PANORAMA ON AX TYPE TRANSCENDENCE RESULTS

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ABSTRACT. We summarize the current situation of the geometric Ax type transcendence results. In particular the Ax-Schanuel conjecture we state here contains all existing geometric Ax type transcendence results. In the case of mixed Shimura varieties, we also prove the refinement of a distribution theorem of positive dimensional weakly special subvarieties by Ullmo (for pure Shimura varieties) and the author (for mixed Shimura varieties).

CONTENTS

1. INTRODUCTION

The goal of this survey is to summarize the current development of functional transcendence results, which we call geometric transcendence results in contrast to classical transcendence number theory. We use the language of bi-algebraic geometry to state these results. This language and these geometric transcendence results have been proven to be very useful in Diophantine geometry, for example the unconditional proof of the André-Oort conjecture for any mixed Shimura variety of abelian type [17].

We start with the simple example of complex algebraic tori in §2. We characterize the geometric bi-algebraic subvarieties, state the Ax-Schanuel theorem and explain the two aspects of this theorem. Then we briefly talk about the arithmetic bi-algebraicity and the Manin-Mumford theorem. Then we pass to mixed Shimura varieties in §3. We give a brief example-based revision of the theory of mixed Shimura variety. Then we give the characterization of the geometric bi-algebraic subvariety (which in this case are precisely the *weakly special subvarieties* defined by Pink) and explain them in details in geometric terms. Next we state the Ax-Schanuel conjecture for mixed Shimura varieties and its current situation. We close this section by proving the refinement of a distribution theorem of positive dimensional weakly special subvarieties by Ullmo's [19, Théorème 4.1] (for pure Shimura varieties) and the author's [6, Theorem 12.2] (for mixed Shimura varieties). In §4 we discuss the geometric bi-algebraic system associated with

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universal vector extensions of abelian varieties. It is necessary to study these universal vector extensions if we want to study arithmetic bi-algebraic systems for abelian varieties. Then we formulate the Ax-Schanuel statement for these universal vector extensions of abelian varieties. In the last section §5 we put together mixed Shimura varieties and vector extensions. We give an example-based introduction to enlarged mixed Shimura varieties, give the characterization of geometric bi-algebraic subvarieties, state the Ax-Schanuel conjecture (which is in its most general form and contains all previous Ax-Schanuel statements as special cases) and explain its situation.

2. First example: algebraic tori

In this section our goal is to explain the topics of this survey by the example of algebraic tori.

Let T be a complex algebraic tori, then $T \simeq (\mathbb{C}^*)^n$ for some n. Take the universal cover of T

$$\mathbf{u}\colon \mathbb{C}^n = \mathrm{Lie}(T) \to T = (\mathbb{C}^*)^n, \quad (x_1, \ldots, x_n) \mapsto (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}).$$

The map $\mathbf{u}$ is not algebraic.

2.1. **Geometric bi-algebraicity.** We say that a closed irreducible subvariety Z of T is *geometric bi-algebraic* if one (and hence any) complex analytic irreducible component of $\mathbf{u}^{-1}(Z)$ is an algebraic subvariety of $\mathbb{C}^n$.

We have the following characterization of geometric bi-algebraic subvarieties of T: the closed irreducible bi-algebraic subvarieties of T are precisely the translates of algebraic subtori. Here we briefly explain an easy proof using the idea of monodromy.

One direction is immediate. For the other direction, let Z be a closed irreducible bi-algebraic subvariety of T and denote by $j\colon Z^{\mathrm{sm}} \hookrightarrow T$ the inclusion. Up to translating Z we may assume that Z contains the neutral element 1 of T. We re-interpret the uniformizing map $\mathbf{u}$ as

$$\pi_1(T, z) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H_1(T, z) \otimes_{\mathbb{Z}} \mathbb{C} \to T.$$

Then we are done if we can prove that the smallest subtorus containing Z is $\mathbf{u}(j_*\pi_1(Z^{\mathrm{sm}}, z) \otimes_{\mathbb{Z}} \mathbb{C})$.

Now up to replacing T by a subtorus we may assume that Z is not contained in any proper subtorus of T. We are done if we can prove $[\pi_1(T, z) : j_*\pi_1(Z^{\mathrm{sm}}, z)] < \infty$. If not, then

$$j_*\pi_1(Z^{\mathrm{sm}}, z) \subset \mathrm{Ker}(\rho\colon \mathbb{Z}^m \twoheadrightarrow \mathbb{Z}) \tag{2.1}$$

for some map ρ. Since the covariant functor $T \mapsto X_*(T)$ ($X_*(T)$ is the co-character group of T) is an equivalence between the category {algebraic tori over $\mathbb{C}$} and the category {free $\mathbb{Z}$-modules of finite rank}, the map ρ corresponds to a surjective map (with connected kernel) of tori $p\colon T \twoheadrightarrow T'$. The composition of the maps $Z^{\mathrm{sm}} \xrightarrow{j} T \xrightarrow{p} T' = \mathbb{G}_{m,\mathbb{C}}$ is dominant by the choice of T. But then we have

$$[\pi_1(T', p(z)) : (p \circ j)_*\pi_1(Z^{\mathrm{sm}}, z)] < \infty$$

([9, 2.10.2]), which contradicts to (2.1) by the canonical isomorphism

$$\psi_T\colon X_*(T) \xrightarrow{\sim} \pi_1(T, 1), \quad \nu \mapsto [\nu \circ i]$$

where i is the inclusion $\{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}^*$.

2.2. **Ax-Schanuel.** Ax [1] proved the following theorem, which is the functional analogue of the Schanuel conjecture. Our statement here is the geometric version formulated by Tsimerman [16].

Theorem 2.1. *Let* $\Delta \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$ *be the graph of* $\mathbf{u}$. *Let* $\mathfrak{Z} = \mathrm{graph}(\widetilde{Z} \xrightarrow{\mathbf{u}} Z)$ *be an irreducible complex subspace of* Δ, *and let* $\mathfrak{B}$ *be its Zariski closure in* $\mathbb{C}^n \times (\mathbb{C}^*)^n$. *Then* $\dim \mathfrak{B} - \dim \mathfrak{Z} \geq \dim F$, *where* F *is the smallest translate of subtorus containing* Z.

We explain this theorem. First replace $\mathfrak{Z}$ be a complex analytic irreducible component of $\mathfrak{B} \cap \Delta$. Then denote by $\widetilde{X} := pr_1(\mathfrak{B}) = \widetilde{Z}^{\mathrm{Zar}}$ and $Y := pr_2(\mathfrak{B}) = Z^{\mathrm{Zar}}$. We have $\mathfrak{B} \subset \widetilde{X} \times Y$. Therefore Theorem 2.1 implies

$$\dim \widetilde{X} + \dim Y - \dim \widetilde{Z} \geq \dim \mathfrak{B} - \dim \mathfrak{Z} \geq \dim F. \tag{2.2}$$

On the other hand let $\widetilde{Y}$ (resp. $\widetilde{F}$) be the complex analytic irreducible component of $\mathbf{u}^{-1}(Y)$ (resp. of $\mathbf{u}^{-1}(F)$) containing $\widetilde{Z}$, then $\widetilde{Z}$ is a complex analytic irreducible component of $\widetilde{X} \cap \widetilde{Y}$ since $\mathfrak{Z}$ is a complex analytic irreducible component of $\mathfrak{B} \cap \Delta$. Hence we always have

$$\dim \widetilde{Z} \geq \dim \widetilde{X} + \dim \widetilde{Y} - \dim \widetilde{F}. \tag{2.3}$$

Now (2.2) and (2.3) together imply

$$\dim \mathfrak{B} = \dim \widetilde{X} + \dim Y, \ \dim \widetilde{Z} = \dim \widetilde{X} + \dim \widetilde{Y} - \dim \widetilde{F},$$

so Theorem 2.1 is equivalent to:

- $\mathfrak{B} = \widetilde{X} \times Y$;
- $\widetilde{X}$ and $\widetilde{Y}$ intersect properly in $\widetilde{F}$.

2.3. **Arithmetic bi-algebraicity.** Note that every closed point of T is geometric bi-algebraic by definition, so for points we need a new notion of bi-algebraicity. This is the arithmetic bi-algebraicity we will discuss in this subsection. To study arithmetic bi-algebraicity, we need $\overline{\mathbb{Q}}$-structures on both T and its universal cover $\mathbb{C}^n$. So in this subsection we assume that T is defined over $\overline{\mathbb{Q}}$.

We say that a point $t \in T(\overline{\mathbb{Q}})$ is *arithmetic bi-algebraic* if $\mathbf{u}^{-1}(t) \subset \overline{\mathbb{Q}}^n$.

We have the following characterization of arithmetic bi-algebraic points of T: $t \in T(\mathbb{C})$ is arithmetic bi-algebraic if and only if t is a torsion coset of T. Let us briefly explain the reason.

It is easy to see that any torsion coset of T is arithmetic bi-algebraic. The other implication follows from the Gel'fond-Schneider theorem: Given complex numbers $\lambda \neq 0$ and β, if e^{λ}, β and $e^{\beta\lambda}$ are all algebraic, then $\beta \in \mathbb{Q}$. It suffices to apply this theorem to $\lambda = 2\pi i$.

2.4. **Manin-Mumford for algebraic tori.** The following theorem is the analogue of the Manin-Mumford conjecture for algebraic tori.

Theorem 2.2. *Let* T *be an algebraic torus over* $\mathbb{C}$. *Then any irreducible component of the the Zariski closure of a subset of* $T(\mathbb{C})_{\mathrm{tor}}$ *is a torsion coset of* T.

As usual, the proof of the theorem can be reduced to those T over $\overline{\mathbb{Q}}$ by specialization argument. But then the theorem becomes:

Theorem 2.3. *Let* T *be an algebraic torus over* $\overline{\mathbb{Q}}$. *Then any irreducible component of the Zariski closure of arbitrarily many arithmetic bi-algebraic points is geometric bi-algebraic.*

3. Mixed Shimura varieties

We generalize the discussion in the last section to mixed Shimura varieties.

3.1. **Examples of mixed Shimura varieties.** Pure (resp. mixed) Shimura varieties are moduli spaces of certain pure (resp. mixed) Hodge structures. Shimura varieties have an extremely rich arithmetic and are central objects in the theory of automorphic forms (Langlands program) and in Diophantine geometry.

The prototype of pure Shimura varieties is the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. A first example of mixed but not pure Shimura varieties is the universal abelian variety $\mathfrak{A}_g$ over a fine moduli space. More concretely we let $N \geq 3$ be an integer and let $\mathcal{A}_g(N)$ be the moduli space of principally polarized abelian varieties of dimension g with a level-N-structure. Then $\mathcal{A}_g(N)$ is a fine moduli space, and thus admits a universal family $\mathfrak{A}_g(N)$. However, $\mathfrak{A}_g(N)$ is NOT the typical example of mixed Shimura varieties. There are several reasons for this, and here let me just point out one: the points of $\mathcal{A}_g$ parametrize not only abelian varieties of dimension g, but also a fixed polarization on each abelian variety. In other words every point of $\mathcal{A}_g$ is of the form $[(A, L)]$, where A is an abelian variety and L is an ample line bundle on A of degree 1. Hence when making the universal family, one should not only consider the abelian varieties themselves, but also the fixed polarizations. However the family $\mathfrak{A}_g(N) \to \mathcal{A}_g(N)$ does not contain any information of the polarizations.

This is why we need to introduce $\mathfrak{L}_g(N)$, the symmetric relatively ample $\mathbb{G}_m$-torsor over $\mathfrak{A}_g(N) \to \mathcal{A}_g(N)$ such that every fiber of $\mathfrak{L}_g(N) \to \mathcal{A}_g(N)$ over $[(A, L)] \in \mathcal{A}_g(N)$ is the total space of the $\mathbb{G}_m$-torsor associated with L, i.e. L with the zero section removed. For technical reasons we let $N \geq 3$ to be even when defining $\mathfrak{L}_g(N)$. The reduction lemma of Pink [, 2.26] suggests that many problems concerning mixed Shimura varieties can be reduced to the product of a pure Shimura variety and copies of $\mathfrak{L}_g(N)$, and the most enlightening case is $\mathfrak{L}_g(N) \times \mathfrak{L}_g(N)$.

When considering geometric transcendence theorems for mixed Shimura varieties, e.g. Ax type transcendence statements, passing from $\mathfrak{L}_g$ to $\mathfrak{L}_g \times \mathfrak{L}_g$ is usually nontrivial and even contains the core of the difficulty.[1] On the other hand it is sometimes more convenient to work with a $\mathbb{G}_m$-torsor over an abelian scheme than with a product of two $\mathbb{G}_m$-torsors. Thus it is sometimes more convenient to work with the universal Poincaré biextension $\mathfrak{P}_g$, i.e. the $\mathbb{G}_m$-torsor over $\mathfrak{A}_g \times \mathfrak{A}_g^\vee$ whose fiber $(\mathfrak{P}_g)_a$ for any point $a \in \mathcal{A}_g$ is the Poincaré biextension over $(\mathfrak{A}_g)_a \times (\mathfrak{A}_g^\vee)_a$. Now $\mathfrak{P}_g$ is an intermediate object between $\mathfrak{L}_g$ and $\mathfrak{L}_g \times \mathfrak{L}_g$.

3.2. **Brief introduction to the Deligne-Pink language of mixed Shimura varieties.** The uniformization of $\mathcal{A}_g$ is $\mathbf{u}\colon \mathcal{H}_g^+ \to \mathcal{A}_g$, where $\mathcal{H}_g^+$ is the Siegel upper half space

$$\mathcal{H}_g^+ = \{Z = X + iY : Z = Z^{\mathrm{t}},\ Y > 0\} \subset \mathrm{Mat}_{2g\times 2g}(\mathbb{C}).$$

The group $\mathrm{GSp}_{2g}(\mathbb{R})^+$ acts on $\mathcal{H}_g^+$ by the law

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ+B)(CZ+D)^{-1},$$

[1]The reason, as revealed by the proof of Pink's reduction lemma, is that the weight -2 part of an arbitrary mixed Shimura variety may a priori NOT give any polarization of its weight -1 part and thus we need to pass to a unipotent extension by $\mathbb{G}_a$.

making $\mathcal{H}_g^+$ a $\mathrm{GSp}_{2g}(\mathbb{R})^+$-orbit. As complex spaces, we have $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g^+$. To sum it up, there is a pair $(\mathrm{GSp}_{2g}, \mathcal{H}_g^+)$ associated with $\mathcal{A}_g$, where GSp_{2g} is a $\mathbb{Q}$-group and $\mathrm{GSp}_{2g}(\mathbb{R})^+$ acts transitively on $\mathcal{H}_g^+$, such that $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g^+$ as complex spaces. By definition $\mathcal{H}_g^+$ is naturally an open semialgebraic subset of $\{Z = X + iY : Z = Z^{\mathrm{t}}\} \simeq \mathbb{C}^{g(g+1)/2}$ in the usual topology.

The discussion above can be generalized to any mixed Shimura varieties. Associated with any connected mixed Shimura variety there is a pair $(P, \mathcal{X}^+)$ consisting of a $\mathbb{Q}$-group P and a complex space $\mathcal{X}^+$ such that $\mathcal{X}^+$ is an open semialgebraic subset of a complex algebraic variety $\mathcal{X}^\vee$, that $P(\mathbb{R})^+U(\mathbb{C})$ acts transitively on $\mathcal{X}^+$,[2] and that $S = \Gamma\backslash\mathcal{X}^+$ as complex spaces for some congruent subgroup Γ of $P(\mathbb{Q})$. We call this the *Deligne-Pink language* of mixed Shimura varieties. The pair $(P, \mathcal{X}^+)$ is called a *mixed Shimura datum*.

As an example let us look at $\mathfrak{A}_g(N)$. The pair associated with $\mathfrak{A}_g(N)$ is $(P_{2g,\mathrm{a}}, \mathcal{X}_{2g,\mathrm{a}}^+)$, where

- $P_{2g,\mathrm{a}} = V_{2g} \rtimes \mathrm{GSp}_{2g}$, where V_{2g} is the vector group of dimension $2g$ and the action of GSp_{2g} on V_{2g} is the natural representation of GSp_{2g};
- $\mathcal{X}_{2g,\mathrm{a}}^+ = \mathrm{Lie}(\mathfrak{A}_{\mathcal{H}_g^+}/\mathcal{H}_g^+)$, where $\mathfrak{A}_{\mathcal{H}_g^+}$ is the pullback of $\mathfrak{A}_g(N) \to \mathcal{A}_g(N)$ by the uniformization $\mathcal{H}_g^+ \to \mathcal{A}_g(N)$ (and hence a family of abelian varieties with a principal polarization).

By the theory of variation of Hodge structures, we have an exact sequence

$$0 \to \mathcal{F}^0\mathcal{H}^1_{\mathrm{dR}}(\mathfrak{A}_{\mathcal{H}_g^+}/\mathcal{H}_g^+)^\vee \to \mathcal{H}^1_{\mathrm{dR}}(\mathfrak{A}_{\mathcal{H}_g^+}/\mathcal{H}_g^+)^\vee \to \mathrm{Lie}(\mathfrak{A}_{\mathcal{H}_g^+}/\mathcal{H}_g^+) \to 0$$

and hence the composite of

$$\mathbb{R}^{2g} \times \mathcal{H}_g^+ \subset \mathbb{C}^{2g} \times \mathcal{H}_g^+ \simeq \mathcal{H}^1_{\mathrm{dR}}(\mathfrak{A}_{\mathcal{H}_g^+}/\mathcal{H}_g^+)^\vee \to \mathrm{Lie}(\mathfrak{A}_{\mathcal{H}_g^+}/\mathcal{H}_g^+) = \mathcal{X}_{2g,\mathrm{a}}^+$$

is a semialgebraic bijection. Now the action of $P_{2g,\mathrm{a}}(\mathbb{R})^+$ on $\mathcal{X}_{2g,\mathrm{a}}^+$ is given by $(v, h) \cdot (v', x) = (v + hv', hx)$. If we denote by $\mathrm{Sp}_{2g}(1 + N\mathbb{Z}) = \{h \in \mathrm{Sp}_{2g}(\mathbb{Z}) : h \equiv 1 \bmod N\}$, then $\mathfrak{A}_g(N) = \mathrm{Sp}_{2g}(1 + N\mathbb{Z})\backslash\mathcal{X}_{2g,\mathrm{a}}^+$.

The underlying group P_{2g} associated with $\mathfrak{L}_g(N)$ is the following group: Let W_{2g} be the Heisenberg group on the symplectic vector space V_{2g}, namely $W_{2g} = \mathbb{G}_a \times V_{2g}$ as sets and the group law on W_{2g} is $(u_1, v_1)(u_2, v_2) = (u_1 + u_2 + \frac{1}{2}\Psi(v_1, v_2), v_1 + v_2)$ where $\Psi\colon V_{2g} \times V_{2g} \to \mathbb{G}_a$ is the symplectic form on V_{2g}. Then $P_{2g} = W_{2g} \rtimes \mathrm{GSp}_{2g}$ where the action of GSp_{2g} on W_{2g} is given by $h \cdot (u, v) = (\nu(h)u, hv)$ with $\nu\colon \mathrm{GSp}_{2g} \to \mathbb{G}_m$ the multiplier.

The underlying group $P_{2g,\mathrm{b}}$ associated with $\mathfrak{P}_g(N)$ is the unipotent extension of $P_{2g,\mathrm{a}}$ by $V_{2g} \oplus \mathbb{G}_a$ via the action $P_{2g,\mathrm{a}}$ on $V_{2g} \oplus \mathbb{G}_a$ defined by $(v, h)(v', u) = (hv', \nu(h)u + \Psi(v, v'))$.

3.3. **Two-step filtration.** Let S be a mixed Shimura variety associated with $(P, \mathcal{X}^+)$. Let W be the unipotent radical of P and let $G := P/W$ be its reductive part. The general theory of mixed Shimura varieties says that W is an extension of a vector group V by a vector group U, so there is an exact sequence of groups

$$1 \to U \to W \to V \to 1.$$

Moreover the groups U and V can be uniquely determined by $(P, \mathcal{X}^+)$, U is a normal subgroup of P and $W = U \times V$ with the group law $(u_1, v_1)(u_2, v_2) = (u_1 + u_2 + \frac{1}{2}\Psi(v_1, v_2), v_1 + v_2)$ for some alternating form $\Psi\colon V \times V \to U$. We call U the weight -2 part and V the weight -1 part. Then the natural projections $P \to P/U$ and $P \to G = P/W$ induces morphisms of Shimura

[2] U is a unipotent normal subgroup of P uniquely determined by S.

data $(P,\mathcal{X}^+) \to (P/U,\mathcal{X}^+_{P/U})$ and $(P,\mathcal{X}^+) \to (G,\mathcal{X}^+_G)$. We say that $(P/U,\mathcal{X}^+_{P/U})$ is the *Kuga part* of $(P,\mathcal{X}^+)$ and that $(G,\mathcal{X}^+_G)$ is the *pure part* of $(P,\mathcal{X}^+)$. For example the Kuga part of $(P_{2g},\mathcal{X}^+_{2g})$ is $(P_{2g,\mathrm{a}},\mathcal{X}^+_{2g,\mathrm{a}})$ and the pure part of $(P_{2g},\mathcal{X}^+_{2g})$ is $(\mathrm{GSp}_{2g},\mathcal{H}^+_g)$. Another example: the Kuga part of $(P_{2g,\mathrm{a}},\mathcal{X}^+_{2g,\mathrm{a}})$ is itself and the pure part of $(P_{2g,\mathrm{a}},\mathcal{X}^+_{2g,\mathrm{a}})$ is $(\mathrm{GSp}_{2g},\mathcal{H}^+_g)$.

Assume $S=\Gamma\backslash\mathcal{X}^+$. Define $\Gamma_{P/U}$ to be the image of Γ under the projection $P\to P/U$ and Γ_G to be the image of Γ under the projection $P\to G=P/W$. Let $S_{P/U}=\Gamma_{P/U}\backslash\mathcal{X}^+_{P/U}$ and let $S_G=\Gamma_G\backslash\mathcal{X}^+_G$. We say that $S_{P/U}$ is the *Kuga part* of S and that S_G is the *pure part* of S. For example the Kuga part of $\mathfrak{L}_g(N)$ is $\mathfrak{A}_g(N)$ and the pure part of $\mathfrak{L}_g(N)$ is $\mathcal{A}_g(N)$. Another example: the Kuga part of $\mathfrak{A}_g(N)$ is itself and the pure part of $\mathfrak{A}_g(N)$ is $\mathcal{A}_g(N)$.

The discussion above can be generalized in the following sense: For any normal subgroup N of P, there is a connected mixed Shimura datum $(P,\mathcal{X}^+)/N$ induced by the group homomorphism $P\to P/N$, which furthermore gives rise to a connected mixed Shimura variety $S_{P/N}$ together with a Shimura morphism $S\to S_{P/N}$.

3.4. **Geometric bi-algebraicity.** We explain in this subsection the geometric bi-algebraic system associated with mixed Shimura varieties and the characterization of geometric bi-algebraic subvarieties. Let S be a connected mixed Shimura variety associated with $(P,\mathcal{X}^+)$. Denote by $\mathbf{u}\colon \mathcal{X}^+\to S$ the uniformization.

We say that a closed irreducible subvariety Z of S is *geometric bi-algebraic* if one (and hence every) complex analytic irreducible component of $\mathbf{u}^{-1}(Z)$ is algebraizable, i.e. its dimension equals the dimension of its Zariski closure in $\mathcal{X}^\vee$.

To give the characterization of geometric bi-algebraic subvarieties of S, we recall the definition of weakly special subvarieties introduced by Pink [15, Definition 4.1(b)] and refined by the author [6, Proposition 5.4]: A subvariety Z of S is called *weakly special* if there exist a connected mixed Shimura subvariety S_Q of S and a Shimura morphism $[p]\colon S_Q\to S_{Q/N}$ for some normal subgroup N of Q such that $Z=[p]^{-1}(t)$ for some point $t\in S_{Q/N}$. We have the following theorem [6, Corollary 8.3].

Theorem 3.1. *The geometric bi-algebraic subvarieties of S are precisely the weakly special subvarieties of S.*

Weakly special subvarieties of S have good geometric descriptions. In order to give this decomposition let us first recall some facts about connected mixed Shimura varieties.

Let $S=\Gamma\backslash\mathcal{X}^+$ be a connected mixed Shimura variety. Denote by S_G its pure part and by $S_{P/U}$ its Kuga part. Then up to replacing Γ by a subgroup of finite index, we have

1. The morphism $\mathrm{ad}\colon G\to G^{\mathrm{ad}}$ induces a Shimura morphism $(G,\mathcal{X}^+)\to(G^{\mathrm{ad}},\mathcal{X}^+)$, and the decomposition $G^{\mathrm{ad}}=\prod_{i=1}^r G_i$ into simple adjoint groups induces a decomposition of connected Shimura varieties $S_G^{\mathrm{ad}}=\prod_{i=1}^r S_i$. See §3.4.1 for more details.
2. The Shimura morphism $S_{P/U}\to S_G$ is an abelian scheme.
3. The Shimura morphism $[\pi_{P/U}]\colon S\to S_{P/U}$ is a T-torsor, where $T=\Gamma_U\backslash U(\mathbb{C})$.

With this preparation, we have the following theorem.

Theorem 3.2. *Let $S=\Gamma\backslash\mathcal{X}^+$ be a connected mixed Shimura variety and let Z be a Hodge-generic irreducible subvariety. Then Z is weakly special if and only if, up to replacing Γ by a subgroup of finite index, Z is of the following form:*

(1) *Under the decomposition* $\mathrm{ad}\colon S_G \to S_G^{\mathrm{ad}} = \prod_{i=1}^r S_i$, *we have* $Z_G = \mathrm{ad}^{-1}(\prod_{i\in I} S_i \times \{z_G\})$ *for some* $I \subset \{1,\ldots,r\}$ *and some* $z_G \in \prod_{j\notin I} S_j$.

(2) *There exists a decomposition* $S_{P/U} = \mathcal{B}\times_{S_G}\mathcal{C}$ *of abelian schemes over* S_G, *where* $\mathcal{C}|_{Z_G} = C\times Z_G$ *is a trivial abelian scheme, such that* $Z_{P/U} = \mathcal{B}|_{Z_G}\times_{Z_G}(\{c\}\times Z_G)$ *for some* $c\in C$.

(3) *There exists a subtorus* T_N *of* T *such that we have the following commutative diagram*

$$\begin{array}{ccc} S|_{Z_{P/U}} & \xrightarrow{T_N\text{-torsor } [\rho]} & Z_{P/U}\times (T/T_N) \\ & {\scriptstyle T\text{-torsor}}\searrow \quad \swarrow{\scriptstyle T/T_N\text{-torsor}} & \\ & Z_{P/U}, & \end{array}$$

and $Z = [\rho]^{-1}(Z_{P/U}\times\{t\})$ *for some* $t\in T/T_N$.

The rest of the subsection is devoted to prove this theorem. The proof will occupy §3.4.1-3.4.3, with each subsection showing one part of the theorem.

3.4.1. *Pure Shimura varieties.* Let S be a connected pure Shimura variety, namely S equals its pure part. In this case weakly special subvarieties of S are precisely the totally geodesic subvarieties of S. See Moonen [11, 4.3].

Another description is as follows. Let G^{ad} be the adjoint group of G, then the group homomorphism $G \xrightarrow{\mathrm{ad}} G^{\mathrm{ad}}$ induces a Shimura morphism $(G,\mathcal{X}^+) \to (G^{\mathrm{ad}},\mathcal{X}^+)$ which is the identity map on the underlying space. Hence there is a finite map $\mathrm{ad}\colon S\to S^{\mathrm{ad}}$ where $S^{\mathrm{ad}} = \Gamma^{\mathrm{ad}}\backslash\mathcal{X}^+$.

Moreover, the adjoint semi-simple group G^{ad} admits a decomposition $G^{\mathrm{ad}} = G_1\times\ldots\times G_r$ into simple adjoint groups. It induces a decomposition of the underlying space $\mathcal{X}^+ = \mathcal{X}_1^+\times\ldots\times\mathcal{X}_r^+$. Up to replacing Γ by a finite index subgroup we may assume that $\Gamma^{\mathrm{ad}} = \Gamma_1\times\ldots\times\Gamma_r$ under the decomposition $G^{\mathrm{ad}} = G_1\times\ldots\times G_r$. Hence $S^{\mathrm{ad}} = S_1\times\ldots\times S_r$ where $S_i = \Gamma_i\backslash\mathcal{X}_i^+$.

Now let Z be a closed irreducible subvariety of S. Assume that Z is Hodge generic in S, namely Z is not contained in any proper connected Shimura subvariety of S. Then Z is weakly special if and only if $Z = \mathrm{ad}^{-1}(S_I\times\{z\})$ where $I\subset\{1,\ldots,r\}$, $S_I = \prod_{i\in I} S_i$ and $z\in\prod_{j\notin I} S_j$. This establishes part (1) of Theorem 3.2.

3.4.2. *Mixed Shimura varieties of Kuga type.* Let S be a connected mixed Shimura variety of Kuga type, namely S equals its Kuga part. Denote by $[\pi_G]\colon S\to S_G$ the Shimura morphism of S to its pure part. Then $[\pi_G]\colon S\to S_G$ is an abelian scheme. Hence for any subvariety Z_G of S_G, the restriction $S|_{Z_G}\to Z_G$ is again an abelian scheme. In this case the author has proven [7, Proposition 1.1] that a subvariety Z of S is weakly special if and only if the followings hold:

(i) The variety $Z_G = [\pi_G](Z)$ is a weakly special subvariety of S_G;

(ii) There exists a finite cover $Z'_G\to Z_G$ such that Z is the image under the natural projection $S|_{Z_G}\times_{Z_G} Z'_G\to S|_{Z_G}$ of an abelian subscheme of $S|_{Z_G}\times_{Z_G} Z'_G/Z'_G$ by a torsion section and then by a constant section.

Now let us assume that Z is weakly special and not contained in any proper connected mixed Shimura subvariety of S. Then Z_G is not contained in any proper connected Shimura subvariety of S_G. So by §3.4.1, condition (i) above becomes $Z_G = \mathrm{ad}^{-1}(S_{G,I}\times\{z_G\})$ for some $I\subset\{1,\ldots,r\}$ and some $z_G\in\prod_{j\notin I} S_j$.

Let us look at condition (ii) above. Up to replacing Γ by a subgroup of finite index we may assume that Z is the translate of the abelian subscheme $\mathcal{B}_Z$ of $S|_{Z_G}/Z_G$. But Z is Hodge generic in S. So by (the proof of) [7, Proposition 3.5], we see that $\mathcal{B}_Z$ extends to an abelian subscheme $\mathcal{B}$ of S/S_G and there exists an abelian subscheme $\mathcal{C}$ of S/S_G such that the followings hold:

- We have $\mathcal{B}+\mathcal{C}=S$ as abelian schemes over S_G;
- We have that $\mathcal{B}\cap\mathcal{C}$ is a finite group over S_G;
- For any $y_G\in\prod_{j\notin I}S_j$, the restriction of $\mathcal{C}$ to $\mathrm{ad}^{-1}(S_{G,I}\times\{y_G\})$ is isotrivial.

Now up to replacing Γ by a subgroup of finite index we may assume $S=\mathcal{B}\times_{S_G}\mathcal{C}$. Since the algebraic monodromy groups of $\mathrm{ad}^{-1}(S_{G,I}\times\{y_G\})$ stay the same when y_G varies, up to replacing Γ by a subgroup of finite index we may assume that $\mathcal{C}|_{\mathrm{ad}^{-1}(S_{G,I}\times\{y_G\})}$ is trivial for any y_G. In particular taking y_G to be z_G, we have that $\mathcal{C}|_{Z_G}=C\times Z_G$ for some abelian variety C. Then again by (the proof of) [7, Proposition 3.5] we have $Z=\mathcal{B}|_{Z_G}\times_{Z_G}(\{c\}\times Z_G)$ for some $c\in C$. This establishes part (2) of Theorem 3.2.

3.4.3. *General mixed Shimura varieties.* Let Z be a weakly special subvariety of S. Assume that Z is not contained in any proper connected mixed Shimura subvariety of S. We want to characterize Z in geometric terms.

Denote by $[\pi_{P/U}]\colon S\to S_{P/U}$ the projection of S to its Kuga part. Then $[\pi_{P/U}]$ is a T-torsor with T being the algebraic torus $\Gamma_U\backslash U(\mathbb{C})$, where $\Gamma_U=\Gamma\cap U(\mathbb{Q})$. By assumption, $Z_{P/U}=[\pi_{P/U}](Z)$ is not contained in any proper connected mixed Shimura subvariety of $S_{P/U}$. Hence we can apply §3.4.2 to $Z_{P/U}$ and get the geometric description of $Z_{P/U}$. Note that $S|_{Z_{P/U}}\to Z_{P/U}$ is again a T-torsor.

On the other hand let us denote by $\mathbf{u}\colon\mathcal{X}^+\to S$ the uniformization. Then by definition of weakly special subvarieties, there exist a normal subgroup N of P whose reductive part is semisimple and a point $\widetilde{z}\in\mathcal{X}^+$ such that

$$Z=\mathbf{u}(N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{z})$$

with $U_N=U\cap N$. Let $V_N=(W\cap N)/U_N$ and let $\Psi\colon V\times V\to U$ be the alternating form defining the group law on W. Since G_N (the reductive part of N) is semisimple, we have that G_N acts trivially on N. The condition $N\lhd P$ implies that $\Psi(V_N,V)\subset U_N$. Recall that there is a semialgebraic bijection $\mathcal{X}^+\simeq U(\mathbb{C})\times V(\mathbb{R})\times\mathcal{X}_G^+$. See [6, equation (4.1)].

Recall that any subgroup of U is normal in P. In particular $U_N\lhd P$. Hence the quotient $P\to P/U_N$ induces a quotient connected mixed Shimura datum $\rho\colon(P,\mathcal{X}^+)\to(P/U_N,\mathcal{X}^+_{P/U_N})$, and it furthermore induces a quotient connected mixed Shimura variety $[\rho]\colon S\to S_{P/U_N}$, which is a T_N-torsor with T_N bing the algebraic torus $(\Gamma_U\cap U_N(\mathbb{Q}))\backslash U_N(\mathbb{C})$. We have the following compatible torsors

$$\begin{array}{ccc} S & \xrightarrow{T_N\text{-torsor}} & S_{P/U_N} \\ & {\scriptstyle T\text{-torsor}}\searrow \quad \swarrow{\scriptstyle T/T_N\text{-torsor}} & \\ & S_{P/U}. & \end{array}$$

Denote by $\mathbf{u}_{P/U_N}\colon\mathcal{X}^+_{P/U_N}\to S_{P/U_N}$ the uniformization, then we have

$$[\rho](Z)=\mathbf{u}_{P/U_N}\left((N/U_N)(\mathbb{R})^+\rho(\widetilde{z})\right)\quad\text{and}\quad Z=[\rho]^{-1}([\rho]Z).$$

Now for $\overline{\Psi}\colon V \times V \to U/U_N$ the composite of Ψ and the quotient $U \to U/U_N$, we have $\overline{\Psi}(V_N, V) = 0$.

The semialgebraic bijection $\mathcal{X}^+_{P/U_N} \simeq (U/U_N)(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}^+_G$ for $\mathcal{X}^+_{P/U_N}$ is compatible with that of $\mathcal{X}^+$, namely the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X}^+ & \xrightarrow{\;\sim\;} & U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}^+_G \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow \\ \mathcal{X}^+_{P/U_N} & \xrightarrow{\;\sim\;} & (U/U_N)(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}^+_G \end{array} \tag{3.1}$$

where the right arrow is induced by the natural projection $U \to U/U_N$ and the identities on V and $\mathcal{X}^+_G$. Write $\widetilde{z} = (\widetilde{z}_U, \widetilde{z}_V, \widetilde{z}_G)$ under the top bijection, then $\rho(\widetilde{z}) = (\overline{\widetilde{z}}_U, \widetilde{z}_V, \widetilde{z}_G)$ under the bottom bijection where $\overline{\widetilde{z}}_U$ is the image of $\widetilde{z}_U$ under the natural projection $U \to U/U_N$.

Let us consider the T/T_N-torsor $S_{P/U_N}|_{Z_{P/U}} \to Z_{P/U}$. We claim that it is a trivial torsor. To prove this, it suffices to find a global section of $S_{P/U_N}|_{Z_{P/U}} \to Z_{P/U}$. But $\overline{\Psi}(V_N, V) = 0$ and G_N acts trivially on U. So by a simple computation, we have that $(N/U_N)(\mathbb{R})^+\rho(\widetilde{z})$ is identified with $\{\overline{\widetilde{z}}_U\} \times (V_N(\mathbb{R}) + \widetilde{z}_V) \times G_N(\mathbb{R})^+\widetilde{z}_G$ under the bottom bijection in (3.1). Hence $[\rho](Z) = \mathbf{u}_{P/U_N}((N/U_N)(\mathbb{R})^+\rho(\widetilde{z}))$ is a desired global section. Moreover $[\rho](Z)$ is a constant section of $S_{P/U_N}|_{Z_{P/U}} \to Z_{P/U}$. This establishes condition (3) of Theorem 3.2.

Conversely it is not hard to show that any Z which is not contained in any proper connected mixed Shimura variety of S and which satisfies the conditions of Theorem 3.2 is weakly special.

3.5. **Distribution of positive dimensional weakly special subvarieties.** Let S be a connected mixed Shimura variety associated with $(P, \mathcal{X}^+)$. In the proof of the André-Oort conjecture, an important step is the establish the distribution of positive dimensional weakly special subvarieties. This distribution is an application of the so called Ax-Lindemann theorem. Here we sketch the proof of a stronger form of the distribution theorem and relate it to a classical result of Bogomolov.

Recall that a subvariety Z of S is called *weakly special* if there exist a connected mixed Shimura subvariety S_Q of S and a Shimura morphism $[p]\colon S_Q \to S_{Q/N}$ for some normal subgroup N of Q such that $Z = [p]^{-1}(t)$ for some point $t \in S_{Q/N}$. When Z is of this form we say that this weakly special subvariety Z is *defined by* N.

We start with the following lemma.

Lemma 3.3. *Let Y be a closed subvariety of S. Let N be a normal subgroup of P. Then the union of weakly special subvarieties which are defined by N and contained in Y is a closed subvariety.*

Proof. This follows immediately from the geometric description of weakly special subvarieties Theorem 3.2: Note that when N is fixed, then the set I in Theorem 3.2.(1), the abelian subscheme $\mathcal{B}$ in Theorem 3.2.(2), the algebraic torus T_N and the map $[\rho]$ in Theorem 3.2.(3) are all fixed, so we are only varying z_G, c and the point $t \in T/T_N$ so that the resulting weakly special subvariety is contained in Y. This is an algebraic condition, so the set of all possible (z_G, c, t) is Zariski closed, from which the result follows. □

Now we are ready to prove the following theorem, which refines Ullmo's [19, Théorème 4.1] (for pure Shimura varieties) and the author's [6, Theorem 12.2] (for mixed Shimura varieties).

Theorem 3.4. *Let Y be a closed irreducible subvariety of S. Let $\mathfrak{W}(Y)$ be the union of all positive dimensional weakly special subvarieties of S which are contained in Y. Then*

(1) The set $\mathfrak{W}(Y)$ is Zariski closed;
(2) Assume that Y is Hodge generic in S, namely Y is not contained in any proper connected mixed Shimura subvariety of S. Then $\mathfrak{W}(Y) = Y$ if and only if there exists a positive dimensional normal subgroup N of P such that $Y = [\rho]^{-1}([\rho](Y))$ for the diagram

$$\begin{array}{ccc} (P,\mathcal{X}^+) & \xrightarrow{\rho} & (P,\mathcal{X}^+)/N \\ \mathbf{u}\downarrow & & \downarrow\mathbf{u}' \\ S & \xrightarrow{[\rho]} & S' \end{array}$$

Remark 3.5. *Bogomolov* [3, Theorem 1] *proved the following result: Let A be a complex abelian variety and let Y be a closed irreducible subvariety. Then for the Ueno locus $Z = \bigcup_{w+B\subset Y}(w+B)$ where $w \in A$ and B runs over all positive dimensional abelian subvarieties of A, there exist finitely many abelian subvarieties of positive dimension $B_1,\ldots,B_r$ such that $Z = \bigcup_{w+B_i\subset Y,\ w\in A}(w+B_i)$. From this it is easy to deduce that the union of translates of abelian subvarieties contained in Y is Zariski closed and equals Y if and only if the stabilizer of Y is of positive dimension. Our theorem 3.4 is its direct generalization to mixed Shimura varities.*

Proof. We may and do replace S by its smallest connected mixed Shimura subvariety containing Y.

For any subgroup N' of P, denote by $\mathfrak{F}(N',Y)$ the set of all weakly special subvarieties of S defined by N which are contained in Y. By [6, equation (12.4)], we know that

$$\mathfrak{W}(Y) = \bigcup_{N'} \bigcup_{Z\in\mathfrak{F}(N',Y)} Z \tag{3.2}$$

which is a finite union on N''s and each N' is of positive dimension.

On the other hand by [6, Proposition 12.1], we know that $\cup_{Z\in\mathfrak{F}(N',Y)}Z$ is contained the union of finitely many connected mixed Shimura subvarieties of S with the same underlying group of which N' is a normal subgroup. We call these mixed Shimura subvarieties $S'_1,\ldots,S'_r$.

Now to prove that $\mathfrak{W}(Y)$ is a closed subvariety of S, it suffices to prove: For any S'_i above $(i = 1,\ldots,r)$, the union of weakly special subvarieties which are defined by N' and contained in $Y \cap S'_i$ is a closed subvariety. But N' is a normal subgroup of the underlying group associated with S'_i, so this is true by Lemma 3.3. This prove part (1).

Part (2) is then a direct consequence of Part (1) and [6, Theorem 12.2]. □

3.6. **Ax-Schanuel.** Let S be a connected mixed Shimura variety with $\mathbf{u}\colon \mathcal{X}^+ \to S$ the uniformization. The Ax-Schanuel conjecture for S is the analogous statement of Theorem 2.1.

Conjecture 3.6. *Let $\Delta \subset \mathcal{X}^+ \times S$ be the graph of $\mathbf{u}$. Let $\mathfrak{Z} = \mathrm{graph}(\widetilde{Z} \xrightarrow{\mathbf{u}} Z)$ be a complex analytic irreducible subvariety of Δ and let $\mathfrak{B}$ be its Zariski closure in $\mathcal{X}^+ \times S$. Let F be the smallest weakly special subvariety which contains Z. Then*

$$\dim \mathfrak{B} - \dim \mathfrak{Z} \geq \dim F.$$

Theorem 3.7. *Conjecture 3.6 is true in the following cases:*

(1) when $\widetilde{Z}$ is algebraic;
(2) when Z is algebraic;
(3) when $S = Y(1)^N$.

It is proven by the author [5, Theorem 8.5 and 8.4] that Conjecture 3.6 is equivalent to Ax-Lindemann when $\widetilde{Z}$ is algebraic and equivalent to logarithmic Ax when Z is algebraic. So when $\widetilde{Z}$ is algebraic, Conjecture 3.6 is proven for $Y(1)^n$ by Pila [12], for projective pure Shimura varieties by Ullmo-Yafaev [20], for $\mathcal{A}_g$ by Pila-Tsimerman [13], for any pure Shimura variety by Klingler-Ullmo-Yafaev [8], and for any mixed Shimura variety by the author [6, Theorem 1.2]. When Z is algebraic, Conjecture 3.6 is proven by the author [6, Theorem 8.1]. When $S = Y(1)^N$, Conjecture 3.6 is proven by Pila-Tsimerman [14].

Remark 3.8. *When this survey is under review, Mok-Pila-Tsimerman* [10] *proved Conjecture 3.6 for all pure Shimura varieties. A version for variations of pure Hodge structures is also proven by Bakker-Tsimerman* [2] *in the mean time.*

4. Universal vector extension of abelian varieties

We turn to abelian varieties A over $\overline{\mathbb{Q}}$ in this section. We wish to endow a $\overline{\mathbb{Q}}$-structure on $\mathrm{Lie}A_{\mathbb{C}}$ so that torsion points of A are precisely the arithmetic bi-algebraic points. However using the Schneider-Lang and Wüstholz' analytic subgroup theorems, Ullmo [18, Proposition 2.6] proved: any torsion point of A, except the origin, becomes transcendental in $\mathrm{Lie}A_{\mathbb{C}}$. A solution to this problem is proposed by Bost: instead of A we study its universal vector extension $A^\natural$. We briefly recall some basic facts about $A^\natural$.

By a *vector extension* of A, we mean an algebraic group E such that there exist a vector group W and an exact sequence $0 \to W \to E \to A \to 0$. There exists a universal vector extension $A^\natural$ of A such that any vector extension E of A is obtained by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_A & \longrightarrow & A^\natural & \longrightarrow & A & \longrightarrow & 0 \\ & & \vdots\downarrow & \ulcorner & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & W & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

In fact $A^\natural$ is constructed as follows: Let $\Gamma := H_1(A(\mathbb{C}), \mathbb{Z}) \subset H_1(A(\mathbb{C}), \mathbb{C})$ be the period lattice of A. For the Hodge decomposition $H_1(A(\mathbb{C}), \mathbb{C}) = H^{0,-1}(A_{\mathbb{C}}) \oplus H^{-1,0}(A_{\mathbb{C}})$, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{0,-1}(A_{\mathbb{C}}) & \longrightarrow & H_1(A(\mathbb{C}), \mathbb{C}) & \longrightarrow & H^{-1,0}(A_{\mathbb{C}}) \simeq \mathrm{Lie}(A_{\mathbb{C}}) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\mathbf{u}^\natural} & & \downarrow{\mathbf{u}} & & \\ 0 & \longrightarrow & \Omega^1_{A^\vee_{\mathbb{C}}} & \longrightarrow & A^\natural(\mathbb{C}) \simeq \Gamma\backslash H_1(A(\mathbb{C}), \mathbb{C}) & \longrightarrow & A(\mathbb{C}) \simeq \Gamma\backslash\mathrm{Lie}(A_{\mathbb{C}}) & \longrightarrow & 0 \end{array}$$

and the bottom line is nowhere split. Take the $\overline{\mathbb{Q}}$-structure $H_1(A(\mathbb{C}), \mathbb{Z}) \otimes \overline{\mathbb{Q}}$ on $H_1(A(\mathbb{C}), \mathbb{C})$. As an application of Wüstholz' analytic subgroup theorem [21, Theorem 1], Ullmo [18, Théorème 2.10] proved

$$z \in H_1(A(\mathbb{C}), \mathbb{Z}) \otimes \overline{\mathbb{Q}} \text{ such that } \mathbf{u}^\natural(z) \in A^\natural(\overline{\mathbb{Q}}) \Leftrightarrow \mathbf{u}^\natural(z) \text{ is a torsion point of } A^\natural.$$

Thus we get an arithmetic bi-algebraic description for the torsion points of A because the projection $A^\natural \to A$ induces a bijection between their torsion points. This suggests that in view of arithmetic bi-algebraicity and Theorem 2.3, the abelian varieties are not the good objects to study. Instead, one should study their universal vector extensions.

4.1. **Geometric bi-algebraicity.** Before describing the geometric bi-algebraic subvarieties of $A^\natural$, let us point out the following rigidity of universal vector extensions: Let B be an abelian subvariety of A, then by the universal property of $B^\natural$ we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_B & \longrightarrow & B^\natural & \longrightarrow & B & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow = & & \\ 0 & \longrightarrow & \Omega^1_A & \longrightarrow & A^\natural|_B & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where the left vertical map is the natural inclusion induced by $B \subset A$. In other words the inclusion $B \subset A$ induces a unique embedding $B^\natural (\subset A^\natural|_B) \subset A^\natural$. So $A^\natural|_B = B^\natural \times \Omega^1_{A/B}$. Now let $a \in A$, then there exists a unique point $a^\natural \in A^\natural$ over a corresponding to the origin of the $\mathbb{C}$-vector space Ω^1_A. Then the inclusion $a + B \subset A$ induces a unique embedding $a^\natural + B^\natural \subset A^\natural$.

We say that a closed irreducible subvariety $Z^\natural$ of $A^\natural$ is *geometric bi-algebraic* if one (and hence every) complex analytic irreducible component of $(\mathbf{u}^\natural)^{-1}(Z^\natural)$ is algebraic in $H_1(A(\mathbb{C}), \mathbb{C}) \simeq \mathbb{C}^{2g}$.

We have the following characterization of geometric bi-algebraic subvarieties: a closed irreducible subvariety $Z^\natural$ of $A^\natural$ is geometric bi-algebraic if and only if

- the image of $Z^\natural$ under the projection $A^\natural \to A$ is the translate of an abelian subvariety, which we denote by $a + B$;
- $Z^\natural = a^\natural + B^\natural + W$, where W is a closed irreducible subvariety of $\Omega^1_{A/B}$.

This can be proven similarly as the case of algebraic tori using monodromy. We refer to [5, Theorem 5.8].

4.2. **Ax-Schanuel for $A^\natural$.** Let $\mathbf{u}^\natural \colon \mathbb{C}^{2g} \to A^\natural$ be the uniformization. Let $\Delta^\natural \subset \mathbb{C}^{2g} \times A^\natural$ be the graph of $\mathbf{u}^\natural$. Let $\mathfrak{Z}^\natural = \mathrm{graph}(\widetilde{Z}^\natural \to Z^\natural)$ be a irreducible complex subspace of $\Delta^\natural$, and let $\mathfrak{B}^\natural$ be its Zariski closure in $\mathbb{C}^{2g} \times A^\natural$. Denote by $\widetilde{X}^\natural = (\widetilde{Z}^\natural)^{\mathrm{Zar}}$ and by $Y^\natural = (Z^\natural)^{\mathrm{Zar}}$.

In order to formulate the Ax-Schanuel theorem for $A^\natural$, let us go back to the discussion below Theorem 2.1. We have seen that the Ax-Schanuel statement contains two aspects: description of $\mathfrak{B}^\natural$, and the intersection behavior of $\widetilde{X}^\natural$ and $(\mathbf{u}^\natural)^{-1}(Y^\natural)$. The naive guess would be $\mathfrak{B}^\natural = \widetilde{X}^\natural \times Y^\natural$ and that $\widetilde{X}^\natural$ intersects properly with $(\mathbf{u}^\natural)^{-1}(Y^\natural)$. However the description of $\mathfrak{B}^\natural$ cannot be so neat: say $Z^\natural$ is in a fiber of the projection $A^\natural \to A$ (which is isomorphic to $\mathbb{C}^g$), assume furthermore that $Z^\natural$ is algebraic, then $\mathfrak{Z}^\natural$ is an algebraic subvariety of $\mathbb{C}^{2g} \times A^\natural$. But then $\mathfrak{B}^\natural = \mathfrak{Z}^\natural \neq \widetilde{Z}^\natural \times Z^\natural$. However this is the only obstacle for the analogous statement of Theorem 2.1 to hold for $A^\natural$. We have (See [5, Theorem 9.1])

Theorem 4.1. *Under the notation above. Let $F^\natural$ be the smallest geometric bi-algebraic subvariety of $A^\natural$ containing $Z^\natural$, and let $\widetilde{F}^\natural$ be the irreducible component of $(\mathbf{u}^\natural)^{-1}(F^\natural)$ containing $\widetilde{Z}^\natural$. Then*

(1) $\dim \widetilde{X}^\natural + \dim Y^\natural - \dim \widetilde{Z}^\natural \geq \dim F^\natural$.

(2) Write $F^\natural = a^\natural + B^\natural + W$ as in the last subsection, and $\widetilde{F}^\natural = \widetilde{a}^\natural + \widetilde{B}^\natural + \widetilde{W}$ correspondingly. Denote by $pr^{\mathrm{ws}}\colon \widetilde{F}^\natural \to \widetilde{a}^\natural + \widetilde{B}^\natural$ and by $[pr]^{\mathrm{ws}}\colon F^\natural \to a^\natural + B^\natural$ the natural projections. Then

$$\dim(pr^{\mathrm{ws}}, [pr]^{\mathrm{ws}})(\boldsymbol{\mathcal{B}}^\natural) - \dim(pr^{\mathrm{ws}}, [pr]^{\mathrm{ws}})(\boldsymbol{\mathcal{Z}}^\natural) \geq \dim[pr]^{\mathrm{ws}}(F^\natural).$$

5. Mixed Shimura varieties plus vector extensions

In this section we turn to more general context, unifying algebraic tori and universal vector extensions of abelian varieties. Moreover we will look into families. The idea is similar to that of mixed Shimura varieties. However mixed Shimura varieties do not allow any vector extension. So we need to enlarge the ambient spaces.

Let $\mathfrak{A}_g$ be the universal family over $\mathcal{A}_g$, where $\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties with level-4-structure, and let $\mathfrak{A}_g^\natural$ be the universal vector extension of the abelian scheme $\mathfrak{A}_g/\mathcal{A}_g$.

Take the universal vectorial bi-extension $\mathfrak{P}_g^\natural$ studied by Coleman [4], which is defined as follows in geometric terms: Let $\mathfrak{P}_g$ be the universal Poincaré biextension as in §3.1 and let $\mathfrak{A}_g^\natural$ be the universal vector extension of the abelian scheme $\mathfrak{A}_g/\mathcal{A}_g$. Then $\mathfrak{P}_g^\natural$ is the pullback of $\mathfrak{P}_g$ by $\mathfrak{A}_g^\natural \times (\mathfrak{A}_g^\vee)^\natural \to \mathfrak{A}_g \times \mathfrak{A}_g^\vee$.

The geometric bi-algebraic system we shall study is associated with *enlarged mixed Shimura varieties*. See [5]. We do not go into details of the definition, but look at examples instead. A first example is $\mathfrak{A}_g^\natural$. And the typical example of enlarged mixed Shimura variety to keep in mind is $(\mathfrak{P}_g^\natural)^{[n]}$, i.e. the n-fiber product of $\mathfrak{P}_g^\natural$ over $\mathfrak{A}_g^\natural \times (\mathfrak{A}_g^\vee)^\natural$.

5.1. Geometric bi-algebraicity.

Notation: for any abelian scheme $\mathfrak{A} \to B$ with unit section ε, denote by $\omega_{\mathfrak{A}/B} := \varepsilon^*\Omega^1_{\mathfrak{A}/B}$.

Let $S^\natural$ be an enlarged mixed Shimura variety, whose uniformization is $\mathbf{u}^\natural\colon \mathcal{X}^\natural \to S^\natural$. There exist an algebraic variety $\mathcal{X}^{\natural,\vee}$ over $\mathbb{C}$ and a natural inclusion $\mathcal{X}^\natural \hookrightarrow \mathcal{X}^{\natural,\vee}$ such that $\mathcal{X}^\natural$ is open semialgebraic in $\mathcal{X}^{\natural,\vee}$.

We say that an irreducible subvariety $Y^\natural$ of $S^\natural$ is *geometric bi-algebraic* if one (and hence every) complex analytic irreducible component of $(\mathbf{u}^\natural)^{-1}(Y^\natural)$ is algebraizable, i.e. its dimension equals the dimension of its Zariski closure in $\mathcal{X}^{\natural,\vee}$.

As we have already seen for universal vector extensions of abelian varieties, the characterization of geometric bi-algebraic subvarieties cannot be as neat as for mixed Shimura varieties. The problem arises in the vector extension part. However we show that this is the only problem: for the exact sequence $0 \to \omega_{\mathfrak{A}_g^\vee/\mathcal{A}_g} \to \mathfrak{A}_g^\natural \to \mathfrak{A}_g \to 0$ of groups over $\mathcal{A}_g$, the "non-linear" part of any geometric bi-algebraic subvariety $Y^\natural$ of $\mathfrak{A}_g^\natural$ can only lie in the TRIVIAL subbundle of the vector bundle part, i.e. of $\omega_{\mathfrak{A}_g^\vee/\mathcal{A}_g}|_{Y_G}$ where Y_G is the image of $Y^\natural$ in $\mathcal{A}_g$. More precisely we have (see [5, Theorem 1.6])

Theorem 5.1 (Characterization of geometrically bi-algebraic subvarieties of enlarged mixed Shimura varieties)**.** *An irreducible subvariety $Y^\natural$ of $S^\natural$ is geometric bi-algebraic if and only if it is quasi-linear.*

We define quasi-linear subvarieties of $S^\natural$. There is a commutative diagram for any connected enlarged mixed Shimura variety (on the right for $(\mathfrak{P}_g^\natural)^{[n]}$):

$$
\begin{array}{ccccc}
S^\natural & \xrightarrow{[\pi^\natural]} & S & & \\
{\scriptstyle [\pi^\natural_{P/U}]}\downarrow & & \downarrow{\scriptstyle [\pi_{P/U}]} & \searrow{\scriptstyle [\pi]} & \\
S^\natural_{P/U} & \xrightarrow[{[\pi^\natural_{P/U}]}]{} & S_{P/U} & \xrightarrow[{[\pi_{P/U}]}]{} & S_G
\end{array}
\qquad
\begin{array}{ccccc}
(\mathfrak{P}_g^\natural)^{[n]} & \longrightarrow & \mathfrak{P}_g^{[n]} & & \\
\downarrow & & \downarrow & \searrow & \\
\mathfrak{A}_g^\natural \times (\mathfrak{A}_g^\vee)^\natural & \longrightarrow & \mathfrak{A}_g \times \mathfrak{A}_g^\vee & \longrightarrow & \mathcal{A}_g
\end{array}
\tag{5.1}
$$

where

- all maps in the diagram are projections defined in some natural way;
- S is a connected mixed Shimura variety, and $S_{P/U}$ is the quotient of S by its weight -2 part (and hence $S_{P/U}$ is an abelian scheme over S_G);
- $S^\natural_{P/U}$ is the universal vector extension of the abelian scheme $S_{P/U} \to S_G$.

To define quasi-linear subvarieties of $S^\natural$ we need some preparation. Let Y_G be a subvariety of S_G. Let $Y_{P/U} \subset S_{P/U}|_{Y_G} := [\pi_{P/U}]^{-1}(Y_G)$ be the translate of an abelian subscheme of $S_{P/U}|_{Y_G} \to Y_G$ be a torsion section and then by a constant section of its isotrivial part. Denote by $Y^{\mathrm{univ}}_{P/U}$ the universal vector extension of the abelian scheme $Y_{P/U} \to Y_G$. Then by the rigidity of the universal vector extension there is a unique embedding $Y^{\mathrm{univ}}_{P/U} \subset S^\natural_{P/U}|_{Y_{P/U}} := [\pi^\natural_{P/U}]^{-1}(Y_{P/U})$ compatible with the embedding $Y_{P/U} \subset S_{P/U}|_{Y_G}$ mentioned above. More concretely there is a unique embedding i (left vertical arrow) of vector groups over Y_G inducing the following push-out:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_{Y^\vee_{P/U}/Y_G} & \longrightarrow & Y^{\mathrm{univ}}_{P/U} & \longrightarrow & Y_{P/U} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle i} & & \vdots & & \big\downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & \omega_{[\pi_G]^{-1}(Y_G)^\vee/Y_G} & \longrightarrow & S^\natural_{P/U}|_{Y_{P/U}} & \longrightarrow & Y_{P/U} & \longrightarrow & 0
\end{array}
$$

Then we obtain another vector extension of $Y_{P/U}$

$$
0 \to \frac{\omega_{[\pi_G]^{-1}(Y_G)^\vee/Y_G}}{\omega_{Y^\vee_{P/U}/Y_G}} \to \frac{S^\natural_{P/U}|_{Y_{P/U}}}{Y^{\mathrm{univ}}_{P/U}} \to Y_{P/U} \to 0,
$$

with the unique map $Y^{\mathrm{univ}}_{P/U} \to \frac{S^\natural_{P/U}|_{Y_{P/U}}}{Y^{\mathrm{univ}}_{P/U}}$ being 0. Hence $\frac{S^\natural_{P/U}|_{Y_{P/U}}}{Y^{\mathrm{univ}}_{P/U}} \simeq Y_{P/U} \times_{Y_G} \frac{\omega_{[\pi_G]^{-1}(Y_G)^\vee/Y_G}}{\omega_{Y^\vee_{P/U}/Y_G}}$. Thus

$$
S^\natural_{P/U}|_{Y_{P/U}} = Y^{\mathrm{univ}}_{P/U} \times_{Y_G} \frac{\omega_{[\pi_G]^{-1}(Y_G)^\vee/Y_G}}{\omega_{Y^\vee_{P/U}/Y_G}}
$$

Denote by $\mathbf{V}^{(0)}|_{Y_G}$ the largest trivial subbundle of $\frac{\omega_{[\pi_G]^{-1}(Y_G)^\vee/Y_G}}{\omega_{Y^\vee_{P/U}/Y_G}}$. For simplicity we use ω^{extr} to denote $\frac{\omega_{[\pi_G]^{-1}(Y_G)^\vee/Y_G}}{\omega_{Y^\vee_{P/U}/Y_G}}$.

If furthermore Y_G is a weakly special subvariety of S_G, then denote by H the connected algebraic monodromy group of Y_G. Then the pullback of ω^{extr} under the universal cover $\widetilde{Y}_G \to Y_G$, which we call $\widetilde{\omega}^{\mathrm{extr}}$, is an $H(\mathbb{R})$-bundle. We say that a subvariety $\mathbf{K}^\natural$ of ω^{extr} is an ***automorphic subvariety*** if it is the image of $H(\mathbb{R})^+\widetilde{K}^\natural$ under the natural projection $\widetilde{\omega}^{\mathrm{extr}} \to \omega^{\mathrm{extr}}$ for some $\widetilde{K}^\natural$ in a fiber of $\widetilde{\omega}^{\mathrm{extr}} \to \widetilde{Y}_G$. Note that $\widetilde{K}^\natural$ can be chosen to be invariant under a maximal compact subgroup of $H(\mathbb{R})^+$.

Now we are ready to define

Definition 5.2. *An irreducible subvariety $Y^\natural$ of $S^\natural$ is called **quasi-linear** if the followings hold: under the following notations for $Y^\natural$ compatible with* (5.1)

$$\begin{array}{ccccc} Y^\natural & \longmapsto & Y & & \\ \downarrow & & \downarrow & \searrow & \\ Y^\natural_{P/U} & \longmapsto & Y_{P/U} & \longmapsto & Y_G \end{array}$$

(1) Y is a weakly special subvariety of S. In particular $Y_{P/U}$ is the translate of an abelian subscheme of $S_{P/U}|_{Y_G} \to Y_G$ by a torsion section and then by a constant section of its isotrivial part.

(2) Under the notations above the theorem, $Y^\natural_{P/U} = Y^{\mathrm{univ}}_{P/U} \times_{Y_G} (L^\natural \times Y_G) \times_{Y_G} \mathbf{K}^\natural$, where $L^\natural$ is an irreducible algebraic subvariety of any fiber of $\mathbf{V}^{(0)}|_{Y_G} \to Y_G$, and $\mathbf{K}^\natural$ is an irreducible automorphic subvariety of the bundle $\frac{\omega_{[\pi_G]^{-1}(Y_G)^\vee/Y_G}}{\omega_{Y^\vee_{P/U}/Y_G}}$ whose intersection with $\mathbf{V}^{(0)}|_{Y_G}$ is contained in the zero section.

(3) $Y^\natural = Y \times_{Y_{P/U}} Y^\natural_{P/U}$ for the cartesian diagram in (5.1).

Now we are ready to explain the terminology in part (2) of Conjecture 5.3. Apply Theorem 5.1 to the bi-algebraic subvariety $F^\natural$ of $S^\natural$ (hence we change every letter "Y" by "F"), then we define

$$(F^\natural)^{\mathrm{ws}} := F \times_{F_{P/U}} F^{\mathrm{univ}}_{P/U}$$

and $[pr]^{\mathrm{ws}}_{F^\natural}$ the natural projection $F^\natural \to (F^\natural)^{\mathrm{ws}}$. Let $pr^{\mathrm{ws}}_{\widetilde{F}^\natural}$ be the natural projection from $\widetilde{F}^\natural$ to $(\widetilde{F}^\natural)^{\mathrm{ws}}$, the uniformization of $(F^\natural)^{\mathrm{ws}}$. Then we define

$$\mathbf{pr}^{\mathrm{ws}}_{F^\natural} := (pr^{\mathrm{ws}}_{\widetilde{F}^\natural}, [pr]^{\mathrm{ws}}_{F^\natural}) \colon \widetilde{F}^\natural \times F^\natural \to (\widetilde{F}^\natural)^{\mathrm{ws}} \times (F^\natural)^{\mathrm{ws}}.$$

5.2. **Ax-Schanuel conjecture.**

Conjecture 5.3 (Ax-Schanuel)**.** *Let $\Delta^\natural \subset \mathcal{X}^{\natural+} \times S^\natural$ be the graph of $\mathbf{u}^\natural$. Let $\mathfrak{Z}^\natural = \mathrm{graph}(\widetilde{Z}^\natural \xrightarrow{\mathbf{u}^\natural} Z^\natural)$ be a complex analytic irreducible subvariety of $\Delta^\natural$. Let $F^\natural$ be the smallest quasi-linear subvariety of $S^\natural$ which contains $Z^\natural$. Let $\widetilde{F}^\natural$ be the complex analytic irreducible component of $(\mathbf{u}^\natural)^{-1}(F^\natural)$ which contains $\widetilde{Z}^\natural$. Then*

(1) $\dim(\widetilde{Z}^\natural)^{\mathrm{Zar}} + \dim(Z^\natural)^{\mathrm{Zar}} - \dim \widetilde{Z}^\natural \geq \dim F^\natural$.

(2) Let $\mathfrak{B}^\natural := (\mathfrak{Z}^\natural)^{\mathrm{Zar}} \subset \mathcal{X}^{\natural+} \times S^\natural$. Then $\dim \mathbf{pr}^{\mathrm{lin}}_{F^\natural}(\mathfrak{B}^\natural) - \dim \mathbf{pr}^{\mathrm{lin}}_{F^\natural}(\mathfrak{Z}^\natural) \geq \dim(F^\natural)^{\mathrm{lin}}$.

Let us explain the terminology in part (2) of Conjecture 5.3. Apply Theorem 5.1 to the bi-algebraic subvariety $F^\natural$ of $S^\natural$ (hence we change every letter "Y" by "F"), then we define

$$(F^\natural)^{\mathrm{lin}} := F \times_{F_{P/U}} (F^{\mathrm{univ}}_{P/U} \times_{F_G} \mathbf{V}^\dagger|_{F_G})$$

and $[pr]^{\rm lin}_{F^\natural}$ the natural projection $F^\natural \to (F^\natural)^{\rm lin}$. Let $pr^{\rm lin}_{\widetilde{F}^\natural}$ be the natural projection from $\widetilde{F}^\natural$ to $(\widetilde{F}^\natural)^{\rm lin}$, the uniformization of $(F^\natural)^{\rm lin}$. Then we define

$$\mathbf{pr}^{\rm lin}_{F^\natural} := (pr^{\rm lin}_{\widetilde{F}^\natural}, [pr]^{\rm lin}_{F^\natural})\colon \widetilde{F}^\natural \times F^\natural \to (\widetilde{F}^\natural)^{\rm lin} \times (F^\natural)^{\rm lin}.$$

This conjecture is open in general. However we can prove several interesting cases, some of which having good applications (e.g. to the André-Oort conjecture). In particular the conjecture for the unipotent part is completely solved.

Theorem 5.4. *Conjecture 5.3 holds in the following cases:*

(1) When $\widetilde{Z}^\natural$ is algebraic.
(2) When $Z^\natural$ is algebraic.

See [5, Theorem 1.5].

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