of real dimension l) contained in Y.

For any l with $\mathfrak{W}_l(Y) \neq \emptyset$, there exist by definition (and Proposition 1.2.4) a subgroup N_l of P^{der} and a point $x_0 \in \mathcal{F}$ such that $\mathrm{unif}(N_l(\mathbb{R})^+ U_{N_l}(\mathbb{C}) x_0)$ is a weakly special subvariety of dimension l contained in Y. Note that the triple $(U_{N_l,\mathbb{R}}, V_{N_l,\mathbb{R}}, G^{+\mathrm{nc}}_{N_l,\mathbb{R}}) \in \mathcal{T}$, where $G^{+\mathrm{nc}}_{N_l,\mathbb{R}}$ is the product of the $\mathbb{R}$-simple factors of $G^+_{N_l,\mathbb{R}}$ which are non-compact. We say that two such subgroups N_l, N_l' of P are equivalent if $(U_{N_l,\mathbb{R}}, V_{N_l,\mathbb{R}}, G^{+\mathrm{nc}}_{N_l,\mathbb{R}}) = (U_{N_l',\mathbb{R}}, V_{N_l',\mathbb{R}}, G^{+\mathrm{nc}}_{N_l',\mathbb{R}})$. By condition

(4) of Definition 1.1.12, $\mathrm{unif}(N_l(\mathbb{R})^+U_{N_l}(\mathbb{C})x_0) = \mathrm{unif}(N_l'(\mathbb{R})^+U_{N_l'}(\mathbb{C})x_0)$ iff N_l and N_l' are equivalent.

Define

$$B(N_{l,\mathbb{R}},Y) := \{(g_U,g_V,g,x)\in\mathcal{G}\times\mathcal{F}|\ \mathrm{unif}((g_UU_{N_l}(\mathbb{C}),g_VV_{N_l}(\mathbb{R}),gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1})x) \text{ is contained in } Y \text{ and is not contained in } \cup_{l'>l}\mathfrak{W}_{l'}(Y)\}.$$

Then by analytic continuation,

$$B(N_{l,\mathbb{R}},Y) = \{(g_U,g_V,g,x)\in\mathcal{G}\times\mathcal{F}|\ \mathrm{unif}|_{\mathcal{F}}((g_UU_{N_l}(\mathbb{R}),g_VV_{N_l}(\mathbb{R}),gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1})x) \text{ is contained in } Y \text{ and is not contained in } \cup_{l'>l}\mathfrak{W}_{l'}(Y)\}. \tag{4.1.2}$$

Lemma 4.1.5. *For any* $(g_U,g_V,g,x)\in B(N_{l,\mathbb{R}},Y)$, *define*

$$\widetilde{Z} := (g_UU_{N_l}(\mathbb{C}),g_VV_{N_l}(\mathbb{R}),gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1})x.$$

Then $\mathrm{unif}(\widetilde{Z})$ *is a weakly special subvariety of* Y.

Proof. The set $\widetilde{Z}$ is a connected irreducible semi-algebraic subset of $\mathcal{X}^+$ which is contained in $\mathrm{unif}^{-1}(Y)$ (see the paragraph before Theorem 3.1.2 for the definition of "connected irreducible semi-algebraic subsets of $\mathcal{X}^+$"). Let $\widetilde{Z}^\dagger$ be a connected irreducible semi-algebraic subset of $\mathcal{X}^+$ which is contained in $\mathrm{unif}^{-1}(Y)$ and which contains $\widetilde{Z}$, maximal for these properties. By Ax-Lindemann (here we use Theorem 3.1.2), $\widetilde{Z}^\dagger$ is complex analytic and each of its complex analytic irreducible component is weakly special. But $\widetilde{Z}$ is smooth, so $\widetilde{Z}$ is contained in one complex analytic irreducible component of $\widetilde{Z}^\dagger$ which we denote by $\widetilde{Z}'$. Now we have

$$\begin{aligned}\dim(\widetilde{Z})-\dim(N_l(\mathbb{R})^+U_{N_l}(\mathbb{C})x_0) &= \dim(gG_{N_l}(\mathbb{R})^+g^{-1}\cdot x_G)-\dim(G_{N_l}(\mathbb{R})^+x_{0,G})\\ &= \dim(\mathrm{Stab}_{G_{N_l}(\mathbb{R})^+}(x_{0,G}))-\dim(\mathrm{Stab}_{gG_{N_l}(\mathbb{R})^+g^{-1}}(x_G))\\ &\geqslant 0\end{aligned}$$

because $\mathrm{Stab}_{gG_{N_l}(\mathbb{R})^+g^{-1}}(x_G)$ is a compact subgroup of $gG_{N_l}(\mathbb{R})^+g^{-1}$ and $\mathrm{Stab}_{G_{N_l}(\mathbb{R})^+}(x_{0,G})$ is a maximal compact subgroup of $G_{N_l}(\mathbb{R})^+$. Hence

$$\dim(\widetilde{Z}')\leqslant l=\dim(N_l(\mathbb{R})^+U_{N_l}(\mathbb{C})x_0)\leqslant\dim(\widetilde{Z})\leqslant\dim(\widetilde{Z}')$$

where the first inequality follows from the definition of $B(N_{l,\mathbb{R}},Y)$. Therefore $\widetilde{Z}=\widetilde{Z}'$ is weakly special. So $\mathrm{unif}(\widetilde{Z})$ is weakly special. □

Define

$$C(N_{l,\mathbb{R}},Y) := \{\underline{t} := (g_UU_{N_l}(\mathbb{R}),g_VV_{N_l}(\mathbb{R}),gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1})|(g_U,g_V,g)\in\mathcal{G} \text{ such that } \exists x\in\mathcal{F} \text{ with } \mathrm{unif}(\underline{t}\cdot x)\subset Y \text{ and is not contained in } \cup_{l'>l}\mathfrak{W}_{l'}(Y)\}.$$

Let ψ_l be the morphism from $B(N_{l,\mathbb{R}},Y)$ to

$$(\mathbb{G}_m(\mathbb{R})^r/\mathrm{Stab}_{\mathbb{G}_m(\mathbb{R})^r}U_{N_l}(\mathbb{R}))\times\mathrm{GSp}_{2g}(\mathbb{R})/\mathrm{Stab}_{\mathrm{GSp}_{2g}(\mathbb{R})}V_{N_l}(\mathbb{R})\times G(\mathbb{R})/N_{G(\mathbb{R})}G_{N_l}(\mathbb{R})^{+\mathrm{nc}},$$

sending $(g_U,g_V,g,x)\mapsto(g_UU_{N_l}(\mathbb{R}),g_VV_{N_l}(\mathbb{R}),gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1})$. Then there is a bijection between $\psi_l(B(N_{l,\mathbb{R}},Y))$ and $C(N_{l,\mathbb{R}},Y)$.

Lemma 4.1.6. *The set $C(N_{l,\mathbb{R}}, Y)$ (hence $\psi_l(B(N_{l,\mathbb{R}}, Y))$) is countable.*

Proof. By Lemma 4.1.5, $\mathrm{unif}((g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1}) \cdot x)$ is weakly special. Hence by Proposition 1.2.4 there exists a $\mathbb{Q}$-subgroup N' of P^{der} such that

$$(g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1}) = (U_{N'}(\mathbb{C}), V_{N'}(\mathbb{R}), G_{N'}(\mathbb{R})^{+\mathrm{nc}}). \tag{4.1.3}$$

But $g_U U_{N_l}(\mathbb{R}) = g_U U_{N_l}(\mathbb{C}) \cap U(\mathbb{R})$ and $U_{N'}(\mathbb{R}) = U_{N'}(\mathbb{C}) \cap U(\mathbb{R})$, so

$$(g_U U_{N_l}(\mathbb{R}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1}) = (U_{N'}(\mathbb{R}), V_{N'}(\mathbb{R}), G_{N'}(\mathbb{R})^{+\mathrm{nc}}).$$

So $C(N_{l,\mathbb{R}}, Y)$, and therefore $\psi_l(B(N_{l,\mathbb{R}}, Y))$ is countable. □

Proposition 4.1.7. *For any $l > 0$ and N_l,*

1. *the set $C(N_{l,\mathbb{R}}, Y)$ (hence $\psi_l(B(N_{l,\mathbb{R}}, Y))$) is finite;*

2. *the set $\cup_{l' \geqslant l} \mathfrak{W}_{l'}(Y)$ is definable;*

Proof. We prove the two statements together by induction on l.

<u>*Step I.*</u> Let d be the maximum of the dimensions of weakly special subvarieties of positive dimension contained in Y. For any N_d, $B(N_{d,\mathbb{R}}, Y)$ is definable by (4.1.2), and hence $\psi_d(B(N_{d,\mathbb{R}}, Y))$ is definable since ψ_d is algebraic. So $\psi_d(B(N_{d,\mathbb{R}}, Y))$, and therefore $C(N_{d,\mathbb{R}}, Y)$, is finite by Lemma 4.1.6.

Consider all the triples

$$\mathfrak{W}_d(Y, \mathcal{T}) := \{(U', V', G') \in \mathcal{T} |\ \exists x \in \mathcal{F} \text{ with } \mathrm{unif}((U'(\mathbb{C}), V'(\mathbb{R}), G'(\mathbb{R})^+)x) \\ \text{weakly special of dimension } d \text{ contained in } Y\}.$$

By Lemma 4.1.4, there exist finitely many triples $(U'_i, V'_i, G'_i) \in \mathcal{T}$ $(i = 1, ..., n)$ such that any $\underline{t} \in \mathfrak{W}_d(Y, \mathcal{T})$ is of the form $\underline{g} \cdot (U'_i, V'_i, G'_i)$ for some $\underline{g} \in \mathcal{G}$ and some i. Furthermore, by Proposition 1.2.4, we may assume

$$(U'_i, V'_i, G'_i) = (U_{N'_i,\mathbb{R}}, V_{N'_i,\mathbb{R}}, G^{+\mathrm{nc}}_{N'_i,\mathbb{R}})$$

for some $N'_i < Q$ $(i = 1, ..., n)$. But we just proved that $C(N'_{i,\mathbb{R}}, Y)$ is finite $(\forall i = 1, ..., n)$. Hence $\mathfrak{W}_d(Y, \mathcal{T})$ is a finite set. Again by Propostition 1.2.4, each triple of $\mathfrak{W}_d(Y, \mathcal{T})$ equals $(U_{N',\mathbb{R}}, V_{N',\mathbb{R}}, G^{+\mathrm{nc}}_{N',\mathbb{R}})$ for some $N' < P$. We shall denote this triple by N' for simplicity.

Hence

$$\mathfrak{W}_d(Y) = \bigcup_{N' \in \mathfrak{W}_d(Y,\mathcal{T})} \bigcup_{\substack{(1,1,1,x) \\ \in B(N'_{\mathbb{R}}, Y)}} \mathrm{unif}((N'(\mathbb{R})^+ U_{N'}(\mathbb{C})x)$$

is definable.

Step II. For any l and N_l, $B(N_{l,\mathbb{R}}, Y)$ is definable by (4.1.2) and induction hypothesis (2). Arguing as in the previous case we get that $C(N_{l,\mathbb{R}}, Y)$ is finite. Define

$$\mathfrak{W}_l(Y,\mathcal{T}) := \{(U',V',G') \in \mathcal{T} \mid \exists x \in \mathcal{F} \text{ with } \mathrm{unif}((U'(\mathbb{C}),V'(\mathbb{R}),G'(\mathbb{R})^+)x) \text{ weakly special of dimension } l \text{ contained in } Y \text{ but not contained in } \cup_{l'>l}\mathfrak{W}_{l'}(Y)\}.$$

Arguing as in the previous case we can get that $\mathfrak{W}_l(Y,\mathcal{T})$ is a finite set and each element of it equals $(U_{N',\mathbb{R}}, V_{N',\mathbb{R}}, G^{+\mathrm{nc}}_{N',\mathbb{R}})$ for some $N' < P$. Hence

$$\bigcup_{l' \geqslant l} \mathfrak{W}_{l'}(Y) = \bigcup_{l'>l} \mathfrak{W}_{l'}(Y) \cup \bigcup_{N' \in \mathfrak{W}_l(Y,\mathcal{T})} \bigcup_{\substack{(1,1,1,x) \\ \in B(N'_{\mathbb{R}},Y)}} \mathrm{unif}(N'(\mathbb{R})^+ U_{N'}(\mathbb{C})x)$$

is definable by induction hypothesis (2). □

From now on, for any connected subgroup $N^\dagger$ of P, we will denote by $\mathfrak{F}(N^\dagger)$ the set of all weakly special subvarieties of S defined by the group $N^\dagger$ (see the beginning of this section) and $\mathfrak{F}(N^\dagger, Y) := \{Z \in \mathfrak{F}(N^\dagger) \text{ s.t. } Z \subset Y\}$. Remark that when proving Proposition 4.1.7, we have also given the following description of $\mathfrak{W}(Y) = \cup_{l=1}^d \mathfrak{W}_l(Y)$:

$$\mathfrak{W}(Y) = \bigcup_{N'} \mathrm{unif}(N'(\mathbb{R})^+ U'_N(\mathbb{C})\text{-orbits contained in } \mathrm{unif}^{-1}(Y)) = \bigcup_{N'} \bigcup_{Z \in \mathfrak{F}(N',Y)} Z \tag{4.1.4}$$

which is a finite union on N''s and each N' is of positive dimension. We have assumed that $\mathfrak{W}(Y)$ is Zariski dense in Y (otherwise there is nothing to prove). Therefore by (4.1.4), there exists an N_1 of positive dimension such that

$$\bigcup_{Z \in \mathfrak{F}(N_1,Y)} Z \tag{4.1.5}$$

is Zariski dense in Y.

We now prove $N_1 \lhd P$. If not, then by Proposition 4.1.1, $\cup_{Z \in \mathfrak{F}(N_1)} Z$ equals a finite union of proper special subvarieties of S. The intersection of this union and Y is not Zariski dense in Y since Y is Hodge generic in S. This is a contradiction. Hence $N_1 \lhd P$.

Consider the diagram

$$\begin{array}{ccc} (P,\mathcal{X}^+) & \xrightarrow{\rho_1} & (P_1,\mathcal{X}_1^+) := (P,\mathcal{X}^+)/N_1 \\ {\scriptstyle \mathrm{unif}}\downarrow & & \downarrow{\scriptstyle \mathrm{unif}_1} \\ S & \xrightarrow{[\rho_1]} & S_1 \end{array} \tag{4.1.6}$$

and let $Y_1 := \overline{[\rho_1](Y)}$, which is Hodge generic in S_1. Since $\dim(N_1) > 0$, $\dim(S_1) < \dim(S)$. It is not hard to prove $[\rho]^{-1}(Y_1) = Y$ by the fact (4.1.5). If the union of positive-dimensional weakly special subvarieties contained in

Y_1 is not Zariski dense in Y_1, then take $N = N_1$. Otherwise by the same argument, there exists a normal subgroup $N_{1,2}$ of P_1 such that $\dim(N_{1,2}) > 0$ and $\cup_{Z\in\mathfrak{F}(N_{1,2},Y_1)} Z$ is Zariski dense in Y_1. Let $N_2 := \rho_1^{-1}(N_{1,2})$, then $N_2 \lhd P$. Draw the same diagram (4.1.6) with N_2 instead of N_1, then we get a mixed Shimura variety S_2 with $\dim(S_2) < \dim(S_1)$ and a Hodge generic subvariety Y_2 of S_2. Continue the process (if the union of positive-dimensional weakly special subvarieties contained in Y_2 is Zariski dense in Y_2).

Since $\dim(S) < \infty$, this process will end in a finite step. Hence there exists a number $k > 0$ such that the union of positive-dimensional weakly special subvarieties contained in Y_k is not Zariski dense in Y_k. Then $N := N_k$ is the desired subgroup of P. □

4.2 Lower bound for Galois orbits of special points

For pure Shimura varieties, Ullmo and Pila-Tsimerman have explained separately in [64, §5] [50, §7] how to deduce the André-Oort Conjecture from Ax-Lindemann with a suitable lower bound for Galois orbits of special points. In this section we prove that in order to get a suitable lower bound for Galois orbits of special points for an arbitrary mixed Shimura variety, it is enough to have one for its pure part.

In this section, we will consider mixed Shimura data (resp. varieties) instead of only connected ones. See Definition 1.1.12.

Let $(P, \mathcal{X})$ be a mixed Shimura datum. Let $\pi\colon (P, \mathcal{X}) \to (G, \mathcal{X}_G)$ be the projection to its pure part. We use the notation of §1.1.2.5. In particular, we fix a Levi decomposition $P = W \rtimes G$ and an embedding $(G, \mathcal{X}_G) \hookrightarrow (P, \mathcal{X})$ as in [71, pp 6].

Let K be an open compact subgroup of $P(\mathbb{A}_f)$ defined as follows: for $M > 3$ even, $K_U := MU(\widehat{\mathbb{Z}})$, $K_V := MV(\widehat{\mathbb{Z}})$, $K_W := K_U \times K_V$ with the group law as in §1.1.2.5, $K_G := \{g \in G(\widehat{\mathbb{Z}}) | g \equiv 1 \bmod M\}$ and $K := K_W \rtimes K_G$.

Let s be a special point of $M_K(P, \mathcal{X})$ which corresponds to a special point $x \in \mathcal{X}$. The group $\mathrm{MT}(x)$ is of the form wTw^{-1} for a torus $T \subset G$ and $w \in W(\mathbb{Q})$. Let $\mathrm{ord}(w) \in \mathbb{Z}_{>0}$ be the smallest integer such that $\mathrm{ord}(w)w \in W(\mathbb{Z})$. Define *the order of* s to be $N(s) := \mathrm{ord}(w)$.

Remark 4.2.1. *It is not hard to show that if the fiber of* $S \xrightarrow{[\pi]} S_G$ *is a semi-abelian variety, then* $N(s)$ *coincides with the order of* s *as a torsion point on the fiber (up to a constant).*

Attached to $(P, \mathcal{X})$ there is a number field $E = E(P, \mathcal{X})$ called the **reflex field** and $M_K(P, \mathcal{X})$ is defined over E (cf. [53, 11.5]). We want a comparison of $|\mathrm{Gal}(\overline{\mathbb{Q}}/E)s|$ and $|\mathrm{Gal}(\overline{\mathbb{Q}}/E)[\pi](s)|$.

Define $(G^w, \mathcal{X}_{G^w}) := (wGw^{-1}, w^{-1} \cdot \mathcal{X}_G)$, $K_{G^w} := G^w(\mathbb{A}_f) \cap K$ and $K'_G :=$

$w^{-1}K_{G^w}w$, then we have the following commutative diagram:

$$\begin{array}{ccc} M_{K_{G^w}}(G^w, \mathcal{X}_{G^w}) & \hookrightarrow & M_K(P, \mathcal{X}) \\ \wr\downarrow [w^{-1}\cdot] & & \downarrow [\pi] \\ M_{K'_G}(G, \mathcal{X}_G) & \xrightarrow{\rho} & M_{K_G}(G, \mathcal{X}_G) \end{array} .$$

All the morphisms in this diagram are defined over E since the reflex field of $(P, \mathcal{X})$, $(G, \mathcal{X}_G)$ and $(G^w, \mathcal{X}_{G^w})$ are all E. Denote by $s' := [w^{-1}\cdot](s)$. Let $T^w := wTw^{-1}$. Let $K'_T := K \cap T^w(\mathbb{A}_f)$ and let $K_T := K \cap T(\mathbb{A}_f)$. The following inequality follows essentially from [66, §2.2] (note that we do not need GRH for this inequality since [66, Lemma 2.13, 2.14] are not used!). We refer to the Appendix of this chapter, or more concretely Theorem 4.4.1, for a more precise version.

$$\begin{aligned} |\mathrm{Gal}(\overline{\mathbb{Q}}/E)s| &= |\mathrm{Gal}(\overline{\mathbb{Q}}/E)s'| \\ &\geqslant B^{i(T)}|K_T/K'_T||\mathrm{Gal}(\overline{\mathbb{Q}}/E)\rho(s')| \\ &= B^{i(T)}|K_T/K'_T||\mathrm{Gal}(\overline{\mathbb{Q}}/E)[\pi](s)| \end{aligned} \tag{4.2.1}$$

for some $B \in (0,1)$ depending only on $(P, \mathcal{X})$.

Write $w = (u, v)$ under the identification $W \simeq U \times V$ in §1.1.2.5. All elements of $w^{-1}Kw$ are of the form

$$(-u, -v, 1)(u', v', g')(u, v, 1) = (u' - (u - g'u) - \Psi(v, v'), v' - (v - g'v), g')$$

with $(u', v', g') \in K$. Since $K'_T = w^{-1}K_{T^w}w = w^{-1}Kw \cap T(\mathbb{A}_f)$, this element is in K'_T iff

- $u' = u - g'u + \Psi(v, v') \in K_U$
- $v' = v - g'v \in K_V$
- $g' \in T(\mathbb{A}_f) \cap K_G = K_T$.

So

$$t \in w^{-1}K_{T^w}w \iff \begin{cases} t \in K_T; \\ v - tv \in K_V = MV(\widehat{\mathbb{Z}}); \\ u - tu + \Psi(v, v - tv) \in K_U = MU(\widehat{\mathbb{Z}}). \end{cases} \tag{4.2.2}$$

Lemma 4.2.2. $|K_T/K'_T| \geqslant \mathrm{ord}(w)\prod_{p|\,\mathrm{ord}(w)}(1 - \frac{1}{p})$.

Proof. Let T' be the image of $\mathbb{G}_{m,\mathbb{R}} \xrightarrow{\omega} \mathbb{S} \xrightarrow{w^{-1}\cdot x} G_{\mathbb{R}}$, then it is an algebraic torus defined over $\mathbb{Q}$ by Remark 1.1.13(1). We always have $T' < T$. If T' is trivial, then $P = G$ is adjoint by reason of weight, and $\mathrm{ord}(w) = 1$. If not, $T' \simeq \mathbb{G}_{m,\mathbb{Q}}$ and

$$T'(M) := \{t' \in T'(\widehat{\mathbb{Z}}) | t' \equiv 1 \bmod (M)\} \subset K_G \cap T(\mathbb{A}_f) = K_T.$$

So
$$T'(M)/(T'(M)\cap w^{-1}K_{T^w}w)\hookrightarrow K_T/w^{-1}K_{T^w}w.$$
Hence it is enough to prove that LHS is of cardinality$\geqslant \mathrm{ord}(w)$.

Since T' acts on V and U via a scalar, $t'\in T'(M)\cap w^{-1}K_{T^w}w$ iff

1. $t'\in T'(M)$
2. $v-t'v\in MV(\widehat{\mathbb{Z}})$
3. $u-t'u\in MU(\widehat{\mathbb{Z}})$.

Let $t'\in T'(M)\subset T'(\widehat{\mathbb{Z}})=\widehat{\mathbb{Z}}^*$. Suppose $\mathrm{ord}(w)=\prod p^{n_p}$ and $M=\prod p^{m_p}$. If $n_p=0$, then condition (2) and (3) are automatically satisfied. If $n_p>0$, then condition (2) and (3) imply that $t'_p=1+a_{n_p+m_p}p^{n_p+m_p}+...\in\mathbb{Z}_p^*$, hence
$$|T'(\mathbb{Z}_p)\cap T'(M)/(T'(\mathbb{Z}_p)\cap T'(M)\cap w^{-1}K_{T^w,p}w)|=p^{n_p-1}(p-1). \quad (4.2.3)$$
To sum up,
$$|T'(M)/(T'(M)\cap w^{-1}K_{T^w}w)|=\mathrm{ord}(w)\prod_{p|\,\mathrm{ord}(w)}(1-\frac{1}{p}). \quad (4.2.4)$$
□

Theorem 4.2.3. *For any $\varepsilon\in(0,1)$, there exist a positive constant C_ε (depending only on $(P,\mathcal{X})$ and ε) such that*
$$|\mathrm{Gal}(\overline{\mathbb{Q}}/E)s|\geqslant C_\varepsilon N(s)^{1-\varepsilon}|\mathrm{Gal}(\overline{\mathbb{Q}}/E)[\pi](s)|.$$

Proof. We have proved in Lemma 4.2.1
$$p|\,\mathrm{ord}(w)\iff K_{T,p}\neq K'_{T,p}. \quad (4.2.5)$$
Hence denoting by $\varsigma(M):=|\{p,p|M\}|$ for any $M\in\mathbb{Z}_{>0}$, we have by Lemma 4.2.1
$$|\mathrm{Gal}(\overline{\mathbb{Q}}/E)s|\geqslant B^{\varsigma(N(s))}N(s)\prod_{p|N(s)}(1-\frac{1}{p})|\mathrm{Gal}(\overline{\mathbb{Q}}/E)\rho(s')|$$
by Lemma 4.2.2. Now the theorem follows from the basic facts of elementary math:
$$\forall\varepsilon\in(0,1),\text{ there exists }C_\varepsilon>0\text{ such that }B^{\varsigma(N(s))}N(s)^\varepsilon\geqslant C_\varepsilon. \quad (4.2.6)$$
$$\forall\varepsilon\in(0,1),\text{ there exists }C'_\varepsilon>0\text{ such that }N(s)^\varepsilon\prod_{p|N(s)}(1-\frac{1}{p})\geqslant C'_\varepsilon. \quad (4.2.7)$$
□

Corollary 4.2.4. *For A an abelian variety over a number field $k \subset \mathbb{C}$ and t a torsion point of $A(\mathbb{C})$, denote by $N(t)$ its order and $k(t)$ the field of definition of t over k.*

Let $g, d \in \mathbb{N}_+$ and let $\varepsilon \in (0,1)$. There exists $c > 0$ such that for all number fields $k \subset \mathbb{C}$ of degree d over $\mathbb{Q}$, all g-dimensional CM abelian varieties A over k and all torsion points t in $A(\mathbb{C})$,

$$[k(t) : k] \geqslant cN(t)^{1-\varepsilon}.$$

Proof. (compare with [59]) By Zarhin's trick, it suffices to give a proof for A principally polarized. Such an A can be realized as a fiber of $\mathfrak{A}_g(4) \to \mathcal{A}_g(4)$, and any torsion point t of A is a special point of $\mathfrak{A}_g(4)$. Now this result is a direct consequence of Proposition 4.2.3. □

Remark 4.2.5. *The lower bound of the Galois orbit of a special point for pure Shimura varieties is given by [64, Conjecture 2.7]. It has been proved under the Generalized Riemann Hypothesis by Ullmo-Yafaev [66]. For the case of $\mathcal{A}_g$, it is equivalent to the following conjectural lower bound (suggested and proved for $g = 2$ by Edixhoven [19, 18]): suppose that $x \in \mathcal{A}_g$ is a special point. Let A_x denote the CM abelian variety parametrised by x and let R_x be the center of $\mathrm{End}(A_x)$, then there exists $\delta(g) > 0$ such that*

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})x| \gg_g |\,\mathrm{disc}(R_x)|^{\delta(g)}. \tag{4.2.8}$$

For their equivalence see [62, Theorem 7.1]. The best unconditional result is given by Tsimerman [62, Theorem 1.1]: (4.2.8) is true when $g \leqslant 6$ (and for $g \leqslant 3$ by a similar method in [68]).

Hence for a mixed Shimura variety of Siegel type of genus g and any special point x, Theorem 4.2.3 tells us that if [64, Conjecture 2.7] is verified for the pure part, then for any $\varepsilon \in (0,1)$, there exists $\delta(g) > 0$ such that

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})x| \gg_{g,\varepsilon} N(x)^{1-\varepsilon}|\,\mathrm{disc}(R_{[\pi](x)})|^{\delta(g)}.$$

4.3 The André-Oort conjecture and its weak form

4.3.1 The André-Oort conjecture

Theorem 4.3.1. *Let S be a connected mixed Shimura variety of abelian type (i.e. its pure part is of abelian type). Let Y be an irreducible subvariety of S containing a Zariski-dense set of special points. If (4.2.8) holds for the pure part of S (this is true if we assume GRH), then Y is special.*

In particular, by [62, Theorem 1.1], the André-Oort Conjecture holds unconditionally for any mixed Shimura variety whose pure part is a subvariety of $\mathcal{A}_6^n$.

Proof. Suppose S is associated with $(P, \mathcal{X}^+)$. Replacing Γ by a neat subgroup does not change the assumption or the conclusion, so we may assume that $\Gamma = \{\gamma \in P(\mathbb{Z}) | \gamma \equiv 1 \bmod M\}$ for some $M > 3$ even. Replacing S by the smallest connected mixed Shimura subvariety does not change the assumption or the conclusion, so we may assume that Y is Hodge generic in S. Since Y contains a Zariski-dense set of special points, we may assume that Y is defined over a number field k. Suppose that Y is not special.

If the set of positive-dimensional weakly special subvarieties of Y is Zariski dense in Y, then let N be the normal subgroup P as in Theorem 4.1.3. Consider the diagram (4.1.1), then Y is special iff $Y' := \overline{[\rho](Y)}$ is. The connected mixed Shimura variety S' is again of abelian type. Replacing (S, Y) by (S', Y'), we may assume that the set of positive-dimensional special subvarieties of Y is not Zariski dense in Y.

Now we are left prove that the set of special points of Y which do not lie in any positive-dimensional special subvariety is finite.

By definition, there exists a Shimura morphism $(G, \mathcal{X}_G^+) \to \prod_{i=1}^r (\mathrm{GSp}_{2g}^{(i)}, \mathbb{H}_g^{+(i)})$ (the upper-index (i) is to distinguish different factors) such that $G \to \prod_{i=1}^r \mathrm{GSp}_{2g}^{(i)}$ has a finite kernel (contained in the center) and $\mathcal{X}_G^+ \hookrightarrow \prod_{i=1}^r \mathbb{H}_g^{+(i)}$. Therefore under Proposition 1.3.3, we can identify $\mathcal{X}^+$ as a subspace of $U(\mathbb{C}) \times V(\mathbb{R}) \times \mathbb{H}_g^{+r}$. Then any special point is contained in $U(\mathbb{Q}) \times V(\mathbb{Q}) \times (\mathbb{H}_g^{+r} \cap M_{2g}(\overline{\mathbb{Q}})^r)$ and hence we can define its height (for $\overline{\mathbb{Q}}$-points, see [12, Definition 1.5.4 multiplicative height]).

Now take $\mathcal{F}$ as in §3.3.1. For any special point $x \in S$, take a representative $\widetilde{x} \in \mathrm{unif}^{-1}(x)$ in $\mathcal{F}$, then by [49, Theorem 3.1], $H(\widetilde{x}_{G,i}) \ll |\operatorname{disc}(R_{[\pi](x)_i})|^{B_g}$ for a constant B_g ($\forall i = 1, ..., r$). By choice of $\mathcal{F}$, $H(\widetilde{x}_V)$, $H(\widetilde{x}_U) \ll N(x)$ (see Remark 1.3.4). If (4.2.8) holds, then by Proposition 4.2.3

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/k)x| \gg_g H(\widetilde{x})^{\varepsilon(g)}$$

for some $\varepsilon(g) > 0$. Hence for $H(\widetilde{x}) \gg 0$, Pila-Wilkie [48, 3.2] implies that $\exists \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/k)$ such that $\widetilde{\sigma(x)}$ is contained in a connected semi-algebraic subset $\widetilde{Z}$ of $\mathrm{unif}^{-1}(Y) \cap \mathcal{F}$ of positive dimension. Let Z' be an irreducible component of $\overline{\mathrm{unif}(\widetilde{Z})}$ containing $\mathrm{unif}(\widetilde{\sigma(x)})$. Theorem 3.1.4 tells us that Z' is weakly special. Hence $\sigma^{-1}(Z')$ is weakly special containing a special point x, and therefore is special. But $\dim(Z') > 0$ since $\dim(\widetilde{Z}) > 0$. Hence $\sigma^{-1}(Z')$ is special of positive dimension. To sum up, the heights of the elements of

$$\{\widetilde{x} \in \mathrm{unif}^{-1}(Y) \cap \mathcal{F} \text{ special and } \mathrm{unif}(\widetilde{x}) \text{ is not contained in any positive-dimensional special subvariety}\}$$

is uniformly bounded, and hence this set is finite by Northcott's theorem [12, Theorem 1.6.8]. ☐

4.3.2 The weak form of the André-Oort conjecture

By the proof of Theorem 4.3.1, we can see that the only obstacle left to claim the whole André-Oort conjecture for mixed Shimura varieties of abelian type is the lower bound (4.2.8). However if we consider a weaker version of the André-Oort conjecture, this obstacle is removed by a series of work of Habegger-Pila [24] and Orr [43]. Thus by a similar proof to Theorem 4.3.1, we can prove the following theorem unconditionally. This theorem generalizes the previous work of Edixhoven-Yafaev [72, 20] (for curves in pure Shimura varieties) and Klingler-Ullmo-Yafaev [66, 30] (for pure Shimura varieties). Its p-adic version for $\mathfrak{A}_g$ has been proved by Scanlon [58] based on the result of Moonen for $\mathcal{A}_g$ [40].

Theorem 4.3.2. *Let S be a connected mixed Shimura variety whose pure part S_G is a subvariety of $\mathcal{A}_g$ for some g. Denote by $S \xrightarrow{[\pi]} S_G$. Let Y be an irreducible subvariety of S and let a be a special point of $\mathcal{A}_g$ whose corresponding abelian variety is denoted by A_a. Consider the set*

$$\Sigma'_a := \{s \in S \text{ special such that } A_{[\pi]s} \text{ is isogenous to } A_a, \text{ where } A_{[\pi]s} \text{ is the abelian variety represented by } [\pi]s\}.$$

If $\overline{Y \cap \Sigma'_a} = Y$, then Y is a special subvariety.

Proof. We may assume $a \in [\pi]Y$. Suppose S is associated with $(P, \mathcal{X}^+)$. Replacing Γ by a neat subgroup does not change the assumption or the conclusion, so we may assume that $\Gamma = \{\gamma \in P(\mathbb{Z}) | \gamma \equiv 1 \bmod M\}$ for some $M > 3$ even. Replacing S by the smallest connected mixed Shimura subvariety does not change the assumption or the conclusion, so we may assume that Y is Hodge generic in S.

Let $(G, \mathcal{X}_G^+) := (P, \mathcal{X}^+)/\mathcal{R}_u(P)$. By Theorem 4.1.3, such a group N (which may be trivial) exists: N is the maximal normal subgroup of P such that the followings hold:

- there exists a diagram of Shimura morphisms

$$\begin{array}{ccccc} (P,\mathcal{X}^+) & \xrightarrow{\rho} & (P',\mathcal{X}'^+) := (P,\mathcal{X}^+)/N & \xrightarrow{\pi'} & (G',\mathcal{X}'^+_G) := (P',\mathcal{X}'^+)/\mathcal{R}_u(P') \\ \downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}'} & & \downarrow{\scriptstyle \mathrm{unif}'_G} \\ S & \xrightarrow{[\rho]} & S' & \xrightarrow{[\pi']} & S'_G \end{array}$$

- the union of positive-dimensional weakly special subvarieties which are contained in $Y' := \overline{[\rho](Y)}$ is not Zariski dense in Y';

- $Y = [\rho]^{-1}(Y')$.

Suppose that Y is not special. Then Y' is not a special subvariety of S'. On the other hand, Y' is defined over a number field since it contains a Zariski dense subset of special points.

Define $W_N := \mathcal{R}_u(N) < W := \mathcal{R}_u(P)$ and $G_N := N/W_N \lhd G < \mathrm{GSp}_{2g}$. The reductive group G decomposes as an almost direct product $Z(G)^\circ H_1 ... H_r$ with all H_i's simple. Without any loss of generality, we may assume that H_1,...,H_l are the simple factors of G which appear in the decomposition of G_N. Define $G_N^\perp := H_{l+1}...H_r$. Define $T := \mathrm{MT}(a)$, then T is a torus since a is a special point of $\mathcal{A}_g$.

Let $G_1 := G_N^\perp T$. This is a subgroup of G (and therefore a subgroup of GSp_{2g}). Moreover, it defines a connected Shimura subdatum $(G_1, \mathcal{X}_{G_1}^+)$ of $(\mathrm{GSp}_{2g}, \mathbb{H}_g^+)$ and hence its associated connected Shimura subvariety S_{G_1} of $\mathcal{A}_g$ such that $a \in S_{G_1}$. Recall that $(P', \mathcal{X}'^+) = (P, \mathcal{X}^+)/N$ and $(G', \mathcal{X}'^+_G) = (G, \mathcal{X}_G^+)/G_N$. Therefore the natural Shimura morphisms

$$(G_1, \mathcal{X}_{G_1}^+) \hookrightarrow (G, \mathcal{X}_G^+) \twoheadrightarrow (G', \mathcal{X}'^+_G)$$

identify $\mathcal{X}_{G_1}^+$ and $\mathcal{X}'^+_G$.

Consider the connected mixed Shimura datum $(P, \mathcal{X}^+)$. Then $W := \mathcal{R}_u(P)$ is a G_1-module such that the action of G_1 on W induces a Hodge-structure of type $\{(-1,0),(0,-1),(-1,-1)\}$ on $\mathrm{Lie}\, W$. Therefore by Proposition 1.1.23, there exists a connected mixed Shimura datum $(P_1, \mathcal{X}_1^+)$ such that $P_1 = W \rtimes G_1$ and $(G_1, \mathcal{X}_{G_1}) = (P_1, \mathcal{X}_1^+)/W$. Now $(P_1, \mathcal{X}_1^+)$ is a connected mixed Shimura subdatum of $(P, \mathcal{X}^+)$. Since $N \lhd P$, we have $W_N \lhd P_1$. Now we have the following diagram of Shimura morphisms:

$$\begin{array}{ccccccc}
(P_2, \mathcal{X}_2^+) := (P_1, \mathcal{X}_1^+)/W_N & \overset{\rho'}{\twoheadleftarrow} & (P_1, \mathcal{X}_1^+) & \overset{j}{\hookrightarrow} & (P, \mathcal{X}^+) & \overset{\rho}{\twoheadrightarrow} & (P', \mathcal{X}'^+) = (P, \mathcal{X}^+)/N \\
\downarrow{\scriptstyle \mathrm{unif}_2} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{unif}'} \\
S_2 & \overset{[\rho']}{\twoheadleftarrow} & S_1 & \overset{[j]}{\longrightarrow} & S & \overset{[\rho]}{\longrightarrow} & S'
\end{array}.$$

Then the map $\rho \circ j \circ \rho'^{-1} \colon (P_2, \mathcal{X}_2^+) \to (P', \mathcal{X}'^+)$ is well-defined and is a Shimura morphism. Hence Y' is a special subvariety of S' iff $Y_2 := ([\rho] \circ [j] \circ [\rho']^{-1})^{-1}(Y')$ is a special subvariety of S_2. Hence it suffices to prove that Y_2 is special. But $\mathcal{X}_2^+$ and $\mathcal{X}'^+$ are identified under $\rho \circ j \circ \rho'^{-1}$ by the discussion in the last paragraph, so the union of positive-dimensional weakly special subvarieties of Y_2 is not Zariski dense in Y_2 by choice of Y'. Therefore we are left to prove that the set of special points of Y_2 which do not lie in any positive-dimensional special subvariety is finite. Remark that Y_2 is defined over a number field (which we call k) since Y' is.

Take the pure part of the diagram above, we get the following diagram of

Shimura morphisms between pure Shimura data and pure Shimura varieties:

$$\begin{array}{ccccccc} (G_2, \mathcal{X}_{G_2}^+) & \xleftarrow[\sim]{\rho'_G} & (G_1, \mathcal{X}_{G_1}^+) & \xhookrightarrow{j_G} & (G, \mathcal{X}_G^+) & \xrightarrow{\rho_G} & (G', \mathcal{X}_G'^+) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S_{G_2} & \xleftarrow[\sim]{[\rho'_G]} & S_{G_1} & \xrightarrow{[j_G]} & S_G & \xrightarrow{[\rho_G]} & S'_G \end{array}.$$

Therefore $\mathcal{X}_{G_2}^+$ can be seen as a subset of $\mathcal{X}_G^+$, and hence of $\mathbb{H}_g^+$. Denote by $[\pi_2]\colon S_2 \to S_{G_2}$. Let

$$\Sigma''_a := \{t \in S_2 \text{ special such that } A_{[\pi_2]t} \text{ is isogenous to } A_a, \text{ where } A_{[\pi_2]t} \text{ is the abelian variety represented by } [\pi_2]t\}.$$

Since $\overline{Y \cap \Sigma'_a} = Y$, we have $\overline{Y' \cap [\rho](\Sigma'_a)} = Y'$. But then by the identification of $\mathcal{X}_2^+$ and $\mathcal{X}'^+$, we get that

$$\overline{Y_2 \cap \Sigma''_a} = Y_2.$$

For any $t \in \Sigma''_a$, take a representative $\widetilde{t} \in \mathrm{unif}_2^{-1}(t)$ in the fundamental set $\mathcal{F}$ as in §3.3.1. Then $\widetilde{t} = (\widetilde{t}_U, \widetilde{t}_V, \widetilde{t}_G) \in U_2(\mathbb{Q}) \times V_2(\mathbb{Q}) \times (\mathbb{H}_g^+ \cap M_{2g}(\overline{\mathbb{Q}}))$ and hence we can define its height. By choice of $\mathcal{F}$, both $H(\widetilde{t}_U)$ and $H(\widetilde{t}_V)$ are bounded by $N(t)$ which is defined as in the paragraph above Remark 4.2.1 (see Remark 1.3.4). But up to constants depending only on a (or more explicitely, only on $H(\widetilde{a})$), $H(\widetilde{t}_G)$ is polynomially bounded from above by the minimum degree of the isogenies $A_{[\pi_2]t} \to A_a$. This follows from [43, Proposition 4.1, Section 4.2]. But the minimum degree of the isogenies $A_{[\pi_2]t} \to A_a$ is polynomially bounded from above by $|\mathrm{Gal}(\overline{\mathbb{Q}}/k)[\pi_2]t|$. This follows from [43, Theorem 5.1]. Hence by Theorem 4.2.3,

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/k)t| \gg_{g,\widetilde{a}} H(\widetilde{t})^{\mu(g,\widetilde{a})}$$

for some $\mu(g, \widetilde{a}) > 0$. Hence for $H(\widetilde{t}) \gg 0$, Pila-Wilkie [48, 3.2] implies that there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/k)$ such that $\widetilde{\sigma(t)}$ is contained in a connected semi-algebraic subset $\widetilde{Z}$ of $\mathrm{unif}_2^{-1}(Y_2) \cap \mathcal{F}$ of positive dimension. Let Z' be an irreducible component of $\overline{\mathrm{unif}(\widetilde{Z})}$ containing $\mathrm{unif}(\widetilde{\sigma(t)})$. Theorem 3.1.4 tells us that Z' is weakly special. Hence $\sigma^{-1}(Z')$ is weakly special containing a special point t, and therefore is special. But $\dim(Z') > 0$ since $\dim(\widetilde{Z}) > 0$. Hence $\sigma^{-1}(Z')$ is special of positive dimension. To sum it up, the heights of the elements of

$$\{\widetilde{t} \in \mathrm{unif}_2^{-1}(Y_2) \cap \mathcal{F} \text{ special and } \mathrm{unif}_2(\widetilde{t}) \text{ is not contained in a positive-dimensional special subvariety of } S_2\}$$

is uniformly bounded from above. Therefore this set is finite by Northcott's theorem. □

4.4 Appendix: comparison of Galois orbits of special points of pure Shimura varieties

Let $(G, \mathcal{X}_G)$ be a pure Shimura datum satisfying

$$\begin{array}{c} Z(G)^\circ \text{ is an almost direct product of a } \mathbb{Q}\text{-split torus } Z_G^s \\ \text{with a torus of compact type } Z_G^c \text{ defined over } \mathbb{Q} \end{array} \tag{SV5}$$

In this case, G is an almost direct product of Z_G^s with $G^c := Z_G^c G^{\mathrm{der}}$. Let $E = E(G, \mathcal{X}_G)$ be its reflex field and let $K' = \prod_p K'_p \subset K = \prod_p K_p$ be two neat open compact subgroups of $G(\mathbb{A}_f)$. We have a natural morphism

$$\rho\colon M_{K'}(G, \mathcal{X}_G) \to M_K(G, \mathcal{X}_G). \tag{4.4.1}$$

By [37, Theorem 5.5, Proposition 5.2], $M_{K'}(G, \mathcal{X}_G)$, $M_K(G, \mathcal{X}_G)$ and ρ can all be defined over E.

Let s be a special point of $M_{K'}(G, \mathcal{X}_G)$, then $s \in M_{K'}(G, \mathcal{X}_G)(\overline{E})$. The goal of this section is to compare $|\mathrm{Gal}(\overline{E}/E)s|$ and $|\mathrm{Gal}(\overline{E}/E)\rho(s)|$. Let $T := \mathrm{MT}(s)$ be the Mumford-Tate group of s. Define $K'_T := K' \cap T(\mathbb{A}_f)$ and $K_T := K \cap T(\mathbb{A}_f)$. Then $K'_T = \prod_p K'_{T,p}$ and $K_T = \prod_p K_{T,p}$. Now we can state our theorem:

Theorem 4.4.1. *There exists a constant $B \in (0, 1)$ depending only on $(G, \mathcal{X})$ such that*

$$|\mathrm{Gal}(\overline{E}/E)s| \geqslant B^{i(T)} |K_T/K'_T| |\mathrm{Gal}(\overline{E}/E)\rho(s)|$$

where $i(T) = |\{p : K_{T,p} \neq K'_{T,p}\}|$.

Proof. This is a direct consequence of Lemma 4.4.4, (4.4.2), Lemma 4.4.6 and Lemma 4.4.7. ☐

Remark 4.4.2. *This theorem has essentially been proved by Ullmo-Yafaev [66, §2.2]: the authors proved this result for a less general $(G, \mathcal{X}_G)$ and a particular K_T, but their proof also works for our $(G, \mathcal{X}_G)$ and arbitrary K_T as long as it is neat. To make the demonstration more clear, we summarize their results and arguments and see how they apply to our $(G, \mathcal{X}_G)$ and a general K_T.*

Lemma 4.4.3. *For any point $y \in M_K(G, \mathcal{X}_G)$, K acts transitively on the right on $\rho^{-1}(y)$ and the stabilizer of any point of $\rho^{-1}(y)$ is K'. By consequence ρ is étale of degree $|K/K'|$.*

Proof. (cf. [66, Lemma 2.11]) Let $y = \overline{(x, g)}$ be a point of $M_K(G, \mathcal{X})$, then $\rho^{-1}(y) = \overline{(x, gK)}$. We first prove that $\forall a \in K$,

$$\overline{(x, ga)} = \overline{(x, gak)} \text{ in } M_{K'}(G, \mathcal{X}) \iff k \in K'.$$

The direction $\Leftarrow$ is trivial. Now let us prove $\Rightarrow$. Suppose

$$\overline{(x, ga)} = \overline{(x, gak)} \in M_{K'}(G, \mathcal{X})$$

with $k \in K$. There exist $q \in G(\mathbb{Q})$ and $k' \in K'$ such that $x = qx$ and $ga = qgakk'$. The second condition implies $q \in gKg^{-1}$.

Define $G' := G/Z_G^s$, then $(G, \mathcal{X}_G)/Z_G^s = (G', \mathcal{X}_G)$ is a Shimura datum such that $Z(G')(\mathbb{R})$ is compact. Now we have $x = \overline{q}x$ and $\overline{q} \in \overline{gKg^{-1}}$ where we add $^{-}$ to denote elements and subsets of G'. The set $\overline{gKg^{-1}}$ is a neat open compact subgroup of $G'(\mathbb{A}_f)$ and $\overline{q} \in G'(\mathbb{Q})$. Since $Z(G')(\mathbb{R})$ is compact, $\mathrm{Stab}_{G'(\mathbb{R})}(x)$ is compact (see e.g. [66, Remark 2.3]). But $G'(\mathbb{Q}) \cap \overline{gKg^{-1}}$ is a lattice of $G'(\mathbb{R})$, so $\mathrm{Stab}_{G'(\mathbb{R})}(x) \cap G'(\mathbb{Q}) \cap \overline{gKg^{-1}}$ is finite. Furthermore the latter intersection must be $\{1\}$ since $\overline{gKg^{-1}}$ is neat. Therefore as an element of the latter intersection, $\overline{q} = 1$. Hence $q \in Z_G^s(\mathbb{Q}) \simeq (\mathbb{Q}^*)^n$. This implies also $q \in Z_G^s(\mathbb{A}_f) \cap gKg^{-1}$, which is a neat open compact subgroup of $Z_G^s(\mathbb{A}_f) \simeq (\mathbb{A}_f^*)^n$. But the intersection of $(\mathbb{Q}^*)^n$ with any neat open compact subgroup of $(\mathbb{A}_f^*)^n$ is trivial, hence $q = 1$.

Now $ga = gakk'$ implies $k = (k')^{-1} \in K'$. So K acts transitively on the right on $\rho^{-1}(y)$ and the stabilizer of any point of $\rho^{-1}(y)$ is K'. □

Lemma 4.4.4. $|\mathrm{Gal}(\overline{E}/E)s| \geqslant |\mathrm{Gal}(\overline{E}/E)s \cap \rho^{-1}\rho(s)| \cdot |\mathrm{Gal}(\overline{E}/E)\rho(s)|$.

Proof. (cf. [66, Lemma 2.12]) Because ρ is defined over E, $|\mathrm{Gal}(\overline{E}/E)s \cap \rho^{-1}(\sigma(\rho(s)))|$ is independent of $\sigma \in \mathrm{Gal}(\overline{E}/E)$. This allows us to conclude. □

To give a lower bound for $|\mathrm{Gal}(\overline{E}/E)s \cap \rho^{-1}\rho(s)|$, we shall work with the Shimura subdatum (T, x) of $(G, \mathcal{X}_G)$. The Shimura subdatum (T, x) is defined as follows: $T = \mathrm{MT}(s)$. By [38, Lemma 5.13], $M_{K'}(G, \mathcal{X}_G) = \coprod \Gamma(g)\backslash\mathcal{X}^+$, where $\Gamma(g) = G(\mathbb{Q})_+ \cap gK'g^{-1}$ is a congruence subgroup of $G(\mathbb{Q})$. Choose $x \in \mathcal{X}^+$ such that s is the image of x under the uniformization. The Shimura datum (T, x) still satisfies (SV5) (see e.g. [66, Remark 2.3]).

Let F be the reflex field of (T, x), then F is a finite extension of E. Define

$$\rho' \colon M_{K'_T}(T, x) \to M_{K_T}(T, x),$$

which is the restriction of ρ, then ρ' is defined over F. We have

$$|\mathrm{Gal}(\overline{E}/E)s \cap \rho^{-1}\rho(s)| \geqslant |\mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)| \tag{4.4.2}$$

Let $\pi_0(M_{K'_T}(T, x))$ be the set of geometric components of $M_{K'_T}(T, x)$. Recall that

$$\pi_0(M_{K'}(T, x)) = T(\mathbb{Q})_+\backslash T(\mathbb{A}_f)/K'_T.$$

This is a finite abelian group. The action of $\mathrm{Gal}(\overline{E}/F)$ on $\pi_0(M_{K'_T}(T, x))$ is given by the reciprocity morphism

$$r \colon \mathrm{Gal}(\overline{E}/F) \to \pi_0(M_{K'_T}(T, x)).$$

Let us describe this action more explicitly. Denote for any $\alpha \in T(\mathbb{A}_f)$ by $\overline{(x, \alpha)}$ the image of (x, α) in $M_{K'_T}(T, x)$. It is a connected component of $M_{K'_T}(T, x)$.

As sets we have the following identification:

$$\begin{array}{ccc} \{\overline{(x,\alpha)} |\ \alpha \in T(\mathbb{A}_f)\} & \xrightarrow{\sim} & \pi_0(M_{K'_T}(T,x)) \\ \overline{(x,\alpha)} & \mapsto & \overline{\alpha} \end{array} .$$

Let $\sigma \in \mathrm{Gal}(\overline{E}/F)$ and let $t \in T(\mathbb{A}_f)$ such that $\overline{t} = r(\sigma)$, then $\forall \alpha \in T(\mathbb{A}_f)$,

$$\sigma(\overline{(x,\alpha)}) = \overline{(x,t\alpha)} = \overline{(x,\alpha t)}. \tag{4.4.3}$$

Recall the following result from Ullmo-Yafaev [66, Proposition 2.9]:

Lemma 4.4.5. *There exists a positive integer A depending only on $(G, \mathcal{X})$ such that $\forall m \in T(\mathbb{A}_f)$, the image of m^A in $\pi_0(M_{K'_T}(T,x))$ is $r(\sigma)$ for some $\sigma \in \mathrm{Gal}(\overline{E}/F)$.*

Proof. [66, Proposition 2.9], which follows from Lemma 2.4-Lemma 2.8 of *loc.cit.*, announces this result when $Z(G)(\mathbb{R})$ is compact. However the only role this hypothesis plays is to guarantee that $T(\mathbb{Q})$ is discrete (hence closed) in $T(\mathbb{A}_f)$ in Lemma 2.8 of *loc.cit.*. Our hypothesis for $Z(G)$ at the beginning of this section implies that T is an almost product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$ (see e.g. [66, Remark 2.3]), and hence $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$ ([38, Theorem 5.26]). □

Lemma 4.4.6. *Let Θ_A be the image of the morphism $k \mapsto k^A$ on K_T/K'_T. We have*

1. $\Theta_A \cdot s \subset \mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)$;
2. $|\mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)| \geqslant |\Theta_A|$.

Proof. (cf. [66, Lemma 2.15 & 2.16])

1. We have $\rho'(\Theta_A \cdot s) = \rho'(s)$. So $\Theta_A \cdot s \subset \rho'^{-1}\rho'(s)$. Moreover similar to Lemma 4.4.3, K_H/K'_H acts simply transitively on $\rho'^{-1}\rho'(s)$. For any $\overline{(x,\alpha)} \in \rho'^{-1}\rho'(s)$ and $k \in K_T/K'_T$, this action is given by

$$\overline{(x,\alpha)}k = \overline{(x,\alpha k)}. \tag{4.4.4}$$

Let $m \in K_T$, then the image of m^A in $\pi_0(M_{K'_T}(T,x))$ is $r(\sigma)$ for some $\sigma \in \mathrm{Gal}(\overline{E}/F)$ by Lemma 4.4.5. It follows that the image of Θ_A in $\pi_0(M_{K'_T}(T,x)) = T(\mathbb{Q})_+ \backslash T(\mathbb{A}_f)/K'_T$ is contained in the image of $\mathrm{Gal}(\overline{E}/F)$. So for $s = \overline{(x,\beta)}$, we have $\Theta_A \cdot s \subset \mathrm{Gal}(\overline{E}/F)s$ by (4.4.4) and (4.4.3). To sum it up,

$$\Theta_A \cdot s \subset \mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s).$$

2. By (1) we have $|\mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)| \geqslant |\Theta_A \cdot s|$. Moreover we have

$$|\rho'^{-1}\rho'(s)| = |(K_T/K'_T) \cdot s| \leqslant \frac{|K_T/K'_T|}{|\Theta_A|}|\Theta_A \cdot s|$$

and

$$|K_T/K'_T| = |\rho'^{-1}\rho'(s)|$$

by the same argument for Lemma 4.4.3. These three (in)equalities yield the desired inequality. Remark that we have also proved $|\Theta_A \cdot s| = |\Theta_A|$.

□

Lemma 4.4.7. *There exists an integer $r > 0$ depending only on $(G, \mathcal{X})$ such that*

$$|\Theta_A| \geqslant \prod_{\{p : K_{T,p} \neq K'_{T,p}\}} \frac{1}{A^r}|K_{T,p}/K'_{T,p}|.$$

Proof. (cf. [66, Lemma 2.18]) Since $K_T/K'_T = \prod_p K_{T,p}/K'_{T,p}$, we have

$$\Theta_A = \prod_{\{p : K_{T,p} \neq K'_{T,p}\}} \Theta_{A,p}.$$

Let L_T be the splitting field of T and let $d := \dim(T)$. Then $[L_T : \mathbb{Q}]$ is the size of the image of the representation of $\mathrm{Gal}(\overline{E}/\mathbb{Q})$ on the character group $X^*(T)$ of T. This is a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$ and hence its size is bounded from above in terms of d only. But d is bounded from above in terms of $\dim(G)$ only, so $[L_T : \mathbb{Q}]$ is bounded from above in terms of $\dim(G)$ only.

Using a basis of the character group of T one can embed T into $\mathrm{Res}_{L_T/\mathbb{Q}}\mathbb{G}_{m,L_T}$. Via this embedding, K_T and K'_T are both subgroups of the product of $(\mathbb{Z}_p \otimes O_{L_T})^*$. The group $(\mathbb{Z}_p \otimes O_{L_T})^*$ is the direct product of the groups of units of E_v, completion of E at the place v with $v|p$. By the local unit theorem, the group of units of such an E_v is a dirct product of a cyclic group and $\mathbb{Z}_p^{[E_v:\mathbb{Q}_p]}$.

It follows that there exists a constant r depending only on $(G, \mathcal{X})$ such that $K_{T,p}/K'_{T,p}$ is a finite abelian group which is the product of at most r cyclic factors. Therefore the size of the kernel of the A-th power map on $K_{T,p}/K'_{T,p}$ is bounded by A^r, i.e.

$$\Theta_{A,p} \geqslant \frac{1}{A^r}|K_{T,p}/K'_{T,p}|.$$

□

Chapter 5

From André-Oort to André-Pink-Zannier

5.1 Main results

5.1.1 Background

In the last chapter we have studied the André-Oort conjecture, which is a subconjecture of the Zilber-Pink conjecture. In particular we have proved a weaker version of the André-Oort conjecture (Theorem 4.3.2). This weaker version corresponds to another important case of the Zilber-Pink conjecture, which we call the André-Pink-Zannier conjecture. The goal of this chapter is to study this André-Pink-Zannier conjecture.

In the whole chapter, we restrict to the case $\mathfrak{A}_g \xrightarrow{[\pi]} \mathcal{A}_g$.

Conjecture 5.1.1. *Let Y be a subvariety of $\mathfrak{A}_g$. Let $s \in \mathfrak{A}_g$ and Σ be the generalized Hecke orbit of s. If $\overline{Y \cap \Sigma} = Y$, then Y is weakly special.*

Several cases of this conjecture had been studied by André before its final form was made by Pink [54, Conjecture 1.6]. It is also closely related to a problem (Conjecture 5.1.3) proposed by Zannier. Pink has also proved [54, Theorem 5.4] that Conjecture 5.1.1 implies that Mordell-Lang conjecture.

Conjecture 5.1.1 for $\mathcal{A}_g$, the pure part of $\mathfrak{A}_g$, has been intensively studied by Orr [43, 42], generalizing the previous work of Habegger-Pila [24, Theorem 3] with the Pila-Zannier method.

The set Σ has good moduli interpretation: by Corollary 5.2.5,

$$\begin{aligned}\Sigma &= \text{division points of the polarized isogeny orbit of } s\\ &= \{t \in \mathfrak{A}_g \mid \exists n \in \mathbb{N} \text{ and a polarized isogeny}\\ &\qquad f\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t}) \text{ such that } nt = f(s)\}.\end{aligned} \tag{5.1.1}$$

There are authors who consider isogenies instead of polarized isogenies. However this does not essentially improve the result because of Zarhin's trick (see [42, Proposition 4.4]): for any isogeny $f\colon A \to A'$ between polarized abelian varieties, there exists $u \in \mathrm{End}(A^4)$ such that $f^4 \circ u\colon A^4 \to A'^4$ is a polarized isogeny. See §5.5 for more details.

Although Conjecture 5.1.1 and the André-Oort conjecture do not imply each other, they do have some overlap, which for $\mathfrak{A}_g$ is precisely Theorem 4.3.2 when $S = \mathfrak{A}_g$.

We shall divide Conjecture 5.1.1 into two cases: when s is a torsion point of $\mathfrak{A}_{g,[\pi]s}$ and when s is not a torsion point of $\mathfrak{A}_{g,[\pi]s}$. The diophantine estimates for both cases are not quite the same.

5.1.2 The torsion case

When s is a torsion point of $\mathfrak{A}_{g,[\pi]s}$, this conjecture is related to a special-point problem proposed by Zannier. We define the following "special topology" proposed by Zannier:

Definition 5.1.2. *Fix a point $a \in \mathcal{A}_g$. Then a corresponds to a principally polarized abelian variety (A_a, λ_a) of dimension g.*

1. *We say that a point $t \in \mathfrak{A}_g$ is A_a-**special** (or a-**special**) if there exists an isogeny $A_a \to \mathfrak{A}_{g,[\pi]t}$ and that t is a torsion point on the abelian variety $\mathfrak{A}_{g,[\pi]t}$. We shall denote by Σ'_a (or Σ' when there is no confusion) the set of a-special points.*

2. *We say that a point $t \in \mathfrak{A}_g$ is (A_a, λ_a)-**special** if there exists a polarized isogeny $(A_a, \lambda_a) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ and that t is a torsion point on the abelian variety $\mathfrak{A}_{g,[\pi]t}$. We shall denote by Σ_a (or Σ when there is no confusion) the set of (A_a, λ_a)-special points.*

3. *We say that a subvariety Z of $\mathfrak{A}_g$ is a-**special** if Z contains an a-special point, $[\pi]Z$ is a totally geodesic subvariety of $\mathcal{A}_g$ and Z is an irreducible component of a subgroup of $[\pi]^{-1}([\pi]Z)$.*

In view of Proposition 1.2.15, every a-special subvariety is weakly special. The following conjecture is proposed by Zannier.

Conjecture 5.1.3. *Let Y be a subvariety of $\mathfrak{A}_g$ and let $a \in \mathcal{A}_g$. If $\overline{Y \cap \Sigma'_a} = Y$, then Y is a-special.*

By (5.1.1), Conjecture 5.1.1 when s is a torsion point of $\mathfrak{A}_{g,[\pi]s}$ is equivalently to a weaker version of Conjecture 5.1.3, i.e. replace Σ'_a by Σ_a in Conjecture 5.1.3. However by [42, Proposition 4.4], Conjecture 5.1.1 for $\mathfrak{A}_{4g}$ also implies Conjecture 5.1.3 for $\mathfrak{A}_g$. Our first main result is:

Theorem 5.1.4. *Conjecture 5.1.3 holds if* $\dim([\pi](Y)) \leqslant 1$.

The proof of this theorem will be presented in §5.3. Remark that by Corollary 5.2.6, the case where $\dim([\pi]Y) = 0$ (i.e. $[\pi](Y)$ is a point) is nothing but the Manin-Mumford conjecture, which has been proved by many people (the first proof was given by Raynaud).

5.1.3 The non-torsion case

The situation becomes more complicated when s is not a torsion point of $\mathfrak{A}_{g,[\pi]s}$. In this case we prove:

Theorem 5.1.5. *Conjecture 5.1.1 holds if $s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$ and Y is a curve.*

As we have seen in Theorem 1.1.34, $\mathfrak{A}_g$ is defined over $\overline{\mathbb{Q}}$. Hence it is reasonable to talk about its $\overline{\mathbb{Q}}$-points. Moreover, if $s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$, then its generalized Hecke orbit Σ is also contained in $\mathfrak{A}_g(\overline{\mathbb{Q}})$ by Corollary 5.2.6. Hence if $\overline{Y \cap \Sigma} = Y$, then Y itself is defined over $\overline{\mathbb{Q}}$. The proof of this theorem will be presented in §5.4.

5.2 Generalized Hecke orbits in $\mathfrak{A}_g$

In this section, we discuss the matrix expression of a polarized isogeny and then compute the generalized Hecke orbit of a point of $\mathfrak{A}_g$.

5.2.1 Polarized isogenies and their matrix expressions

Let $b \in \mathcal{A}_g$. Denote by $A_b = \mathfrak{A}_{g,b}$ and denote by $\lambda_b \colon A_b \xrightarrow{\sim} A_b^\vee$ the principal polarization induced by $\mathfrak{L}_{g,b}$. Then the point b corresponds to the polarized abelian variety (A_b, λ_b). Let $\mathcal{B}$ be a symplectic basis of $H_1(A_b, \mathbb{Z})$ w.r.t. the polarization λ_b. Let $\widetilde{b} \in \mathbb{H}_g^+$ be the period matrix of A_b w.r.t. the basis $\mathcal{B}$. In this subsection, we fix $\mathcal{B}$ to be the $\mathbb{Q}$-basis of V_{2g}.

Consider all points $b' \in \mathcal{A}_g$ such that there exists a polarized isogeny

$$f \colon (A_b, \lambda_b) \to (A_{b'}, \lambda_{b'})$$

where $(A_{b'}, \lambda_{b'}) = (\mathfrak{A}_{g,b'}, A_{b'} \xrightarrow{\sim} A_{b'}^\vee$ induced by $\mathfrak{L}_{g,b'})$. Let $\mathcal{B}'$ be a symplectic basis of $H_1(A_{b'}, \mathbb{Z})$ w.r.t. the polarization $\lambda_{b'}$ and let $\widetilde{b}' \in \mathbb{H}_g^+$ be the period matrix of $A_{b'}$ w.r.t. the basis $\mathcal{B}'$.

Definition 5.2.1. *The matrix $\alpha \in \mathrm{GSp}_{2g}(\mathbb{Q})^+ \cap \mathrm{M}_{2g\times 2g}(\mathbb{Z})$ associated to*

$$f_* \colon H_1(A_b, \mathbb{Z}) \to H_1(A_{b'}, \mathbb{Z})$$

in terms of $\mathcal{B}$ and $\mathcal{B}'$ is called the ***rational representation of*** *f w.r.t. $\mathcal{B}$ and $\mathcal{B}'$.*

The periods $\widetilde{b}$ and $\widetilde{b}'$ are related by α in the following way:

$$\widetilde{b} = \alpha^t \cdot \widetilde{b}' = (A\widetilde{b}'+B)(C\widetilde{b}'+D)^{-1}, \text{ where } \alpha^t = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } \widetilde{b}, \widetilde{b}' \in \mathbb{H}_g^+ \subset M_{g\times g}(\mathbb{C}).$$

Under the $\mathbb{Q}$-basis $\mathcal{B}$ of V_{2g}, the matrix α^t corresponds to the dual isogeny of f, i.e. the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{X}_{2g,\mathrm{a}}^+)_{\widetilde{b}'} & \xrightarrow{\alpha^t\cdot} & (\mathcal{X}_{2g,\mathrm{a}}^+)_{\widetilde{b}}, \quad (v, \widetilde{b}') \mapsto \left(\alpha^t v, \alpha^t \widetilde{b}'\right) = \left(\alpha^t v, \widetilde{b}\right) \\ \Big\downarrow{\scriptstyle \mathrm{unif}} & & \Big\downarrow{\scriptstyle \mathrm{unif}} \\ A_{b'} & & A_b \\ \Big\downarrow{\scriptstyle \lambda_b}\,\wr & & \Big\downarrow{\scriptstyle \lambda_{b'}}\,\wr \\ A_{b'}^\vee & \xrightarrow{f^\vee} & A_b^\vee \end{array} \quad . \tag{5.2.1}$$

However, since f is a polarized isogeny, $f^*\mathfrak{L}_{g,b'} = \mathfrak{L}_{g,b}^{\otimes(\deg f)^{1/g}}$. So the following diagram commutes:

$$\begin{array}{ccc} A_b & \xrightarrow{\quad f \quad} & A_{b'} \\ {\scriptstyle [(\deg f)^{1/g}]\circ\lambda_b}\downarrow & & \downarrow{\scriptstyle \lambda_{b'}}\ \wr \\ A_b^\vee & \xleftarrow{\quad f^\vee \quad} & A_{b'}^\vee \end{array}. \tag{5.2.2}$$

Therefore by (5.2.1) and (5.2.2), we get the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{X}_{2g,\mathrm{a}}^+)_{\widetilde{b}} & \xrightarrow{(\deg f)^{1/g}(\alpha^t)^{-1}\cdot} & (\mathcal{X}_{2g,\mathrm{a}}^+)_{\widetilde{b'}} \\ {\scriptstyle \mathrm{unif}}\downarrow & & \downarrow{\scriptstyle \mathrm{unif}} \\ A_b & \xrightarrow{\qquad f \qquad} & A_{b'} \end{array}. \tag{5.2.3}$$

Definition 5.2.2. *The matrix* $(\deg f)^{1/g}(\alpha^t)^{-1}$ *is called the* ***matrix expression of*** f *in coordinates* $\mathcal{B}$ *w.r.t.* $\mathcal{B}'$.

Remark 5.2.3.

1. *The two bases* $\mathcal{B}$ *and* $\mathcal{B}'$ *play different roles for the matrix expression of* f*: the matrix expression of* f *depends on both bases because it depends on the period matrices determined by these bases, but its dependence on* $\mathcal{B}$ *is more important because we fix* $\mathcal{B}$ *to be the* $\mathbb{Q}$*-basis for* V_{2g} *when writing the matrix expression.*

2. *It is good to give the matrix* $(\deg f)^{1/g}(\alpha^t)^{-1}$ *a name because we will use it several times in the proof of Theorem 5.1.5. The name "matrix expression" is given by the author. Remark that this definition only works for polarized isogenies because* (5.2.2) *fails for general non-polarized isogenies.*

5.2.2 Generalized Hecke orbits in $\mathfrak{A}_g$

Lemma 5.2.4. *Let* $\varphi \in \mathrm{Aut}\left((P_{2g,\mathrm{a}}, \mathcal{X}_{2g,\mathrm{a}}^+)\right)$. *Then there exist* $g' \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ *and* $v_0 \in V_{2g}(\mathbb{Q})$ *such that the action of* φ *on* $\mathcal{X}_{2g,\mathrm{a}}^+$ *is given by*

$$\varphi\left((v,x)\right) = (g'v + v_0, g'x).$$

Proof. Notice that $\varphi(V_{2g}) = \varphi(\mathcal{R}_u(P_{2g,\mathrm{a}})) \subset \mathcal{R}_u(P_{2g,\mathrm{a}}) = V_{2g}$. Since every two Levi decompositions of $P_{2g,\mathrm{a}}$ differ by the conjugation by an element $v_0 \in V_{2g}(\mathbb{Q})$, there exists a $v_0 \in V_{2g}(\mathbb{Q})$ such that $\psi := \mathrm{Int}(v_0)^{-1} \circ \varphi$ maps $(\{0\} \times \mathrm{GSp}_{2g}, \{0\} \times \mathbb{H}_g^+)$ to itself. Now ψ maps V_{2g} and $(\mathrm{GSp}_{2g}, H_g^+)$ to themselves. So ψ can be written as (A, B), where $A \in \mathrm{GL}_{2g}(\mathbb{Q})$ and $B \in \mathrm{Aut}\left((\mathrm{GSp}_{2g}, \mathbb{H}_g^+)\right) = \mathrm{GSp}_{2g}(\mathbb{Q})^+$. Remark that $\psi \in \mathrm{Aut}(P_{2g,\mathrm{a}})$, so we can do the following computation:

For any $v \in V_{2g}(\mathbb{Q})$ and $h \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$,

$$\begin{aligned}(Ahv, BhB^{-1}) &= \psi((hv, h)) = \psi((0,h)(v,1)) = \psi(0,h)\psi(v,1) \\ &= (0, BhB^{-1})(Av, 1) = (BhB^{-1}Av, BhB^{-1}).\end{aligned}$$

Because v is an arbitrary element of $V_{2g}(\mathbb{Q})$, this implies that $Ah = BhB^{-1}A$ for any $h \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$. But this tells us that $A^{-1}B$ commutes with any element of $\mathrm{GSp}_{2g}(\mathbb{Q})^+$, and hence $A^{-1}B \in \mathbb{G}_m(\mathbb{Q})$. So ψ acts on the group $P_{2g,\mathrm{a}}$ as $\psi((v,h)) = (cBv, BhB^{-1})$ where $c \in \mathbb{Q}^*$ and $B \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$. Therefore ψ acts on $\mathcal{X}^+_{2g,\mathrm{a}}$ as $\psi((v,x)) = (cBv, Bx) = (cBv, cBx)$. Denote by $g' := cB \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$, then the action of φ on $\mathcal{X}^+_{2g,\mathrm{a}}$ is given by

$$\varphi((v,x)) = (g'v + v_0, g'x).$$

□

Let $s \in \mathfrak{A}_g$, then $[\pi]s \in \mathcal{A}_g$ corresponds to the polarized abelian variety $(\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})$.

Corollary 5.2.5. *Let $s \in \mathfrak{A}_g$. Then a point t is in the generalized Hecke orbit of s iff there exist a polarized isogeny $f\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ and $n' \in \mathbb{N}$ such that $f(s) = n't$.*

Proof. Let $(v,x) \in \mathcal{X}^+_{2g,\mathrm{a}}$ (resp. $(v_t, x_t) \in \mathcal{X}^+_{2g,\mathrm{a}}$) be such that $s = \mathrm{unif}((v,x))$ (resp. $t = \mathrm{unif}((v_t,x_t))$). Then by Proposition 1.1.31 and Lemma 5.2.4, t is in the generalized Hecke orbit of s iff

$$(v_t, x_t) = (g'v + v_0, g'x) \tag{5.2.4}$$

for some $g' \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ and $v_0 \in V_{2g}(\mathbb{Q})$.

If (5.2.4) is satisfied, then there exists $c \in \mathbb{G}_m(\mathbb{Q}) = \mathbb{Q}^*$ s.t $h := c^{-1}g' \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ is a $\mathbb{Z}$-coefficient matrix. Hence h corresponds to a polarized isogeny $f\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$. We have $t = \mathrm{unif}((chv + v_0, x_t))$ by (5.2.4), and therefore

$$n't = m'f(s) + \mathrm{unif}((v_0, x_t))$$

where $c = m'/n'$. But $\mathrm{unif}((v_0,x_t))$ is a torsion point of $\mathfrak{A}_{g,[\pi]t}$ since $v_0 \in V_{2g}(\mathbb{Q})$, and therefore can be removed by replacing m' and n' by sufficient large multiples. On the other hand $m'f$ is still a polarized isogeny, and hence replacing f by $m'f$, we may assume $m' = 1$. Finally we may assume $n' \in \mathbb{N}$ by possibly replacing f by $-f$.

Conversely, suppose that there exist a polarized isogeny $f\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ and $n' \in \mathbb{N}$ such that $f(s) = n't$. Let $\mathcal{B}_s$ (resp. $\mathcal{B}_t$) be a symplectic basis of $H_1(\mathfrak{A}_{g,[\pi]s}, \mathbb{Z})$ (resp. $H_1(\mathfrak{A}_{g,[\pi]t}, \mathbb{Z})$) and let h be the matrix expression of f in coordiante $\mathcal{B}_s$ w.r.t. $\mathcal{B}_t$. Then $h \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ and there exists $(\gamma_V, \gamma_G) \in \Gamma$ such that

$$(n'v_t, x_t) = (\gamma_V, \gamma_G)(hv, hx) = (\gamma_V + \gamma_G hv, \gamma_G hx).$$

Now $g' := \gamma_G h/n' \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ and $v_0 := \gamma_V/n' \in V_{2g}(\mathbb{Q})$ satisfy (5.2.4). □

Corollary 5.2.6. *Let $s \in \mathfrak{A}_g$ and t be a point in the generalized Hecke orbit of s. Let $f_t \colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ be a polarized isogeny of minimal degree. Then there exist*

- *a point $s_0 \in \mathfrak{A}_{g,[\pi]s}$;*
- *$\varphi \in \mathrm{End}\big((\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})\big)$;*
- *$n_0 \in \mathbb{N}$*

such that $s = n_0 s_0$ and

$$f_t(\varphi(s_0) + p) = t$$

for some torsion point $p \in \mathfrak{A}_{g,[\pi]s}$.

Proof. By Corollary 5.2.5, there exist a polarized isogeny $f \colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ and $m', n' \in \mathbb{N}$ such that $p_1 := m'f(s) - n't$ is a torsion point of $\mathfrak{A}_{g,[\pi]t}$. Now $f_t^{-1} \circ f \in \mathrm{End}\big((\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})\big) \otimes \mathbb{Q}$, i.e. there exist $\varphi' \in \mathrm{End}\big((\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})\big)$ and $n_0' \in \mathbb{N}$ such that $f_t^{-1} \circ f = \varphi' \otimes (1/n_0)$. So $n_0' \circ f = f_t \circ \varphi'$ and hence

$$m' f_t(\varphi'(s)) = m'n_0' f(s) = n_0'(n't + p_1) = n_0'n't + n_0 p_1.$$

Let $\varphi := m' \circ \varphi' \in \mathrm{End}\big((\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})\big)$ and $n_0 := n_0'n' \in \mathbb{N}$, then there exists a torsion point $p_2 \in \mathfrak{A}_{g,[\pi]t}$ such that

$$f_t(\varphi(s)) = n_0 t + p_2.$$

Hence the conclusion follows. □

5.3 Proof for the torsion case

5.3.1 Preliminary

In this subsection, we fix some definitions and notation for the proof of Theorem 5.1.4.

Let $a \in \mathcal{A}_g$. The point $a \in \mathcal{A}_g$ corresponds to the polarized abelian variety $(A_a, \lambda_a) := (\mathfrak{A}_{g,a}, \lambda_a)$. We use Σ instead of Σ_a to denote the set of all (A_a, λ_a)-special points of $\mathfrak{A}_g$. Let $\mathrm{unif} \colon \mathcal{X}^+_{2g,\mathrm{a}} \to \mathfrak{A}_g$ be the uniformization map and let $\mathcal{F}$ be the fundamental set in $\mathcal{X}^+_{2g,\mathrm{a}}$ defined as in Theorem 1.1.34.(3). Let

$$\widetilde{Y} := \mathrm{unif}^{-1}(Y) \cap \mathcal{F} \text{ and } \widetilde{\Sigma} := \mathrm{unif}^{-1}(\Sigma) \cap \mathcal{F}.$$

Let $\mathcal{B}$ be a symplectic basis for $H_1(A_a, \mathbb{Z})$ w.r.t. the polarization λ_a. Let $\widetilde{a}$ be the period matrix of A_a w.r.t. the chosen basis $\mathcal{B}$. In the rest of the paper, we shall sometimes identify $\widetilde{a} \in \mathbb{H}_g^+$ and $(0, \widetilde{a}) \in \{0\} \times \mathbb{H}_g^+ \subset V_{2g}(\mathbb{R}) \times \mathbb{H}_g^+ \simeq \mathcal{X}^+_{2g,\mathrm{a}}$.

For any $t \in \Sigma$, there exists by definition of Σ_a a polarized isogeny $(A_a, \lambda_a) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$. Besides, t is a torsion point of $A_{[\pi]t} := \mathfrak{A}_{g,[\pi]t}$, whose order we denote by $N(t)$.

Definition 5.3.1. *For any $t \in \Sigma$, define its* ***complexity*** *to be*

$$\max\left(\text{minimum degree of polarized isogenies } (A_a, \lambda_a) \to (A_{[\pi]t}, \lambda_{[\pi]t}), N(t)\right).$$

Besides, define the ***complexity*** *of any point of $\widetilde{\Sigma}$ to be the complexity of its image in Σ.*

5.3.2 Application of Pila-Wilkie

The goal of this subsection is to prove the following proposition:

Proposition 5.3.2. *Let Y, $\widetilde{a}$ be as in the last subsection. Let $\varepsilon > 0$. There exists a constant $c = c(Y, \widetilde{a}, \varepsilon) > 0$ with the following property:*

For every $n \geqslant 1$, there exist at most cn^{ε} definable blocks $B_i \subset \widetilde{Y}$ such that $\cup B_i$ contains all points of complexity at most n in $\widetilde{Y} \cap \widetilde{\Sigma}$.

Lemma 5.3.3. *There exist constants c', κ depending only on g and $\widetilde{a}$ such that*

For any $\widetilde{t} \in \widetilde{Y} \cap \widetilde{\Sigma}$ of complexity n, there exists $(v, h) \in P_{2g}(\mathbb{Q})^+$ such that $(v, h)\widetilde{a} = \widetilde{t}$ and $H((v, h)) \leqslant c'n^{\kappa}$.

Proof. Let $t = \mathrm{unif}(\widetilde{t})$. By [43, Proposition 4.1], there exist

- a polarized isogeny $f \colon \mathfrak{A}_{g,[\pi]t} \to A_a$;
- a symplectic basis $\mathcal{B}'$ for $H_1(\mathfrak{A}_{g,[\pi]t}, \mathbb{Z})$ w.r.t. the polarization $\lambda_{[\pi]t}$

such that the rational representation h_1 of f w.r.t. the chosen bases satisfies that $H(h_1)$ is polynomially bounded by $\deg(f)$.

But $\mathrm{unif}_G(h_1^t \widetilde{a}) = [\pi]t$ by (5.2.3). Hence there exists a $h_2 \in \Gamma_G$ such that $h_2 h_1^t \widetilde{a} = \pi(\widetilde{t}) \in \mathcal{F}_G$. By [49, Lemma 3.2], $H(h_2)$ is polynomially bounded by the norm of $h_1^t \cdot \widetilde{a}$.

Now define $h := h_2 h_1^t$. We have then $h\widetilde{a} = \pi(\widetilde{t})$ and

$$H(h) \leqslant c_0 \deg(f)^{\kappa_0}$$

where $c_0 > 0$ and $\kappa_0 > 0$ depend only on g and $\widetilde{a}$.

Next write $\widetilde{t} = (\widetilde{t}_V, \pi(\widetilde{t})) \in \mathcal{F}$. Let $v := \widetilde{t}_V$, then $v \in V_{2g}(\mathbb{Q})$ since t is a torsion point of $\mathfrak{A}_{g,[\pi]t}$. Besides, the denominator of v is precisely the order of the torsion point t. But by choice, $\mathcal{F} \simeq [0, N)^{2g} \times \mathcal{F}_G \subset V_{2g}(\mathbb{R}) \times \mathbb{H}_g^+ \simeq \mathcal{X}_{2g,\mathrm{a}}^+$ (see Theorem 1.1.34.(3)). Therefore up to a constant depending on nothing, $H(v)$ is bounded by its denominator, i.e. the order of the torsion point t of $\mathfrak{A}_{g,[\pi]t}$.

To sum it up, (v, h) is the element of $P_{2g}(\mathbb{Q})^+$ which we desire. □

Now we can prove Proposition 5.3.2 with the help of Lemma 5.3.3.

Proof of Proposition 5.3.2. Let

$$\begin{aligned} \sigma\colon P_{2g}(\mathbb{R})^+ &\to \mathcal{X}^+_{2g,\mathrm{a}} \\ (v,h) &\mapsto (v,h)\widetilde{a} \end{aligned}$$

The set $R := \sigma^{-1}(\widetilde{Y}) = \sigma^{-1}(\mathrm{unif}^{-1}(Y) \cap \mathcal{F})$ is definable because σ is semi-algebraic and $\mathrm{unif}\,|_{\mathcal{F}}$ is definable. Hence we can apply the family version of the Pila-Wilkie theorem ([48, 3.6]) to the definable set R: for every $\varepsilon > 0$, there are only finitely many definable block families $B^{(j)}(\varepsilon) \subset R \times \mathbb{R}^m$ and a constant $C_1(R,\varepsilon)$ such that for every $T \geqslant 1$, the rational points of R of height at most T are contained in the union of at most $C_1 T^{\varepsilon}$ definable blocks $B_i(T,\varepsilon)$, taken (as fibers) from the families $B^{(j)}(\varepsilon)$. Since σ is semi-algebraic, the image under σ of a definable block in R is a finite union of definable blocks in $\widetilde{Y}$. Furthermore the number of blocks in the image is uniformly bounded in each definable block family $B^{(j)}(\varepsilon)$. Hence $\sigma(B_i(T,\varepsilon))$ is the union of at most $C_2 T^{\varepsilon}$ blocks in $\widetilde{Y}$, for some new constant $C_2(Y,\widetilde{a},\varepsilon) > 0$.

By Lemma 5.3.3, for any point $\widetilde{t} \in \widetilde{Y} \cap \widetilde{\Sigma}$ of complexity n, there exists a rational element $\gamma \in R$ such that $\sigma(\gamma) = \widetilde{t}$ and $H(\gamma) \leqslant c' n^{\kappa}$. By the discussion in the last paragraph, all such γ's are contained in the union of at most $C_1(c'n^{\kappa})^{\varepsilon}$ definable blocks. Therefore all points of $\widetilde{Y} \cap \widetilde{\Sigma}$ of complexity n are contained in the union of at most $C_1 C_2 c'^{\varepsilon} n^{\kappa\varepsilon}$ blocks in $\widetilde{Y}$. □

5.3.3 Galois orbit

In this section we shall deal with the Galois orbit. We handle the case of $\overline{\mathbb{Q}}$-points at first and then use the standard specialization argument to prove the result for general points of $\Sigma \cap Y$.

Proposition 5.3.4. *Suppose $a \in \mathcal{A}_g(\overline{\mathbb{Q}})$. There exist positive constants $c'_1 = c'_1(g)$, $c'_2 = c'_2(g, k(a))$ and $c'_3 = c'_3(g)$ satisfying the following property:*

For any point $t \in \Sigma \cap Y \cap \mathfrak{A}_g(\overline{\mathbb{Q}})$ of complexity n,

$$[k(t):\mathbb{Q}] \geqslant c'_1 \frac{n^{c'_2}}{\max\left(1, h_F(A_a)\right)^{c'_3}}$$

where $k(t)$ is the definition field of t.

Proof. Define (as Gaudron-Rémond [21])

$$\kappa(\mathfrak{A}_{g,[\pi]t}) := ((14g)^{64g^2}[k([\pi]t):\mathbb{Q}]\max(h_F(\mathfrak{A}_{g,[\pi]t}), \log[k([\pi]t):\mathbb{Q}],1)^2)^{1024g^3}.$$

Take a point $t \in \Sigma \cap Y \cap \mathfrak{A}_g(\overline{\mathbb{Q}})$ of complexity n. Denote by $k([\pi]t)$ the definition field of $[\pi]t$. Denote by $N(t)$ the order of t as a torsion point of $A_{[\pi]t} := \mathfrak{A}_{g,[\pi]t}$. There are two cases.

Case i $n=$ minimum degree of polarized isogenies $(A_a,\lambda_a)\to(A_{[\pi]t},\lambda_{[\pi]t})$. Then by [21, Théorème 1.4] and [42, Theorem 5.6],

$$n\leqslant\kappa(\mathfrak{A}_{g,[\pi]t}).$$

On the other hand, by a result of Faltings [16, Chapter II, §4, Lemma 5],

$$h_F(\mathfrak{A}_{g,[\pi]t})\leqslant h_F(A_a)+(1/2)\log n.$$

Now the conclusion for this case follows from the two inequalities above and the easy fact $[k(t):\mathbb{Q}]\geqslant[k([\pi]t):\mathbb{Q}]$.

Case ii $n=N(t)$. By [21, Théorème 1.2], there exist positive natural numbers l, simple abelian varieties A_1,...,A_l over a finite extension k' of $k([\pi]t)$ (A_i and A_j can be isogenious to each other over $\overline{\mathbb{Q}}$ for $i\neq j$) and an isogeny

$$\varphi\colon\mathfrak{A}_{g,[\pi]t}\to\prod_{i=1}^{l}A_i \tag{5.3.1}$$

such that φ is defined over k', $\deg\varphi\leqslant\kappa(\mathfrak{A}_{g,[\pi]t})$ and $[k':k([\pi]t)]\leqslant\kappa(\mathfrak{A}_{g,[\pi]t})^g$. Call $p_i\colon A\to A_i$ the composite of φ and the i-th projection $\prod_{i=1}^{l}A_i\to A_i$ ($\forall i=1,...,l$).

Now $t\in A$ is a torsion point of order $N(t)$. Without any loss of generality we may assume

$$N(p_1(t))\geqslant N(p_i(t))$$

where $N(p_i(t))$ is the order of $p_i(t)$ as a torsion point of A_i.

Lemma 5.3.5.

$$N(t)\leqslant\kappa(\mathfrak{A}_{g,[\pi]t})N(p_1(t))^g \text{ and } [k(t):\mathbb{Q}]\geqslant[k(p_1(t)):\mathbb{Q}]/\kappa(\mathfrak{A}_{g,[\pi]t})^{2g}.$$

where $k(p_1(t))$ is the definition field of $p_1(t)$.

Proof. Denote by $N(\varphi(t))$ the order of $\varphi(t)$ as a torsion point of $\prod_{i=1}^{l}A_i$. We have

$$N(\varphi(t))\geqslant N(t)/\deg\varphi\geqslant N(t)/\kappa(\mathfrak{A}_{g,[\pi]t}).$$

On the other hand, $N(\varphi(t))=\mathrm{lcd}(N(p_1(t)),...,N(p_l(t)))\leqslant N(p_1(t))^g$. Now the first inequality follows.

For the second inequality, first of all since φ and $\prod_{i=1}^{l}A_i$ are both defined over k', we have

$$[k(\varphi(t)):\mathbb{Q}]\leqslant[k(t)k':\mathbb{Q}]=[k(t):\mathbb{Q}][k(t)k':k(t)]\leqslant[k(t):\mathbb{Q}][k':k]\leqslant[k(t):\mathbb{Q}]\kappa(\mathfrak{A}_{g,[\pi]t})^g.$$

Next since all abelian varieties A_1,...,A_l are defined over k', we have then

$$[k(\varphi(t))k':\mathbb{Q}]\geqslant[k(p_1(t)):\mathbb{Q}].$$

But

$$\begin{aligned}[k(\varphi(t))k' : \mathbb{Q}] &= [k(\varphi(t))k' : k'][k' : k][k : \mathbb{Q}] \\ &\leqslant [k(\varphi(t)) : k][k' : k][k : \mathbb{Q}] \\ &= [k(\varphi(t)) : \mathbb{Q}][k' : k] \\ &\leqslant [k(\varphi(t)) : \mathbb{Q}]\kappa(\mathfrak{A}_{g,[\pi]t})^g.\end{aligned}$$

Now the second inequality follows from the three inequalities above. □

By [17, Corollaire 1.5],

$$[k(p_1(t)) : \mathbb{Q}] \geqslant c_0'(g)\frac{N(p_1(t))^{1/(2g)}}{\log N(p_1(t))(h_F(A_1) + \log N(p_1(t)))}. \tag{5.3.2}$$

By the comment below [21, Corollaire 1.5], we have

$$h_F(A_1) \leqslant h_F(\mathfrak{A}_{g,[\pi]t}) + \frac{1}{2}\log\kappa(\mathfrak{A}_{g,[\pi]t}). \tag{5.3.3}$$

By assumption, there exists an isogeny $A_a \to \mathfrak{A}_{g,[\pi]t}$ of degree $\leqslant n$. So by Faltings [16, Chapter II, §4, Lemma 5],

$$h_F(\mathfrak{A}_{g,[\pi]t}) \leqslant h_F(A_a) + (1/2)\log n. \tag{5.3.4}$$

Now because $[k(t) : \mathbb{Q}] \geqslant [k([\pi]t) : \mathbb{Q}]$, the conclusion of *Case ii* now follows from Lemma 5.3.5, (5.3.2), (5.3.3) and (5.3.4). □

Corollary 5.3.6. *Suppose that a is defined over a finitely generated field k. There exist positive constants $c_1 = c_1(A_a, k)$ and $c_2 = c_2(A_a, k)$ satisfying the following property:*

For any point $t \in \Sigma \cap Y$ of complexity n defined over a finitely extension $k(t)$ of k,

$$[k(t) : k] \geqslant c_1 n^{c_2}.$$

Proof. This follows from Proposition 5.3.4 and a specialization argument. The case where n = minimum degree of polarized isogenies $(A_a, \lambda_a) \to (A_{[\pi]t}, \lambda_{[\pi]t})$ is proved by Orr [43, Theorem 5.1] (possibly combined with [42, Theorem 5.6]). The case where $n = N(t)$, the order of t as a torsion point of $\mathfrak{A}_{g,[\pi]t}$, follows from the standard specialization argument introduced by Raynaud (see [43, Section 5] and [56, Section 7]). □

5.3.4 End of the proof for the torsion case

In this section, Y is always an irreducible subvariety of $\mathfrak{A}_g$, $a \in \mathcal{A}_g$ and Σ is the set of all a-strongly special points of $\mathfrak{A}_g$.

Theorem 5.3.7. *If $\overline{Y \cap \Sigma} = Y$, then the union of all positive-dimensional weakly special subvarieties contained in Y is Zariski dense in Y.*

Proof. Let Σ_1 be the set of points $t \in Y \cap \Sigma$ such that there is a positive-dimensional block $B \subset \widetilde{Y}$ with $t \in \mathrm{unif}(B)$. Let Y_1 be the Zariski closure of Σ_1. Let k be the finitely generated field $k(a)$. Enlarge k if necessary such that both Y and Y_1 are defined over k.

Let t be a point in $Y \cap \Sigma$ of complexity n. By Corollary 5.3.6, there exist positive constants c_1 and c_2 depending only on g, A_a and k such that

$$[k(t) : k] \geqslant c_1 n^{c_2/2}.$$

But all $\mathrm{Gal}(\overline{k}/k)$-conjugates of t are contained in $Y \cap \Sigma$ and have complexity n. By Proposition 5.3.2, the preimages in $\mathcal{F}$ of these points are contained in the union of $c(Y, \widetilde{a}, c_2/4)n^{c_2/4}$ definable blocks, each of these blocks being contained in $\widetilde{Y}$.

For n large enough, $c_1 n^{c_2/2} > cn^{c_2/4}$. Hence for $n \gg 0$, there exists a definable block $B \subset \widetilde{Y}$ such that $\mathrm{unif}(B)$ contains at least two Galois conjugates of t, and therefore $\dim B > 0$ since blocks are connected. So being in $\mathrm{unif}(B)$, those conjugates of t are in Σ_1. But Y_1 is defined over k, so $t \in Y_1$.

In summary, all points of $Y \cap \Sigma$ of large enough complexity are in Σ_1. This excludes only finitely many points of $Y \cap \Sigma$. So $Y_1 = Y$.

Let Σ_2 be the set of points $t \in Y \cap \Sigma$ such that there is a connected positive-dimensional semi-algebraic set $B' \subset \widetilde{Y}$ with $t \in \mathrm{unif}(B')$. Let Y_2 be the Zariski closure of Σ_2. By definition of blocks, $\Sigma_2 = \Sigma_1$, and hence $Y_2 = Y_1 = Y$.

But for any connected semi-algebraic set $B' \subset \widetilde{Y}$, the Ax-Lindemann theorem (in the form of Theorem 3.1.4) implies that every irreducible component of $\overline{\mathrm{unif}(B')}$, whose dimension is positive if $\dim(B') > 0$, is weakly special. Now the conclusion follows. ☐

Proof of Theorem 5.1.4. Let S be the smallest connected mixed Shimura subvariety containing Y. Assume S is associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$. Let $(G, \mathcal{X}_G^+) := (P, \mathcal{X}^+)/\mathcal{R}_u(P)$. By Theorem 4.1.3 and Theorem 5.3.7, such a non-trivial group N exists: N is the maximal normal subgroup of P such that the followings hold:

- there exists a diagram of Shimura morphisms

$$\begin{array}{ccccc}
(P, \mathcal{X}^+) & \xrightarrow{\rho} & (P', \mathcal{X}'^+) := (P, \mathcal{X}^+)/N & \xrightarrow{\pi'} & (G', \mathcal{X}'^+_G) := (P', \mathcal{X}'^+)/\mathcal{R}_u(P') \\
\downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}'} & & \downarrow{\scriptstyle \mathrm{unif}'_G} \\
S & \xrightarrow{[\rho]} & S' & \xrightarrow{[\pi']} & S'_G
\end{array}$$

  (then S' is by definition a connected Shimura variety of Kuga type)

- the union of positive-dimensional weakly special subvarieties which are contained in $Y' := \overline{[\rho](Y)}$ is not Zariski dense in Y';

- $Y = [\rho]^{-1}(Y')$.

We prove the theorem by induction on g. When $g = 1$, the only non-trivial case is when Y is a curve. But then Y must be weakly special by Theorem 4.1.3 (Or more simply, one can use Theorem 2.3.3 to avoid using the Ax-Lindemann theorem). Remark that this case has also been proved by André [3, Lecture 4] when he proposed the mixed André-Oort conjecture.

When $\dim([\pi](Y)) = 0$, this is the Manin-Mumford conjecture by Corollary 5.2.6. Hence we only have to treat the case $\dim([\pi](Y)) = 1$. Remark that in this case $[\pi](Y)$ is weakly special by the main result of [43], and hence equals $\mathrm{unif}_G\,(G''(\mathbb{R})^+\widetilde{y})$ for some $G'' < \mathrm{GSp}_{2g}$ of positive dimension and $\widetilde{y} \in \mathbb{H}_g^+$. Now there are two cases:

If $\dim([\pi'](Y')) = 0$, then $[\pi'](Y')$ is a point. In this case Y' is a subvariety of an abelian variety. The hypothesis $\overline{Y \cap \Sigma} = Y$ implies that Y' contains a Zariski dense subset of torsion points. Therefore by the result of the Manin-Mumford conjecture, Y' is a special subvariety, i.e. the translate of an abelian subvariety by a torsion point. But the union of positive-dimensional weakly special subvarieties which are contained in Y' is not Zariski dense in Y', so Y' is a point. Therefore Y is weakly special by definition.

If $\dim([\pi'](Y')) = 1$, then $N/\mathcal{R}_u(N)$ is trivial because the dimension of $[\pi](Y) = \mathrm{unif}_G\,(G''(\mathbb{R})^+\widetilde{y})$ is 1. Therefore $V_N := \mathcal{R}_u(N) < V_{2g}$ is non-trivial since N is non-trivial.

Denote for simplicity by $B := [\pi'](Y') = \mathrm{unif}'_G(G''(\mathbb{R})^+\rho(\widetilde{y}))$ and $X := [\pi']^{-1}(B)$. Then $X \to B$ is a family of abelian varieties of dimension g'. We have $g' < g$ since V_N is non-trivial. Besides, $X \to B$ is non-isotrivial because otherwise G'' acts trivially on V_{2g}/V_N, and therefore $G'' \lhd P'$. This contradicts the maximality of N. Hence there exists, up to taking finite covers of $X \to B$, a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & \mathfrak{A}_{g'} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\;i_B\;} & \mathcal{A}_{g'} \end{array}$$

such that both i and i_B are finite. Apply induction hypothesis to $i(Y') \subset \mathfrak{A}_{g'}$, we get that $i(Y')$ is weakly special. By the geometric interpretation of weakly special subvarieties (Proposition 1.2.15), $i^{-1}(i(Y'))$ is irreducible. Therefore $Y' = i^{-1}(i(Y'))$ since they are of the same dimension. So Y' is a weakly special subvariety of S' (again by Proposition 1.2.15). But then Y' must be a point because the union of the positive-dimensional weakly special subvarieties contained in Y' is not Zariski dense in Y'. Hence Y is weakly special by definition. □

5.4 Proof for the non-torsion case

We prove Theorem 5.1.5 in this section. Let Y be a curve over $\overline{\mathbb{Q}}$ in $\mathfrak{A}_g$, let $s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$ and let Σ be the generalized Hecke orbit of s. Then $\Sigma \subset \mathfrak{A}_g(\overline{\mathbb{Q}})$.

For simplicity, we will denote by $(A,\lambda) := (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})$ the polarized abelian variety attached to $[\pi](s)$ in this section. Assume that s is not a torsion point of A. Through all this section, we assume that Y is not contained in a fiber of $[\pi]\colon \mathfrak{A}_g \to \mathcal{A}_g$ (otherwise this is a special case of the Mordell-Lang conjecture, which is proved by a series of work of Vojta, Faltings and Hindry).

We fix some notation here. Let $\mathcal{B}$ be a symplectic basis of $H_1(A,\mathbb{Z})$ w.r.t. the polarization λ. Let $\widetilde{s}_G \in \mathbb{H}_g^+$ be the period matrix of (A,λ) w.r.t. the basis $\mathcal{B}$, then $\mathrm{unif}_G(\widetilde{s}_G) = [\pi]s$. Now let $\widetilde{s} = (\widetilde{s}_V, \widetilde{s}_G) \in V_{2g}(\mathbb{R}) \times \mathbb{H}_g^+ \simeq \mathcal{X}_{2g,\mathrm{a}}^+$ be a point in $\pi^{-1}(\widetilde{s}_G) \cap \mathrm{unif}^{-1}(s)$. In the whole section, we will fix $\mathcal{B}$ to be the $\mathbb{Q}$-basis of V_{2g} as in §5.2.1.

Denote by k the definition field of s. Then A is defined over the number field k.

5.4.1 Complexity of points in a generalized Hecke orbit

Let $\mathrm{unif}\colon \mathcal{X}_{2g,\mathrm{a}}^+ \to \mathfrak{A}_g$ be the uniformization map and let $\mathcal{F}$ be the fundamental set in $\mathcal{X}_{2g,\mathrm{a}}^+$ defined in Theorem 1.1.34.(3). Let

$$\widetilde{Y} := \mathrm{unif}^{-1}(Y) \cap \mathcal{F} \text{ and } \widetilde{\Sigma} := \mathrm{unif}^{-1}(\Sigma) \cap \mathcal{F}.$$

Let $t \in \Sigma$. Let f_t be as in Corollary 5.2.6 (i.e. a polarized isogeny $(A,\lambda) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ of minimum degree). Define

$$n_t := \min\{n \in \mathbb{N} \mid \exists \varphi \in \mathrm{End}\,((A,\lambda)) \text{ such that } nt \in f_t\big(\varphi(s) + A(\overline{\mathbb{Q}})_{\mathrm{tor}}\big)\}.$$

The existence of such an n_t is guaranteed by Corollary 5.2.6. Furthermore, let $s_t := \mathrm{unif}\,((\widetilde{s}_V/n_t, \widetilde{s}_G)) \in \mathfrak{A}_{g,[\pi]s} = A$. Then there exist by definition of n_t

- $\varphi_t \in \mathrm{End}\,((A,\lambda))$;
- δ_t a torsion point of A

such that

$$f_t\left(\varphi_t(s_t) + \delta_t\right) = t. \tag{5.4.1}$$

The notation n_t, f_t, φ_t, s_t and δ_t will be used through the whole section.

Definition 5.4.1. *Define the* ***complexity*** *of $t \in \Sigma$ to be*

$$\max\left(n_t, N(\delta_t)\right)$$

where $N(\delta_t)$ is the order of δ_t. Besides, define the ***complexity*** *of any point of $\widetilde{\Sigma}$ to be the complexity of its image in Σ.*

The fact that this complexity is a "good enough" parameter will be proved in §5.4.3.

5.4.2 Galois orbit

In contrast to the torsion case, we deal with the Galois orbit at first for the non-torsion case. Keep the notation of the beginning of this section and §5.4.1.

Proposition 5.4.2. *Let $t \in \Sigma$ be of complexity n, then*

$$[k(t):\mathbb{Q}] \geqslant c_3 n^{c_4}$$

where $c_3 = c_3(A,\lambda,s)$ and $c_4 = c_4(A,\lambda,s)$ are two positive constants.

Proof. By [21, Théorème 1.2] and [42, Theorem 5.6], there exist positive constants $c_5 = c_5(A,\lambda)$ and $c_6 = c_6(A,\lambda)$ such that

$$\deg(f_t) \leqslant c_5 [k(t):\mathbb{Q}]^{c_6} \tag{5.4.2}$$

The abelian variety A is defined over k. By the main result of [34], there exist two positive constants c_9 and c_{10} depending only on A and k such that for any torsion point $q \in A$ of order $N(q)$, we have

$$[k(q):\mathbb{Q}] \geqslant c_9 N(q)^{c_{10}}. \tag{5.4.3}$$

Case i $N(\delta_t)^{c_{10}/2} \geqslant n_t^{2g^2+4g+1}$. By [26, Proposition 1] or [36, Theorem 2.1.2], there exists a positive constant $c_{11} = c_{11}(A,s,k)$ such that

$$\mathrm{Gal}\left(k(\varphi_t(s_t), A[n_t])/k(A[n_t])\right) \leqslant c_{11} n_t^{2g}.$$

Hence

$$[k(\varphi_t(s_t)):\mathbb{Q}] \leqslant c'_{11} n_t^{2g^2+4g+1} \tag{5.4.4}$$

for another positive constant c'_{11} depending only on A, s and k. Now by (5.4.4), (5.4.3) and the assumption for this case,

$$[k(\varphi_t(s_t),\delta_t):k(\varphi_t(s_t))] \geqslant c_{12} \frac{N(\delta_t)^{c_{10}}}{n_t^{2g^2+4g+1}} \geqslant c_{12} N(\delta_t)^{c_{10}/2} \tag{5.4.5}$$

for a positive constant $c_{12} = c_{12}(A,s,k)$.

Since A is defined over the number field k, every element of $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ induces a homomorphism $A(\overline{\mathbb{Q}}) \to A(\overline{\mathbb{Q}})$, and hence a homomorphism $A \to A$. It is not hard to prove the following claim:

Claim 5.4.3. *For any $\sigma_1, \sigma_2 \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/k(\varphi_t(s_t))\right)$, $\sigma_1(\varphi_t(s_t)+\delta_t) = \sigma_2(\varphi_t(s_t)+\delta_t)$ iff $\sigma_2^{-1}\sigma_1 \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/k(\varphi_t(s_t),\delta_t)\right)$.*

This claim implies $[k(\varphi_t(s_t)+\delta_t):\mathbb{Q}] \geqslant [k(\varphi_t(s_t),\delta_t):k(\varphi_t(s_t))]$. Hence by (5.4.5),

$$[k(\varphi_t(s_t)+\delta_t):\mathbb{Q}] \geqslant c_{12} N(\delta_t)^{c_{10}/2}.$$

Since $t = f_t(\varphi_t(s_t)+\delta_t)$, we have therefore

$$[k(t):\mathbb{Q}] \geqslant c_{12} \frac{N(\delta_t)^{c_{10}/2}}{\deg(f_t)}. \tag{5.4.6}$$

Now the conclusion for this case follows from (5.4.2), (5.4.6) and the definition of complexity (recall that k is the definition field of s, and therefore depends only on s).

Case ii $N(\delta_t)^{c_{10}/2} \leqslant n_t^{2g^2+4g+1}$. Roughly speaking, this case follows from the Kummer theory [26, Appendix 2]. Here are the details of the proof:

Let $\Delta := \mathrm{End}\,((A,\lambda))\,s$ and let $\overline{\Delta} := \mathrm{End}(A)s \subset A$. Then $\overline{\Delta}$ is a finitely generated subgroup of A. Let k' be the smallest number field over which all points of $\overline{\Delta}$ are defined, then k' depends only on A and s. Then by the Mordell-Weil theorem, $A(k')$ is a finitely generated subgroup of A. By definition of k', $\overline{\Delta} \subset A(k')$. Let $\Delta' := \mathbb{Q}\Delta \cap A(k')$ and let $\overline{\Delta}' := \mathbb{Q}\overline{\Delta} \cap A(k')$. Then $\overline{\Delta}'$ is again a finitely generated subgroup of A. It contains $\overline{\Delta}$ and $\mathrm{rank}\,\overline{\Delta}' = \mathrm{rank}\,\overline{\Delta}$. Therefore $[\overline{\Delta}' : \overline{\Delta}]$ is a finite number depending only on k', and hence only on A and s. On the other hand, $\Delta \subset \overline{\Delta} \cap \Delta' \subset \Delta + A(k')_{\mathrm{tor}}$. So $[\overline{\Delta} \cap \Delta' : \Delta]$ is a finite number depending only on k', and hence only on A and s. Therefore by

$$[\Delta' : \Delta] = [\Delta' : \overline{\Delta} \cap \Delta'][\overline{\Delta} \cap \Delta' : \Delta] \leqslant [\overline{\Delta}' : \overline{\Delta}][\overline{\Delta} \cap \Delta' : \Delta],$$

there exists $c_{13} > 0$ depending only on A and s such that $[\Delta' : \Delta] = c_{13}$.

For each $t \in \Sigma$, define another number $n'_t := \min\{n \in \mathbb{N} |\ nt \in f_t\big(A(k') + A(\overline{\mathbb{Q}})_{\mathrm{tor}}\big)\}$. Let $s' \in A(k')$ be such that $n'_t t = f_t(s' + A(\overline{\mathbb{Q}})_{\mathrm{tor}})$. Then because $t = f_t(\varphi_t(s_t) + \delta_t)$, we have

$$s^{\dagger} := s' - n'_t \varphi_t(s_t) \in A(\overline{\mathbb{Q}})_{\mathrm{tor}}.$$

So $s' \in n'_t\varphi_t(s_t) + A(\overline{\mathbb{Q}})_{\mathrm{tor}} \subset \mathbb{Q}\Delta$, and therefore $n'_t\varphi_t(s_t) + s^{\dagger} = s' \in \Delta'$. So

$$n'_t = \min\{n \in \mathbb{N} |\ nt \in f_t(\Delta' + A(\overline{\mathbb{Q}})_{\mathrm{tor}})\}. \tag{5.4.7}$$

However by definition,

$$n_t = \min\{n \in \mathbb{N} |\ nt \in f_t(\Delta + A(\overline{\mathbb{Q}})_{\mathrm{tor}}). \tag{5.4.8}$$

Compare (5.4.7) and (5.4.8), we get

$$n_t/n'_t \leqslant [\Delta' : \Delta] = c_{13}. \tag{5.4.9}$$

By [26, Lemma 14] or [36, Corollary 2.1.5], there exists a positive constant $c_{14} = c_{14}(A, k') = c_{14}(A, s)$ such that

$$\mathrm{Gal}\Big(k'\big(\varphi_t(s_t), A[n'_t N(\delta_t)]\big)/k'\big(A[n'_t N(\delta_t)]\big)\Big) \geqslant c_{14} n'_t.$$

Hence

$$[k(t) : \mathbb{Q}] \geqslant \frac{[k'(\varphi_t(s_t) + \delta_t) : \mathbb{Q}]}{\deg(f_t)[k' : k]} \geqslant \frac{c_{14} n'_t}{\deg(f_t)[k' : k]}. \tag{5.4.10}$$

Now the conclusion follows from (5.4.2), (5.4.9) and (5.4.10) (remark that $[k' : k]$ is a constant depending only on A and s). ∎

5.4.3 Néron-Tate height in family

Next we prove that the complexity defined in Definition 5.4.1 is a good parameter. More explicitly we have the following proposition:

Proposition 5.4.4. *Let Y be as in the beginning of this section. Let $t \in Y(\overline{\mathbb{Q}}) \cap \Sigma$. Let f_t, n_t, s_t, φ_t and δ_t be as in §5.4.1. Then*

$$\deg(\varphi_t) \leqslant c_7 n_t^{c_8} \quad \text{and} \quad \deg(f_t) \leqslant c_7' n_t^{c_8'}$$

for some positive constants $c_7 = c_7(g, Y, s)$, $c_7' = c_7'(g, Y, s)$ and $c_8 = c_8(g, Y, s)$, $c_8' = c_8'(g, Y, s)$.

We shall prove this proposition with help of a well-chosen family of Néron-Tate heights, i.e. the one related to the $\mathbb{G}_m$-torsor $\mathfrak{L}_g$ defined in Theorem 1.1.34. Then we shall use a theorem of Silverman-Tate [60, Theorem A].

By Theorem 1.1.34(2), $\mathfrak{L}_g \to \mathfrak{A}_g$ is a symmetric and relatively ample $\mathbb{G}_m$-torsor w.r.t. $\mathfrak{A}_g \to \mathcal{A}_g$. Now consider the Néron-Tate height $\widehat{h}_{\mathfrak{L}_{g,b}}$ on A_b for each $b \in \mathcal{A}_g(\overline{\mathbb{Q}})$. For any $s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$, we shall denote by

$$\widehat{h}_{\mathfrak{L}_g}(s) := \widehat{h}_{\mathfrak{L}_{g,[\pi]s}}(s).$$

Lemma 5.4.5. *Let s_1 and s_2 be two points of $\mathfrak{A}_g(\overline{\mathbb{Q}})$. Assume that there exists a polarized isogeny*

$$f \colon (\mathfrak{A}_{g,[\pi]s_1}, \lambda_{[\pi]s_1}) \to (\mathfrak{A}_{g,[\pi]s_2}, \lambda_{[\pi]s_2})$$

such that $s_1 = f(s_2)$. Then $\widehat{h}_{\mathfrak{L}_g}(s_2) = (\deg f)^{1/g} \widehat{h}_{\mathfrak{L}_g}(s_1)$.

Proof. By the moduli interpretation of $\mathfrak{L}_g$ (Theorem 1.1.34(3)), $f^*\mathfrak{L}_{g,[\pi]s_2} = \mathfrak{L}_{g,[\pi]s_1}^{\otimes(\deg f)^{1/g}}$. So we have

$$\begin{aligned} \widehat{h}_{\mathfrak{L}_g}(s_2) &= \widehat{h}_{\mathfrak{L}_{g,[\pi]s_2}}(f(s_1)) \\ &= \widehat{h}_{\mathfrak{L}_{g,[\pi]s_1}^{\otimes(\deg f)^{1/g}}}(s_1) \\ &= (\deg f)^{1/g} \widehat{h}_{\mathfrak{L}_{g,[\pi]s_1}}(s_1) \\ &= (\deg f)^{1/g} \widehat{h}_{\mathfrak{L}_g}(s_1). \end{aligned}$$

□

Now we begin the proof of Proposition 5.4.4.

Proof of Proposition 5.4.4. Denote by $\varepsilon \colon \mathcal{A}_g \to \mathfrak{A}_g$ the zero section.

By abuse of notation we denote also by $\mathfrak{L}_g$ the relative ample line bundle associated to the $\mathbb{G}_m$-torsor. Let $\mathcal{M}$ be an ample line bundle over $\overline{\mathbb{Q}}$ over $\mathcal{A}_g$ which extends over $\overline{\mathbb{Q}}$ to an ample line bundle $\overline{\mathcal{M}}$ over $\overline{\mathcal{A}_g}$. For $a \gg 0$, the line bundle $\mathfrak{L} := \mathfrak{L}_g \otimes [\pi]^* \mathcal{M}^{\otimes a}$ over $\mathfrak{A}_g$ is ample. Moreover the *canonical height* $\widehat{h}_{\mathfrak{L}}$ defined by Silverman [60, §2, pp 200] satisfies

$$\widehat{h}_{\mathfrak{L}} = \widehat{h}_{\mathfrak{L}_g}.$$

By Theorem 1.1.34(6), we can apply [60, Theorem A]: there exist constants $c_{15} = c_{15}(g) > 0$ and $c_{16} = c_{16}(g)$ such that

$$|\widehat{h}_{\mathfrak{L}_g}(t) - h_{\mathfrak{A}_g, \mathfrak{L}}(t)| < c_{15} h_{\mathcal{A}_g, \varepsilon^* \mathfrak{L}}([\pi]t) + c_{16} \tag{5.4.11}$$

for any $t \in \mathfrak{A}_g(\overline{\mathbb{Q}})$.

We need the following lemma, which uses the fact that Y is a curve in an essential way:

Lemma 5.4.6. *There exist two constants $c_{17} > 0$ and c_{18} depending only on Y such that*

$$h_{\mathfrak{A}_g, \mathfrak{L}}(t) \leqslant c_{17} h_{\mathcal{A}_g, \varepsilon^* \mathfrak{L}}([\pi]t) + c_{18}$$

Proof. The idea is due to Lin-Wang [32, Proof of Proposition 2.1]. The following notation will be used only in this proof: denote by $B = [\pi](Y)$ and $X = [\pi]^{-1}(B)$. By abuse of notation, we will not distinguish $[\pi]$ and $[\pi]|_X$. Remark that $X \to B$ is a non-isotrivial family of abelian varieties.

Let Y' be a smooth resolution of $Y \subset \mathfrak{A}_g$, then $X \times_B Y' \to Y'$ is also a non-isotrivial family of abelian varieties of dimension g and we write $\varepsilon_{Y'} \colon Y' \to X \times_B Y'$ to be the zero-section. Let $f \colon Y' \to \mathfrak{A}_g$ be the natural morphism. Consider the following commutative diagram

$$\begin{array}{ccc} X \times_B Y' & \underset{p_2}{\overset{\varepsilon_{Y'}}{\rightleftarrows}} & Y' \\ \Big\downarrow{\scriptstyle p_1} & \lrcorner & \Big\downarrow{\scriptstyle [\pi]\circ f} \\ X & \xrightarrow{[\pi]} & B \end{array}.$$

Now let $t' \in Y'(\overline{\mathbb{Q}})$ be such that $f(t') = t$. Then up to bounded functions,

$$\begin{aligned} h_{\mathfrak{A}_g, \mathfrak{L}}(t) &= h_{X, \mathfrak{L}_g|_X}(t) \\ &= h_{X, \mathfrak{L}|_X}(f(t')) \\ &= h_{Y', f^* \mathfrak{L}|_X}(t') \end{aligned} \qquad \begin{aligned} h_{\mathcal{A}_g, \varepsilon^* \mathfrak{L}}([\pi]t) &= h_{B, \varepsilon^* \mathfrak{L}|_X}([\pi]t) \\ &= h_{B, \varepsilon^* \mathfrak{L}|_X}(f \circ [\pi](t')) \\ &= h_{Y', (f \circ [\pi])^* \varepsilon^* \mathfrak{L}|_X}(t') \\ &= h_{Y', \varepsilon_{Y'}^* p_1^* \mathfrak{L}|_X}(t'). \end{aligned}$$

Since Y is a curve, the morphism $[\pi] \circ f \colon Y' \to B$ is finite. Therefore $p_1^* \mathfrak{L}|_X$ is ample. So $\varepsilon_{Y'}^* p_1^* \mathfrak{L}|_X$ is ample. Hence there exist two constants $c_{17} > 0$ and c_{18} depending only on Y' (and hence only on Y) such that

$$h_{Y', f* \mathfrak{L}|_X}(t') \leqslant c_{17} h_{Y', \varepsilon_{Y'}^* p_1^* \mathfrak{L}|_X}(t') + c_{18} \tag{5.4.12}$$

for any $t' \in Y'(\overline{\mathbb{Q}})$. Now the conclusion follows. □

Now for any $t \in Y \cap \Sigma \cap \mathfrak{A}_g(\overline{\mathbb{Q}})$, by (5.4.1) and Lemma 5.4.5,

$$\widehat{h}_{\mathfrak{L}_g}(t) = \frac{\deg(f_t)^{1/g}\deg(\varphi_t)^{1/g}}{n_t^2}\widehat{h}_{\mathfrak{L}_g}(s). \tag{5.4.13}$$

But for any $t \in \Sigma \cap \mathfrak{A}_g(\overline{\mathbb{Q}})$, we have the following result of Faltings [16, Chapter II, §4, Lemma 5]

$$|h_F(A_{[\pi]t}) - h_F(A)| \leqslant \frac{1}{2}\log\deg(f_t). \tag{5.4.14}$$

Besides by [44, Corollary 1.3], there exists a positive constant $c_{19} = c_{19}(g, \mathcal{M})$ such that

$$|\frac{1}{2}h_F(A_{[\pi]t}) - h_{\mathcal{A}_g,\varepsilon^*\mathfrak{L}}([\pi]t)| \leqslant c_{19}\log\Big(\max\big(1, h_F(A_{[\pi]t})\big) + 2\Big) \tag{5.4.15}$$

for any $t \in \mathfrak{A}_g(\overline{\mathbb{Q}})$.

Now (5.4.11), Lemma 5.4.6, (5.4.13), (5.4.14) and (5.4.15) together imply

$$\frac{\deg(\varphi_t)^{1/g}}{n_t^2}\deg(f_t)^{1/g}\widehat{h}_{\mathfrak{L}_g}(s) \leqslant (c_{15}+c_{17})c_{19}\log\Big(\max\Big(1, h_F(A) + \frac{1}{2}\log\deg(f_t)\Big) + 2\Big)$$
$$+\frac{c_{15}+c_{17}}{4}\log\deg(f_t) + \frac{c_{15}+c_{17}}{2}h_F(A) + c_{16} + c_{18}.$$

Since $\deg(\varphi_t) \geqslant 1$, we get that $\deg(f_t)$ is polynomially bounded by n_t from above.

On the other hand, letting $\deg(f_t) \to \infty$, we see that there exist two positive constants M_0 and c_{20} depending on nothing such that $\deg(\varphi_t)^{1/g} \leqslant c_{20}n_t^2$ for any $t \in Y(\overline{\mathbb{Q}}) \cap \Sigma$ with $\deg(f_t) > M_0$. But if $\deg(f_t) \leqslant M_0$, then $\deg(f_t)$ takes value in a finite set $\{1, ..., M_0\}$. So $\deg(\varphi_t)$ is bounded by n_t from above. □

5.4.4 Application of Pila-Wilkie

Keep the notation of the beginning of this section and §5.4.1.

Proposition 5.4.7. *Let Y and $\widetilde{s}$ be as in the beginning of this section. Let $\varepsilon > 0$. There exists a constant $C = C(Y, s, \varepsilon) > 0$ with the following property:*

For every $n \geqslant 1$, there exist at most Cn^ε definable blocks $B_i \subset \widetilde{Y}$ such that $\cup B_i$ contains all point of complexity n of $\widetilde{Y} \cap \widetilde{\Sigma}$.

Proof. The proof starts with the following lemma:

Lemma 5.4.8. *There exist constants C' and κ' depending only on g and $\widetilde{s}$ such that*

For any $\widetilde{t} \in \widetilde{Y} \cap \widetilde{\Sigma}$ of complexity n, there exists a $(v, h) \in P_{2g}(\mathbb{Q})^+$ such that $(v, h) \cdot \widetilde{s} = \widetilde{t}$ and $H((v, h)) \leqslant C'n^{\kappa'}$.

Proof. Let $t := \mathrm{unif}(\widetilde{t})$. Then $t \in \Sigma$ and therefore we have a relation as (5.4.1). Let $f'_t := f_t \circ \varphi_t$, then $f'_t\colon (A, \lambda) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ is a polarized isogeny. Moreover, there exists a $\delta'_t \in A(\overline{\mathbb{Q}})_{\mathrm{tor}}$ such that $N(\delta'_t) \leqslant N(\delta_t)\deg(\varphi_t)$ and

$$t = f'_t(s_t + \delta'_t). \tag{5.4.16}$$

Claim 5.4.9. *There exists a symplectic basis $\mathcal{B}'$ for $H_1(\mathfrak{A}_{[\pi]t}, \mathbb{Z})$ w.r.t. the polarization $\lambda_{[\pi]t}$ such that the height of $\gamma_{f'} \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ (the matrix expression of f'_t in coordinate $\mathcal{B}$ w.r.t. $\mathcal{B}'$) is polynomially bounded by $\deg(f'_t) = \deg(\varphi_t)\deg(f_t)$ from above (see the beginning of this section for $\mathcal{B}$).*

This claim follows from [43, Proposition 4.1]: remark that f'_t is a polarized isogeny instead of an arbitrary isogeny, hence the endomorphism $q \in \mathrm{End}(A)$ in [43, 4.3] equals $[\deg \varphi_t]^{1/g}$, and therefore the $u \in (\mathrm{End}\, A)^*$ in [43, 4.6] can be taken to be 1_A.

Then $\mathrm{unif}_G(\gamma_{f'} \cdot \widetilde{s}_G) = [\pi]s$. Besides let $\widetilde{\delta}'_t = (\widetilde{\delta}'_{t,V}, \widetilde{s}_G) \in \mathcal{F}$ be such that $\mathrm{unif}(\widetilde{\delta}'_t) = \delta'_t$. Then $\widetilde{\delta}'_{t,V} \in V_{2g}(\mathbb{Q})$ and, by (5.4.16) and (5.2.3),

$$\mathrm{unif}\left(\gamma_{f'}\big(\frac{\widetilde{s}_V}{n_t} + \widetilde{\delta}'_{t,V}, \widetilde{s}_G\big)\right) = t.$$

So there exists an element $\gamma = (\gamma_V, \gamma_G) \in \Gamma$ such that

$$\gamma\gamma_{f'}\big(\frac{\widetilde{s}_V}{n_t} + \widetilde{\delta}'_{t,V}, \widetilde{s}_G\big) = \widetilde{t},$$

i.e.

$$\widetilde{t} = \left(\gamma_V + \gamma_G\gamma_{f'}\big(\frac{\widetilde{s}_V}{n_t} + \widetilde{\delta}'_{t,V}\big), \gamma_G\gamma_{f'}\widetilde{s}_G\right) = \big(\gamma_V + \gamma_G\gamma_{f'}\widetilde{\delta}'_{t,V}, \frac{\gamma_G\gamma_{f'}}{n_t}\big) \cdot \widetilde{s}.$$

Denote by

$$(v, h) := \big(\gamma_V + \gamma_G\gamma_{f'}\widetilde{\delta}'_{t,V}, \frac{\gamma_G\gamma_{f'}}{n_t}\big),$$

then (v, h) is an element of $P_{2g}(\mathbb{Q})^+$ such that $(v, h)\widetilde{s} = \widetilde{t}$. Now we prove that $H((v, h))$ is polynomially bounded by the complexity n of $\widetilde{t}$. To prove this, it suffices to prove that n_t, $H(\widetilde{\delta}'_{t,V})$, $H(\gamma_{f'})$, $H(\gamma_G)$ and $H(\gamma_V)$ are all polynomially bounded by n.

The fact that n_t is bounded by n follows directly from the definition of complexity.

For $H(\widetilde{\delta}'_{t,V})$: because $\widetilde{\delta}'_t \in \mathcal{F} \simeq [0, N)^{2g} \times \mathcal{F}_G$ (where N is the level structure, and hence depend on nothing), we have $\widetilde{\delta}'_{t,V} \in [0, N)^{2g}$. Therefore $H(\widetilde{\delta}'_{t,V})$ is bounded up to a constant by the denominator of $\widetilde{\delta}'_{t,V}$, which equals $N(\delta'_t)$. But $N(\delta'_t) \leqslant \deg(\varphi_t)N(\delta_t)$, hence it suffices to bound both $\deg(\varphi_t)$ and $N(\delta_t)$ by n. Now $\deg(\varphi_t)$ is polynomially bounded by n_t, and hence by n, by Proposition 5.4.4. By definition of complexity, $N(\delta_t) \leqslant n$.

For $H(\gamma_{f'})$: by choice, $H(\gamma_{f'})$ is polynomially bounded by $\deg(f_t)\deg(\varphi_t)$, which is polynomially bounded by n_t by Proposition 5.4.4. Hence $H(\gamma_{f'})$ is polynomially bounded by n by definition of complexity.

For $H(\gamma_G)$: remark $\gamma_G\gamma_{f'}\widetilde{s}_G = \pi(\widetilde{t}) \in \mathcal{F}_G$. By [49, Lemma 3.2], $H(\gamma_G)$ is polynomially bounded by $||\gamma_{f'}\widetilde{s}_G||$. Therefore $H(\gamma_G)$ is polynomially bounded, with constants depending on $||\widetilde{s}_G||$, by n.

For $H(\gamma_V)$: remark $\gamma_V + \gamma_G\gamma_{f'}\widetilde{\delta}'_{t,V} + \gamma_G\gamma_{f'}\widetilde{s}_V/n_t = \widetilde{t}_V \in [0,N)^{2g}$ (where N is the level structure, and hence depend on nothing). Therefore $H(\gamma_V)$ is polynomially bounded by $||\gamma_G\gamma_{f'}\widetilde{\delta}_{t,V} + \gamma_G\gamma_{f'}\widetilde{s}_V/n_t||$. Therefore $H(\gamma_V)$ is polynomially bounded, with constants depending on $||\widetilde{s}_V||$, by n. ☐

Let $\sigma\colon P_{2g}(\mathbb{R})^+ \to \mathcal{X}^+_{2g,\mathrm{a}}$ be the map $(v,h) \mapsto (v,h)\cdot\widetilde{s}$.

The set $R = \sigma^{-1}(\widetilde{Y}) = \sigma^{-1}(\mathrm{unif}^{-1}(Y)\cap\mathcal{F})$ is definable because σ is semi-algebraic and $\mathrm{unif}|_{\mathcal{F}}$ is definable. Hence we can apply the family version of the Pila-Wilkie theorem ([48, 3.6]) to the definable set R: for every $\varepsilon > 0$, there are only finitely many definable block families $B^{(j)}(\varepsilon) \subset R\times\mathbb{R}^m$ and a constant $C'_1(R,\varepsilon)$ such that for every $T \geqslant 1$, the rational points of R of height at most T are contained in the union of at most C'_1T^{ε} definable blocks $B_i(T,\varepsilon)$, taken (as fibers) from the families $B^{(j)}(\varepsilon)$. Since σ is semi-algebraic, the image under σ of a definable block in R is a finite union of definable blocks in $\widetilde{Y}$. Furthermore the number of blocks in the image is uniformly bounded in each definable block family $B^{(j)}(\varepsilon)$. Hence $\sigma(B_i(T,\varepsilon))$ is the union of at most C'_2T^{ε} blocks in $\widetilde{Y}$, for some new constant $C'_2(Y,\widetilde{a},\varepsilon) > 0$.

By Lemma 5.4.8, for any point $\widetilde{t} \in \widetilde{Y}\cap\widetilde{\Sigma}$ of complexity n, there exists a rational element $\gamma \in R$ such that $\sigma(\gamma) = \widetilde{t}$ and $H(\gamma) \leqslant C'n^{\kappa'}$. By the discussion in the last paragraph, all such γ's are contained in the union of at most $C'_1(C'n^{\kappa'})^{\varepsilon}$ definable blocks. Therefore all points of $\widetilde{Y}\cap\widetilde{\Sigma}$ of complexity n are contained in the union of at most $C'_1C'_2C'^{\varepsilon}n^{\kappa'\varepsilon}$ blocks in $\widetilde{Y}$. ☐

5.4.5 End of proof of Theorem 5.1.5

Now we are ready to finish the proof of Theorem 5.1.5.

Let Σ_1 be the set of points $t \in Y\cap\Sigma$ such that there is a positive-dimensional block $B \subset \widetilde{Y}$ with $t \in \mathrm{unif}(B)$. Let Y_1 be the Zariski closure of Σ_1. Let k be a number field such that both Y and Y_1 are defined over k.

Let t be a point in $Y\cap\Sigma$ of complexity n. By Proposition 5.4.2, there exist positive constants c_5 and c_6 depending only on (A,λ) and s such that

$$[k(t):k] \geqslant \frac{c_5}{[k:\mathbb{Q}]}n^{c_6}.$$

But all $\mathrm{Gal}(\overline{k}/k)$-conjugates of t are contained in $Y\cap\Sigma$ and have complexity n. By Proposition 5.4.7, the preimages in $\mathcal{F}$ of these points are contained in the union of $C(Y,s,c_6/2)n^{c_6/2}$ definable blocks, each of these blocks being contained in $\widetilde{Y}$.

For n large enough, $(c_5/[k:\mathbb{Q}])n^{c_6} > Cn^{c_6/2}$. Hence for $n \gg 0$, there exists a definable block $B \subset \widetilde{Y}$ such that $\mathrm{unif}(B)$ contains at least two Galois conjugates of t, and therefore $\dim B > 0$ since blocks are connected. So being in $\mathrm{unif}(B)$, those conjugates of t are in Σ_1. But Y_1 is defined over k, so $t \in Y_1$.

In summary, all points of $Y \cap \Sigma$ of large enough complexity are in Σ_1. This excludes only finitely many points of $Y \cap \Sigma$. So $Y_1 = Y$.

Let Σ_2 be the set of points $t \in Y \cap \Sigma$ such that there is a connected positive-dimensional semi-algebraic set $B' \subset \widetilde{Y}$ with $t \in \mathrm{unif}(B')$. Let Y_2 be the Zariski closure of Σ_2. By definition of blocks, $\Sigma_2 = \Sigma_1$, and hence $Y_2 = Y_1 = Y$.

Now the mixed Ax-Lindemann theorem (Theorem 3.1.4) yields the conclusion since $\dim(Y) = 1$. Alternatively, let $\widetilde{Y}'$ be a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$. Then since $Y = Y_2$, there exists a positive-dimensional irreducible algebraic subset (in the sense of Definition 1.3.5) $\widetilde{Z}$ of $\mathcal{X}_{2g,\mathrm{a}}$ contained in $\widetilde{Y}'$ by [49, Lemma 4.1]. But $\dim \widetilde{Y}' = \dim \widetilde{Z} = 1$, therefore $\widetilde{Y}' = \widetilde{Z}$ is algebraic in the sense of Definition 1.3.5. In other words, Y is algebraic and a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$ is also algebraic. Hence by Theorem 2.3.3, Y is weakly special.

5.5 Variants of the André-Pink-Zannier conjecture

In the previous sections we have discussed the intersection of a subvariety of $\mathfrak{A}_g$ with the set of division points of the polarized isogeny orbit of a given point (5.1.1). The goal of this section is twofold: one is to replace the given point by a finitely generated subgroup of one fiber of $\mathfrak{A}_g \to \mathcal{A}_g$ (remark that the fiber is an abelian variety), the other is to replace the polarized isogeny orbit by the isogeny orbit. In particular we will prove that although these changes to Conjecture 5.1.1 a priori seem to generalize the conjecture, both can actually be implied by Conjecture 5.1.1 itself.

In the rest of the section, fix a point $b \in \mathcal{A}_g$, which corresponds to a polarized abelian variety $(A, \lambda) := (\mathfrak{A}_{g,b}, \lambda_b)$. Let Λ be any finitely generated subgroup of A.

Theorem 5.5.1. *Let Y be an irreducible subvariety of $\mathfrak{A}_g$. Let Σ_0 be the set of division points of the polarized isogeny orbit of Λ, i.e.*

$$\Sigma_0 = \{t \in \mathfrak{A}_g \mid \exists n \in \mathbb{N} \text{ and a polarized isogeny } f\colon (A,\lambda) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t}) \text{ with } nt \in f(\Lambda)\}.$$

Assume that Conjecture 5.1.1 holds for all g. If $\overline{Y \cap \Sigma_0} = Y$, then Y is weakly special.

Proof. The proof is basically the same as Pink [54, Theorem 5.4] (how Conjecture 5.1.1 implies the Mordell-Lang conjecture).

Suppose $\mathrm{rank}\,\Lambda = r - 1$. Let V_{2g}^r be the direct sum of r copies of V_{2g} as a representation of GSp_{2g}. Then the connected mixed Shimura variety associated with $V_{2g}^r \rtimes \mathrm{GSp}_{2g}$ is the r-fold fiber product of $\mathfrak{A}_g$ over $\mathcal{A}_g$, and so its fiber over b is A^r. Denote by

$$\sigma\colon \mathfrak{A}_g \times_{\mathcal{A}_g} ... \times_{\mathcal{A}_g} \mathfrak{A}_g \to \mathfrak{A}_g$$

the summation map (remark that both varieties are abelian schemes over $\mathcal{A}_g$).

Now the homomorphisms

$$\begin{array}{llll} P_{2g,\mathrm{a}} & = V_{2g} \rtimes \mathrm{GSp}_{2g} & \hookrightarrow V_{2g}^r \rtimes \mathrm{GSp}_{2g} \hookrightarrow & V_{2gr} \rtimes \mathrm{GSp}_{2gr} \\ & \quad (v,h) & \mapsto ((v,...,v),h)) \mapsto & ((v,...,v),(h,...,h)) \end{array}$$

induce Shimura immersions

$$\begin{array}{ccccc} \mathfrak{A}_g & \longrightarrow & \mathfrak{A}_g \times_{\mathcal{A}_g} ... \times_{\mathcal{A}_g} \mathfrak{A}_g & \longrightarrow & \mathfrak{A}_{gr} \\ {\scriptstyle [\pi]}\downarrow & & \downarrow & & \downarrow \\ \mathcal{A}_g & \xrightarrow{=} & \mathcal{A}_g & \hookrightarrow & \mathcal{A}_{gr} \end{array}$$

For simplicity we shall not distinguish a point in $\mathfrak{A}_g$ (resp. $\mathcal{A}_g$) and its image in $\mathfrak{A}_{gr}$ (resp. $\mathcal{A}_{gr}$). Then $\mathfrak{A}_{gr,b} = A^r$.

Fix generators a_1,...,a_{r-1} of Λ and set $a_r := -a_1 - ... - a_{r-1}$. Let Λ' be the division group of Λ, i.e. $\Lambda' = \{s|\ \exists n \in \mathbb{N} \text{ such that } ns \in \Lambda\} \subset A$. Then [54, Lemma 5.3] asserts that

$$\Lambda' = \Lambda_{a_1}^* + ... + \Lambda_{a_r}^* = \sigma(\Lambda_{a_1}^* \times ... \times \Lambda_{a_r}^*) \tag{5.5.1}$$

where (as Pink defined) $\Lambda_{a_i}^* := \{s \in A|\ \exists m,n \in \mathbb{Z}\setminus\{0\} \text{ such that } ns = ma_i\}$.

Now consider

$$\Lambda^\dagger := \sigma^{-1}(Y) \cap \{f^r(\Lambda_{a_1}^* \times ... \times \Lambda_{a_r}^*)|\ f\colon (A,\lambda) \to (\mathfrak{A}_{g,b'},\lambda_{b'}) \text{ a polarized isogeny}\}.$$

We have

$$\begin{aligned} \sigma(\Lambda^\dagger) &= Y \cap \sigma(\{f^r(\Lambda_{a_1}^* \times ... \times \Lambda_{a_r}^*)|\ f\colon (A,\lambda) \to (\mathfrak{A}_{g,b'},\lambda_{b'}) \text{ a polarized isogeny}\}) \\ &= Y \cap \{f^r\left(\sigma(\Lambda_{a_1}^* \times ... \times \Lambda_{a_r}^*)\right) \mid f\colon (A,\lambda) \to (\mathfrak{A}_{g,b'},\lambda_{b'}) \text{ a polarized isogeny}\} \\ &= Y \cap \{f^r(\Lambda')|\ f\colon (A,\lambda) \to (\mathfrak{A}_{g,b'},\lambda_{b'}) \text{ a polarized isogeny}\} \qquad (5.5.1). \end{aligned}$$

Because $\overline{Y \cap \Sigma_0} = Y$, $Y \cap \{f(\Lambda')|\ f\colon (A,\lambda) \to (\mathfrak{A}_{g,b'},\lambda_{b'}) \text{ a polarized isogeny}\}$ is Zariski dense in Y (as subsets of $\mathfrak{A}_g$). Therefore $\sigma(\Lambda^\dagger)$ is Zariski dense in Y (as subsets of $\mathfrak{A}_g \times_{\mathcal{A}_g} ... \times_{\mathcal{A}_g} \mathfrak{A}_g$, and hence as subsets of $\mathfrak{A}_{gr}$). Let $Y^\dagger$ be the Zariski closure of $\Lambda^\dagger$ in $\mathfrak{A}_g \times_{\mathcal{A}_g} ... \times_{\mathcal{A}_g} \mathfrak{A}_g$. Then $Y^\dagger$ is also a subvariety of $\mathfrak{A}_{gr}$. Since taking Zariski closures commutes with taking images under proper morphisms, we deduce that $\sigma(Y^\dagger) = Y$. So there exists an irreducible component Y' of $Y^\dagger$ such that $\sigma(Y') = Y$.

For any polarized isogeny $f\colon (A,\lambda) \to (\mathfrak{A}_{g,b'},\lambda_{b'})$, the generalized Hecke orbit of $(a_1,...,a_r) \in A^r$ as a point on $\mathfrak{A}_{gr}$ contains $f^r(\Lambda_{a_1}^* \times ... \times \Lambda_{a_r}^*)$ by Corollary 5.2.5. Therefore the intersection of Y' with generalized Hecke orbit of $(a_1, ..., a_r)$ in $\mathfrak{A}_{gr}$ is Zariski dense in Y'. Hence Conjecture 5.1.1 for $\mathfrak{A}_{gr}$ implies that Y' is weakly special. Therefore $Y = \sigma(Y')$ is also weakly special by the geometric interpretation of weakly special subvarieties of $\mathfrak{A}_g$ and of $\mathfrak{A}_{gr}$ (Proposition 1.2.15). □

Corollary 5.5.2. *Let Y be an irreducible subvariety of $\mathfrak{A}_g$. Let Σ'_0 be the set of division points of the isogeny orbit of Λ, i.e.*

$$\Sigma'_0 = \{t \in \mathfrak{A}_g | \ \exists n \in \mathbb{N} \text{ and an isogeny } f\colon A \to \mathfrak{A}_{g,[\pi]t} \text{ such that } nt \in f(\Lambda)\}.$$

Assume that Conjecture 5.1.1 holds for all g. If $\overline{Y \cap \Sigma'_0} = Y$, then Y is weakly special.

Proof. Recall Zarhin's trick (see Orr [42, Proposition 4.4]): for any isogeny $f\colon A \to A'$ between polarized abelian varieties, there exists $u \in \mathrm{End}(A^4)$ such that $f^4 \circ u\colon A^4 \to (A')^4$ is a polarized isogeny.

Now let $i\colon \mathfrak{A}_g \hookrightarrow \mathfrak{A}_{4g}$ be the natural embedding. Then $\Lambda_4 := \mathrm{End}(A^4)i(\Lambda)$ is a finitely generated subgroup of $A^4 = \mathfrak{A}_{4g,i(b)}$ and hence

$$\begin{aligned} \Sigma'_0 \quad \subset \{t \in \mathfrak{A}_{4g} | \ \exists n \in \mathbb{N} & \text{ and a polarized isogeny} \\ f\colon (A^4, \lambda^{\boxtimes 4}) \to (\mathfrak{A}_{4g,[\pi]t}, \lambda_{[\pi]t}) & \text{ such that } nt \in f(\Lambda_4)\}. \end{aligned}$$

Now the conclusion follows from Theorem 5.5.1. □

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Résumé

La conjecture de Zilber-Pink est une conjecture diophantienne concernant les intersections atypiques dans les variétés de Shimura mixtes. C'est une généralisation commune de la conjecture d'André-Oort et de la conjecture de Mordell-Lang. Le but de cette thèse est d'étudier Zilber-Pink. Plus concrètement, nous étudions la conjecture d'André-Oort, selon laquelle une sous-variété d'une variété de Shimura mixte est spéciale si son intersection avec l'ensemble des points spéciaux est dense, et la conjecture d'André-Pink-Zannier, selon laquelle une sous-variété d'une variété de Shimura mixte est faiblement spéciale si son intersection avec une orbite de Hecke généralisée est dense. Cette dernière conjecture généralise Mordell-Lang comme expliqué par Pink.

Dans la méthode de Pila-Zannier, un point clef pour étudier la conjecture de Zilber-Pink est de démontrer le théorème d'Ax-Lindemann qui est une généralisation du théorème classique de Lindemann-Weierstrass dans un cadre fonctionnel. Un des résultats principaux de cette thèse est la démonstration du théorème d'Ax-Lindemann dans sa forme la plus générale, c'est-à-dire le théorème d'Ax-Lindemann mixte. Ceci généralise les résultats de Pila, Pila-Tsimerman, Ullmo-Yafaev et Klingler-Ullmo-Yafaev concernant Ax-Lindemann pour les variétés de Shimura pures.

Un autre résultat de cette thèse est la démonstration de la conjecture d'André-Oort pour une grande collection de variétés de Shimura mixtes : inconditionnellement pour une variété de Shimura mixte arbitraire dont la partie pure est une sous-variété de $\mathcal{A}_6^N$ (par exemple les produits des familles universelles des variétés abéliennes de dimension 6 et le fibré de Poincaré sur $\mathcal{A}_6$) et sous GRH pour toutes les variétés de Shimura mixtes de type abélien. Ceci généralise des théorèmes connus de Klinger-Ullmo-Yafaev, Pila, Pila-Tsimerman et Ullmo pour les variétés de Shimura pures.

Quant à la conjecture d'André-Pink-Zannier, nous démontrons plusieurs cas valables lorsque la variété de Shimura mixte ambiante est la famille universelle des variétés abéliennes. Tout d'abord nous démontrons l'intersection d'André-Oort et André-Pink-Zannier, c'est-à-dire que l'on étudie l'orbite de Hecke généralisée d'un point spécial. Ceci généralise des résultats d'Edixhoven-Yafaev et Klingler-Ullmo-Yafaev pour $\mathcal{A}_g$. Nous prouvons ensuite la conjecture dans le cas suivant : une sous-variété d'un schéma abélien au dessus d'une courbe est faiblement spéciale si son intersection avec l'orbite de Hecke généralisée d'un point de torsion d'une fibre non CM est Zariski dense. Finalement pour une orbite de Hecke généralisée d'un $\overline{\mathbb{Q}}$-point arbitraire, nous démontrons la conjecture pour toutes les courbes. Ces deux derniers cas généralisent des résultats de Habegger-Pila et Orr pour $\mathcal{A}_g$.

Dans toutes les démonstrations, la théorie o-minimale, en particulier le théorème de comptage de Pila-Wilkie, joue un rôle important.

Abstract

The Zilber-Pink conjecture is a diophantine conjecture concerning unlikely intersections in mixed Shimura varieties. It is a common generalization of the André-Oort conjecture and the Mordell-Lang conjecture. This dissertation is aimed to study the Zilber-Pink conjecture. More concretely, we will study the André-Oort conjecture, which predicts that a subvariety of a mixed Shimura variety having dense intersection with the set of special points is special, and the André-Pink-Zannier conjecture which predicts that a subvariety of a mixed Shimura variety having dense intersection with a generalized Hecke orbit is weakly special. The latter conjecture generalizes the Mordell-Lang conjecture as explained by Pink.

In the Pila-Zannier method, a key point to study the Zilber-Pink conjecture is to prove the Ax-Lindemann theorem, which is a generalization of the functional analogue of the classical Lindemann-Weierstrass theorem. One of the main results of this dissertation is to prove the Ax-Lindemann theorem in its most general form, i.e. the mixed Ax-Lindemann theorem. This generalizes results of Pila, Pila-Tsimerman, Ullmo-Yafaev and Klingler-Ullmo-Yafaev concerning the Ax-Lindemann theorem for pure Shimura varieties.

Another main result of this dissertation is to prove the André-Oort conjecture for a large class of mixed Shimura varieties: unconditionally for any mixed Shimura variety whose pure part is a subvariety of $\mathcal{A}_6^N$ (e.g. products of universal families of abelian varieties of dimension 6 and the Poincaré bundle over $\mathcal{A}_6$) and under GRH for all mixed Shimura varieties of abelian type. This generalizes existing theorems of Klinger-Ullmo-Yafaev, Pila, Pila-Tsimerman and Ullmo concerning pure Shimura varieties.

As for the André-Pink-Zannier conjecture, we prove several cases when the ambient mixed Shimura variety is the universal family of abelian varieties. First we prove the overlap of André-Oort and André-Pink-Zannier, i.e. we study the generalized Hecke orbit of a special point. This generalizes results of Edixhoven-Yafaev and Klingler-Ullmo-Yafaev for $\mathcal{A}_g$. Secondly we prove the conjecture in the following case: a subvariety of an abelian scheme over a curve is weakly special if its intersection with the generalized Hecke orbit of a torsion point of a non CM fiber is Zariski dense. Finally for the generalized Hecke orbit of an arbitrary $\overline{\mathbb{Q}}$-point, we prove the conjecture for curves. These generalize existing results of Habegger-Pila and Orr for $\mathcal{A}_g$.

In all these proofs, the o-minimal theory, in particular the Pila-Wilkie counting theorems, plays an important role.

Samenvatting

Het Zilber-Pink vermoeden is een diophantisch vermoeden over zogenaamde "onwaarschijnlijke intersecties" in gemengde Shimura variëteiten. Het is een gemeenschappelijke generalisatie van de vermoedens van André-Oort en Mordell-Lang. In dit proefschrift wordt het Zilber-Pink vermoeden bestudeerd. Preciezer, we bestuderen het André-Oort vermoeden, dat zegt dat in een gemengde Shimura variëteit iedere deelvariëteit waarin de speciale punten dicht liggen zelf speciaal is, en het André-Pink-Zannier vermoeden dat zegt dat in een gemengde Shimura variëteit iedere deelvariëteit met een dichte doorsnede met een gegeneraliseerde Hecke baan zwak speciaal is. Zoals uitgelegd door Pink generaliseert dit laatste vermoeden het Mordell-Lang vermoeden.

Een essentieel punt in de benadering van het Zilber-Pink vermoeden door Pila en Zannier is het bewijzen van de Ax-Lindemann stelling, die een generalisatie is van een functionaal analogon van de klassieke Lindemann-Weierstrass stelling. Één van de hoofdresultaten van dit proefschrift is een bewijs van de Ax-Lindemann stelling in zijn meest algemene vorm, dat wil zeggen, de gemengde Ax-Lindemann stelling. Dit generaliseert resultaten van Pila, Pila-Tsimerman, Ullmo-Yafaev en Klingler-Ullmo-Yafaev over de Ax-Lindemann stelling voor pure Shimura variëteiten.

Een ander hoofdresultaat in dit proefschrift is een bewijs van het André-Oort vermoeden voor een grote klasse van gemengde Shimura variëteiten: onvoorwaardelijk voor elke gemengde Shimura variëteit waarvan het pure quotiënt een deelvariëteit is van $\mathcal{A}_6^N$ (d.w.z., producten van universele families van abelse variëteiten van dimensie 6 en de Poincaré bundel over $\mathcal{A}_6$) en onder de gegeneraliseerde Riemann hypothese (GRH) voor alle gemengde Shimura variëteiten van abels type. Dit generaliseert stellingen van Klinger-Ullmo-Yafaev, Pila, Pila-Tsimerman and Ullmo betreffende pure Shimura variëteiten.

Wat het André-Pink-Zannier vermoeden betreft, bewijzen we een aantal gevallen waarin de ambiënte gemengde Shimura variëteit een universele familie van abelse variëteiten is. Eerst bewijzen we de overlap tussen André-Oort en André-Pink-Zannier, d.w.z., we bestuderen de gegeneraliseerde Hecke baan van een speciaal punt. Dit generaliseert resultaten van Edixhoven-Yafaev en Klingler-Ullmo-Yafaev voor $\mathcal{A}_g$. Daarna bewijzen we het vermoeden in het volgende geval: een deelvariëteit van een abels schema over een kromme is zwak speciaal als zijn doorsnede met de gegeneraliseerde Hecke baan van een torsiepunt van een niet CM-vezel Zariski dicht is. Tenslotte bewijzen we het vermoeden voor krommen en de gegeneraliseerde Hecke baan van een $\overline{\mathbb{Q}}$-punt. Deze resultaten generaliseren resultaten van Habegger-Pila en Orr voor $\mathcal{A}_g$.

In al deze bewijzen speelt o-minimale theorie, en in het bijzonder de telstelling van Pila-Wilkie, een belangrijke rol.

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J'ai présenté les résultats de cette thèse dans plusieurs exposés, notamment à la conférence *Diamant symposium* à Lunteren le 28 novembre 2013, au *Séminaire de théorie des nombres de Jussieu-PRG* le 24 février 2014, à la rencontre *Autour des conjectures de Lang et Vojta* au CIRM le 6 mars 2014 et à la conférence *Second ERC Research Period on Diophantine Geometry* à Cetraro le 22 juillet 2014. Je tiens à en remercier les organisateurs, notamment Robin de Jong, Mathilde Herblot, Erwan Rousseau et Umberto Zannier. J'ai été convié en tant que chercher invité au BICMR (Beijing International Center for Mathematical Research) pendant les étés 2013 et 2014. Je remercie Tian Qingchun et Liu Ruochuan pour leurs invitations.

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I have presented the results of this dissertation in many talks, notably in the conference *Diamant symposium* in Lunteren on November 28, 2013, in *Séminaire de théorie des nombres de Jussieu-PRG* on February 24, 2014, in the conference *On Lang and Vojta's conjectures* at CIRM on March 6, 2014 and in the conference *Second ERC Research Period on Diophantine Geometry* in Cetraro on July 22, 2014. I would like to thank the organizers of the meetings, especially Robin de Jong, Mathilde Herblot, Erwan Rousseau and Umberto Zannier. I was invited as visiting scholar at BICMR (Beijing International Center for Mathematical Research) in the summers of 2013 and 2014. I would like to thank Tian Qingchun and Liu Ruochuan for their invitations.

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Curriculum Vitae

Ziyang Gao was born on 29th June 1988 in Dandong, Liaoning, China.

In 2006, he moved to Beijing to start his bachelor in mathematics at Peking University. In 2010, he finished his bachelor and started the Erasmus-Mundus master-Algant program. He spent his first master year in Milan and his second in Orsay, where he wrote his master thesis entitled "La conjecture d'André-Oort pour le produit des courbes modulaires" under the supervision of Emmanuel Ullmo. He received his "master Algant" degree at the Algant graduation ceremony in Padova in 2012. He also received his master degrees from Università degli Studi di Milano and Université Paris-Sud in the same year.

In 2012 he was awarded an Algant-Doc joint PhD fellowship at Université Paris-Sud and Universiteit Leiden, under the supervision of Emmanuel Ullmo and Bas Edixhoven. He was invited as speaker to quite a few conferences and seminars during his PhD. He was invited as visiting scholar to BICMR (Beijing International Center for Mathematical Research) in the summers of 2013 and 2014. In Fall 2014 he gave a course "Introduction to Algebraic Topology" at Universiteit Leiden.