An Introduction to Shimura varieties and their compactifications

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Contents

Chapter 1

Preparation on Hodge theory

1.1 Hodge structure and polarizations

Take $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Let $n \in \mathbb{Z}$.

1.1.1 Hodge decomposition and Hodge filtration

Definition 1.1.1. *An R-**Hodge structure of weight** n is a torsion-free R-module of finite type V endowed with a bigrading (called the **Hodge decomposition**)*

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \qquad \text{with} \qquad \overline{V^{q,p}} = V^{p,q}.$$

*For a subset $A \subseteq \mathbb{Z} \oplus \mathbb{Z}$, we say that V has **Hodge type** A if $V^{p,q} = 0$ for all $(p,q) \notin A$.*

*An R-linear map $\varphi \colon V \to W$ between two Hodge structures of weight n is said to be a **morphism of Hodge structures** if $\varphi(V^{p,q}) \subseteq W^{p,q}$ for all p, q.*

We thus have the *category of R-Hodge structures of weight n*, denoted by HS_R^n. One can define direct sums in HS_R^n, and hence makes it into an *abelian category.*

We can also consider the *category of R-Hodge structures*, denoted by HS_R. The objects are R-Hodge structures of any weight. Then we can define tensor products, duals, and internal homs in HS_R as follows. Let $V \in \mathrm{HS}_R^n$ and $W \in \mathrm{HS}_R^m$,

(i) the bigrading on $V \otimes W \in \mathrm{HS}_R^{n+m}$ is given by $(V \otimes W)^{p,q} = \bigoplus_{r+r'=p,\ s+s'=q} V^{r,s} \otimes W^{r',s'}$;

(ii) the bigrading on $V^\vee \in \mathrm{HS}_R^{-n}$ is given by $(V^\vee)^{p,q} = (V^{-p,-q})^\vee$;

(iii) $\mathrm{Hom}(V, W) := V^\vee \otimes W$.

Here are some examples.

Example 1.1.2 (Tate twist)**.** *For each $m \in \mathbb{Z}$, set $R(m) \in \mathrm{HS}_R^{-2m}$ to be*

$$R(m) = (2\pi i)^m R, \qquad R(m)_{\mathbb{C}} = R^{-m,-m}.$$

Then $R(0) = R$, $R(m) = R(1)^{\otimes m}$ with $R(-1) = R(1)^\vee$.

Example 1.1.3 (cohomology from geometry)**.** *Let X be a connected smooth projective variety defined over $\mathbb{C}$. For each $n \geq 0$, the Betti cohomology $H^n(X, \mathbb{Z})/\mathrm{tor}$ admits a $\mathbb{Z}$-Hodge structure of weight n via the Betti–de Rham comparison $H^n(X, \mathbb{C}) \simeq H^n_{\mathrm{dR}}(X)$ and the decomposition of $H^n_{\mathrm{dR}}(X)$ into the direct sum of subspaces arising from (p,q)-forms.*

Example 1.1.4 (Complex tori). *We explain in this example the following equivalence of categories:*

$$\{complex\ tori\} \xrightarrow{\sim} \{\mathbb{Z} - Hodge\ structures\ of\ type\ (-1,0)+(0,-1)\}\,.$$

The direction $\to$ *is by sending* $T \mapsto H_1(T,\mathbb{Z})$*. Let* T *be a complex torus of dimension* $g \geq 1$*. Set*

$$V_{\mathbb{Z}} := H_1(T,\mathbb{Z}).$$

As a real manifold, we then have $T \simeq V_{\mathbb{R}}/V_{\mathbb{Z}}$*. Moreover, as a real space* $V_{\mathbb{R}}$ *is isomorphic to* $\mathrm{Lie}(T_{\mathbb{R}})$*, the Lie algebra with* $T_{\mathbb{R}}$ *seen as a real Lie group. The complex structure on* T *gives an action of* J *on* $V_{\mathbb{R}}$*, with*

$$J := \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix},$$

and hence the desired Hodge decomposition

$$V_{\mathbb{C}} = V^{-1,0} \bigoplus V^{0,-1}$$

with $V^{-1,0}$ *the eigenspace of* $\sqrt{-1}$ *and* $V^{0,-1}$ *the eigenspace of* $-\sqrt{-1}$*.*

The direction $\leftarrow$ *is given as follows. Let* $V_{\mathbb{Z}}$ *be a* $\mathbb{Z}$*-Hodge structure of type* $(-1,0)+(0,-1)$*. Then* $V_{\mathbb{C}}/V^{0,-1}$ *is a complex space of dimension* $\frac{1}{2}\mathrm{rank}V_{\mathbb{Z}}$*. Thus we obtain the desired complex torus*

$$V_{\mathbb{Z}}\backslash V_{\mathbb{C}}/V^{0,-1} \simeq V_{\mathbb{Z}}\backslash V^{-1,0}.$$

Notice that we have implicitly an isomorphism of real vector spaces $V_{\mathbb{R}} \simeq V_{\mathbb{C}}/V^{0,-1} = V^{-1,0}$ *given as the composite* $V_{\mathbb{R}} \subseteq V_{\mathbb{C}} \to V_{\mathbb{C}}/V^{0,-1} = V^{-1,0}$*.*

An alternative way to see the Hodge decomposition is the following Hodge filtration. It is of particular importance when we consider families of Hodge structures.

Definition 1.1.5. *Let* V *be an* R*-Hodge structure of weight* n*. The* **Hodge filtration** *is the decreasing chain* $\cdots \supseteq F^pV_{\mathbb{C}} \supseteq F^{p+1}V_{\mathbb{C}} \supseteq \cdots$ *with*

$$F^pV_{\mathbb{C}} := \bigoplus\nolimits_{r\geq p} V^{r,s}. \tag{1.1.1}$$

Conversely, the Hodge decomposition can be recovered by the Hodge filtration via

$$V^{p,q} = F^pV_{\mathbb{C}} \cap \overline{F^qV_{\mathbb{C}}}. \tag{1.1.2}$$

1.1.2 Polarizations

Let V be an R-Hodge structure of weight n.

The *Weil operator* $C \in \mathrm{End}(V_{\mathbb{C}})$ is defined as follows: It acts on $V^{p,q}$ by multiplication by $\sqrt{-1}^{q-p}$. We have $Cx = \overline{Cx}$ for all $x \in V_{\mathbb{R}}$.[1] So $C \in \mathrm{End}(V_{\mathbb{R}})$. A more elegant way to define the Weil operator will be given above Proposition 1.2.5, in terms of Deligne torus.

Definition 1.1.6. *A* **polarization** *on* V *is a morphism of Hodge structures*

$$\psi\colon V \otimes V \to R(-n)$$

such that the bi-linear map

$$V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}, \qquad (x,y) \mapsto \psi_C(x,y) := (2\pi\sqrt{-1})^n\psi(x,Cy) \tag{1.1.3}$$

is symmetric and positive definite.

[1]Indeed, for $x = \sum_{p,q} x_{p,q} \in V_{\mathbb{R}}$, we have $\overline{x_{p,q}} = x_{q,p}$ because $\overline{V^{p,q}} = V^{q,p}$. So $\overline{Cx} = \sum_{p,q} \overline{\sqrt{-1}^{q-p}}\,\overline{x_{p,q}} = \sum_{p,q} \sqrt{-1}^{p-q} x_{q,p} = \sum_{p,q} \sqrt{-1}^{p-q} x_{q,p} = Cx$, and hence $Cx \in V_{\mathbb{R}}$.

The Hermitian pairing associated with the bi-linear map (1.1.3) is $(x, y) \mapsto \psi_C(x, \overline{y})$.

Lemma 1.1.7. *Let $V \in \mathrm{HS}^n_R$, and let ψ be a polarization. Then*

(i) ψ is $(-1)^n$-symmetric, i.e. is alternating if n is odd and is symmetric if n is even.

(ii) the decomposition $V_{\mathbb{C}} = \bigoplus V^{p,q}$ is orthogonal with respect to the Hermitian pairing associated with (1.1.3).

Proof. We start by proving (ii). Take $x \in V^{p,q}$ and $y \in V^{r,s}$. Then

$$(2\pi\sqrt{-1})^{-n}\psi_C(x, \overline{y}) = \psi(x, C\overline{y}) = \psi(x, \sqrt{-1}^{r-s}\overline{y}) = \sqrt{-1}^{r-s}\psi(x, \overline{y})$$

Now $(x, \overline{y}) \in V^{p,q} \times V^{s,r} \subseteq (V \times V)^{p+s,q+r}$. So $\psi(x, \overline{y}) \in R(-n)^{p+s,q+r}$ since ψ is a morphism of Hodge structures. Assume $\psi(x, \overline{y}) \neq 0$. Then $p + s = q + r = n$. But $p + q = r + s = n$. So $p = r$ and $q = s$. Thus $\psi_C(V^{p,q}, \overline{V^{r,s}}) = 0$ unless $p = r$ and $q = s$. This establishes (ii).

Now we turn to (i). The proof will be much easier and more elegant if we apply Proposition 1.2.5. Here we give a direct computation without using this proposition.

For each $x, y \in V_{\mathbb{R}}$, write $x = \sum_{p,q} x_{p,q}$ and $y = \sum_{p,q} y_{p,q}$ under $V_{\mathbb{C}} = \bigoplus V^{p,q}$. Then $(y_{p,q}, x_{r,s}) \subseteq (V \times V)^{p+r,q+s}$, and hence $\psi(y_{p,q}, x_{r,s}) \in R(-n)^{p+r,q+s}$ is 0 unless $p + r = q + s = n$. So

$$\psi(y, x) = \sum_{p,q} \psi(y_{p,q}, x_{p,q}).$$

On the other hand, $x_{p,q} = \overline{x_{q,p}}$ and $y_{p,q} = \overline{y_{q,p}}$ since $\overline{V^{p,q}} = V^{q,p}$. So

$$\begin{aligned}
\psi_C(Cy, x) &= \psi_C(\sum_{p,q} \sqrt{-1}^{q-p} y_{p,q}, \sum_{p,q} x_{p,q}) \\
&= \psi_C(\sum_{p,q} \sqrt{-1}^{q-p} y_{p,q}, \sum_{p,q} \overline{x_{p,q}}) \\
&= \sum_{p,q} \psi_C(\sqrt{-1}^{q-p} y_{p,q}, \overline{x_{p,q}}) \\
&= \sum_{p,q} \psi_C(\sqrt{-1}^{q-p} y_{p,q}, x_{q,p}) \\
&= (2\pi\sqrt{-1})^n \sum_{p,q} \psi(\sqrt{-1}^{q-p} y_{p,q}, Cx_{q,p}) \\
&= (2\pi\sqrt{-1})^n \sum_{p,q} \psi(\sqrt{-1}^{q-p} y_{p,q}, \sqrt{-1}^{p-q} x_{q,p}) \\
&= (2\pi\sqrt{-1})^n \sum_{p,q} \psi(y_{p,q}, x_{q,p}).
\end{aligned}$$

Therefore

$$\psi(y, x) = (2\pi\sqrt{-1})^{-n}\psi_C(Cy, x).$$

Since ψ_C is symmetric, we furthermore have

$$\psi(y, x) = (2\pi\sqrt{-1})^{-n}\psi_C(x, Cy) = \psi(x, C^2 y).$$

Notice that C^2 acts on $V^{p,q}$ by multiplication by $(-1)^{q-p} = (-1)^{q+p} = (-1)^n$ for all p, q. Thus C^2 acts on V as multiplication by $(-1)^n$. So we have

$$\psi(y, x) = (-1)^n\psi(x, y).$$

This establishes (i). $\square$

Example 1.1.8 (Complex abelian varieties)**.** *We continue with Example 1.1.4 and prove*

$$\{\textit{complex abelian varieties}\} \xrightarrow{\sim} \{\textit{polarizable } \mathbb{Z}\textit{-Hodge structures of type } (-1,0)+(0,-1)\}\,.$$

Let T be a complex torus which corresponds to $V_{\mathbb{Z}} = H_1(T,\mathbb{Z})$. Then $T \simeq V_{\mathbb{R}}/V_{\mathbb{Z}}$ as real manifolds. Thus $\bigwedge^2 V_{\mathbb{Z}}^{\vee} \simeq \bigwedge^2 H^1(T,\mathbb{Z}) = H^2(T,\mathbb{Z})$. Therefore the set of alternating pairings

$$\psi\colon V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}(1)$$

is in bijection with $H^2(T,\mathbb{Z}(1))$.

The short exact sequence of sheaves $0 \to \underline{\mathbb{Z}}(1) \to \mathcal{O}_T \xrightarrow{\exp} \mathcal{O}_T^* \to 0$ induces

$$\mathrm{Pic}(T) = H^1(T,\mathcal{O}_T^*) \xrightarrow{c_1} H^2(T,\mathbb{Z}(1)) \to H^2(T,\mathcal{O}_T).$$

Assume T is an abelian variety. Then there exists an ample line bundle L on T. The Ampell–Hubert data for L then gives an alternating pairing $\psi \in H^2(T,\mathbb{Z}(1))$ such that the Hermitian pairing $(x,y) \mapsto \psi(x,\sqrt{-1}\overline{y})$ is the c_1 of L for a suitable Hermitian metric on L. But $V_{\mathbb{Z}}$ has Hodge type $(-1,0)+(0,-1)$ and the complex structure on $V_{\mathbb{R}}/V_{\mathbb{Z}}$ is by identifying $V_{\mathbb{R}} \simeq V^{-1,0}$. So $c_1(L)$ is precisely ψ_C. The ampleness of L implies that ψ_C is positive-definite. Thus ψ is a polarization on $V_{\mathbb{Z}}$.

Conversely assume ψ is a polarization on $V_{\mathbb{Z}}$. Then ψ can be seen as an element in $H^2(T,\mathbb{Z}(1))$, and ψ_C equals $(x,y) \mapsto \psi(x,\sqrt{-1}\overline{y})$ as above. So the Ampell–Hubert Theorem gives a line bundle L on T such that $c_1(L) = \psi_C$. The positivity of ψ_C thus implies the ampleness of L by Kodaira embedding. So T is an abelian variety.

Example 1.1.9 (Primitive cohomology and Lefschetz)**.** *We continue with Example 1.1.3. Assume $d = \dim X$. Let ω be a Kähler form on X^{an}, which is a closed $(1,1)$-form. It induces a homomorphism $L\colon H^n(X,\mathbb{Q}) \to H^{n+2}(X,\mathbb{Q})$, $[\alpha] \mapsto [\omega \wedge \alpha]$; here we are using $H^n(X,\mathbb{Q}) \subseteq H^n(X,\mathbb{C}) \simeq H^n_{\mathrm{dR}}(X)$. The Hard Lefschetz Theorem says that $L^r\colon H^{d-r}(X,\mathbb{Q}) \xrightarrow{\sim} H^{d+r}(X,\mathbb{Q})$ for all $r \geq 0$. Now let $r = d-n$. Set $H^n_{\mathrm{prim}}(X,\mathbb{Q})$ to be the kernel of $L^{r+1}\colon H^n(X,\mathbb{Q}) \to H^{2d-n+2}(X,\mathbb{Q})$. We have a morphism of Hodge structures*

$$\psi\colon H^n(X,\mathbb{Q})\otimes H^n(X,\mathbb{Q}) \xrightarrow[\sim]{1\otimes L^r} H^n(X,\mathbb{Q})\otimes H^{2d-n}(X,\mathbb{Q})(\dim X - n) \xrightarrow{\cup} H^{2d}(X,\mathbb{Q})(d-n) = \mathbb{Q}(-n).$$

The restriction of ψ to $H^n_{\mathrm{prim}}(X,\mathbb{Q})$ is a polarization. Thus we obtain a polarization on $H^n(X,\mathbb{Q})$ by the Lefschetz decomposition $H^n(X,\mathbb{Q}) = \bigoplus_{0\leq s\leq \lfloor n/2\rfloor} L^s(H^{n-2s}_{\mathrm{prim}}(X,\mathbb{Q}))$.

1.2 Mumford–Tate group

1.2.1 Revision on algebraic tori

Let k be a field. A *linear algebraic group* defined over k is an affine group scheme G/k of finite type; it can be embedded as a closed subgroup scheme of GL_N for some N. If $\mathrm{char} k = 0$, then G is reduced and smooth. As an example, we have $\mathbb{G}_{\mathrm{m},k} := \mathrm{GL}_{1,k}$ which is defined by: for any k-algebra R, we have $\mathbb{G}_{\mathrm{m},k}(R) = R^{\times}$. When k is clear in the context, we simply write $\mathbb{G}_{\mathrm{m}}$.

Let k^{s} be a separable closure of k. If $\mathrm{char} k = 0$, then k^{s} is an algebraic closure of k.

Definition 1.2.1. *An* **algebraic torus** *defined over k is a linear algebraic group T defined over k such that its base change to k^{s} is isomorphic to $\mathbb{G}^r_{\mathrm{m},k^{\mathrm{s}}}$ for some $r \geq 1$.*

The *group of characters* (resp. *group of cocharacters*) of T is

$$X^*(T) := \mathrm{Hom}(T_{k^{\mathrm{s}}}, \mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}), \quad (\text{resp. } X_*(T) := \mathrm{Hom}(\mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}, T_{k^{\mathrm{s}}})).$$

Both $X^*(T)$ and $X_*(T)$ are isomorphic (as groups) to $\mathbb{Z}^{\dim T}$ and are naturally endowed with a $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action. We also have a *perfect pairing* as $\mathrm{Gal}(k^{\mathrm{s}}/k)$-modules

$$X^*(T) \times X_*(T) \to \mathbb{Z} = \mathrm{End}(\mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}), \qquad (\chi, \mu) \mapsto \langle \chi, \mu \rangle := \chi \circ \mu. \tag{1.2.1}$$

By definition, $T_{k'} \simeq \mathbb{G}_{\mathrm{m},k'}$ for some finite separable extension k'/k. So the Galois action of $\mathrm{Gal}(k^{\mathrm{s}}/k)$ on $X^*(T)$ factors through $\mathrm{Gal}(k'/k)$ which is a finite group. Therefore the $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action on $X^*(T)$ is continuous. Same for the $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action on $X_*(T)$. Thus the functor $T \mapsto X_*(T)$ gives an equivalence from the category of *algebraic tori defined over* k to the category of *free abelian groups of finite rank endowed with a continuous* $\mathrm{Gal}(k^{\mathrm{s}}/k)$*-action*.

Next we turn to the representations of algebraic tori $\rho\colon T \to \mathrm{GL}(V)$. Passing to k', ρ becomes $T_{k'} \simeq \mathbb{G}^r_{\mathrm{m},k'} \to \mathrm{GL}(V_{k'})$. Then $V_{k'}$ can be decomposed into

$$V_{k'} = \bigoplus\nolimits_{\chi \in X^*(T)} V_\chi = \bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r} V^{n_1,\ldots,n_r} \tag{1.2.2}$$

where $V_\chi = \{v \in V_{k'} : \rho(t)v = \chi(t)v\}$ and $V^{n_1,\ldots,n_r} = \{v \in V_{k'} : \rho(z_1,\ldots,z_r)v = z_1^{-n_1}\cdots z_r^{-n_r}v\}$. On the base field k, the decomposition is Galois compatible, *i.e.* $\sigma(V_\chi) = V_{\chi^\sigma}$ for all $\sigma \in \mathrm{Gal}(k'/k)$.

1.2.2 Deligne torus

View $\mathbb{C}$ as an $\mathbb{R}$-algebra using the inclusion $\mathbb{R} \subseteq \mathbb{C}$. Let $\mathbb{S}$ be the algebraic group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m}}$ defined over $\mathbb{R}$, *i.e.* for any $\mathbb{R}$-algebra R, we have

$$\mathbb{S}(R) = (R \otimes_{\mathbb{R}} \mathbb{C})^\times.$$

Then

$$\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times = \big((\mathbb{R} \oplus \sqrt{-1}\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}\big)^\times = (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})^\times \times (\sqrt{-1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})^\times = \mathbb{C}^\times \times \mathbb{C}^\times.$$

Hence $\mathbb{S}$ is an algebraic torus defined over $\mathbb{R}$, and $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ acts on $\mathbb{S}(\mathbb{C})$ by $\sigma(z_1, z_2) = (\overline{z}_2, \overline{z}_1)$. Thus $\mathbb{S}(\mathbb{R}) = \{z \in \mathbb{S}(\mathbb{C}) : z = \sigma(z)\} = \{(z_1, z_2) \in \mathbb{C}^\times \times \mathbb{C}^\times : z_2 = \overline{z}_1\}$. In other words, the natural inclusion $\mathbb{S}(\mathbb{R}) \subseteq \mathbb{S}(\mathbb{C})$ is given by $z \mapsto (z, \overline{z})$.

Definition 1.2.2. *The algebraic torus* $\mathbb{S}$ *is called the* **Deligne torus**.

The character group of the Deligne torus is

$$X^*(\mathbb{S}) = \mathrm{Hom}(\mathbb{S}(\mathbb{C}), \mathbb{C}^\times) = \mathrm{Hom}(\mathbb{C}^\times \times \mathbb{C}^\times, \mathbb{C}^\times) = \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \oplus \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \simeq \mathbb{Z} \oplus \mathbb{Z}, \tag{1.2.3}$$

where the last isomorphism is obtained from the inverse of

$$\mathbb{Z} \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times), \qquad p \mapsto (z \mapsto z^{-p}). \tag{1.2.4}$$

The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ acts on $X^*(\mathbb{S})$ by $\sigma(p, q) = (q, p)$.

Among the cocharacters of $\mathbb{S}$, two are particularly important:

- the *weight cocharacter* $w\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$, $z \mapsto (z,z)$, which descends to $\mathbb{R}$ (namely it is the base change to $\mathbb{C}$ of a morphism $\mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathbb{S}$).

- the *principal cocharacter* $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$, $z \mapsto (z,1)$.

An important character of $\mathbb{S}$ is the *norm character* $\mathrm{Nm}\colon \mathbb{S} \to \mathbb{G}_{\mathrm{m}}$, $z \mapsto z\sigma(z)$. It fits into the following short exact sequence:

$$0 \to U(1) \to \mathbb{S} \xrightarrow{\mathrm{Nm}} \mathbb{G}_{\mathrm{m}} \to 0. \tag{1.2.5}$$

Notice that $\mathrm{Nm} \circ w$ sends each $z \in \mathbb{G}_{\mathrm{m}}(\mathbb{R}) = \mathbb{R}^{\times}$ to z^2.

1.2.3 Hodge structures as representations of the Deligne torus

Now let V be an R-Hodge structure of weight n. Recall the Hodge decomposition $V_{\mathbb{C}} = \bigoplus V^{p,q}$. It gives rise to an action of $\mathbb{S}_{\mathbb{C}}$ on $V_{\mathbb{C}}$ by setting $V^{p,q}$ to be the eigenspace of the character $(p,q) \in X^*(\mathbb{S})$. More precisely, for each $(z_1,z_2) \in \mathbb{S}(\mathbb{C}) = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$ and each $v = (v_{p,q})_{p,q} \in V_{\mathbb{C}} = \bigoplus V^{p,q}$, we have

$$(z_1,z_2) \cdot v = (z_1^{-p} z_2^{-q} v_{p,q})_{p,q}. \tag{1.2.6}$$

This action of $\mathbb{S}_{\mathbb{C}}$ on $V_{\mathbb{C}}$ induces a morphism

$$h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}}). \tag{1.2.7}$$

Lemma 1.2.3. *The morphism h descends to $\mathbb{R}$, i.e. it is the base change to $\mathbb{C}$ of a morphism $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$.*

Proof. For $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1,\sigma\}$, we can do the following computation. Let $(z_1,z_2) \in \mathbb{S}(\mathbb{C})$ and $v = (v_{p,q})_{p,q} \in V_{\mathbb{C}}$.

Recall that the Hodge decomposition satisfies $\overline{V^{p,q}} = V^{q,p}$. So $\overline{v_{p,q}} \in \overline{V^{p,q}} = V^{q,p}$. Hence the decomposition of $\overline{v} = \sigma(v)$ under $V_{\mathbb{C}} = \bigoplus V^{p,q}$ is $\overline{v} = (\overline{v_{q,p}})_{p,q}$. In particular, $\overline{v}_{p,q} = \overline{v_{q,p}}$.

Now we have

$$h(\sigma(z_1,z_2))v = (\overline{z}_2,\overline{z}_1) \cdot v = (\overline{z}_2^{-p}\overline{z}_1^{-q} v_{p,q})_{p,q}$$

and

$$\sigma\left(h(z_1,z_2)\right)v = \sigma\left(h(z_1,z_2)\overline{v}\right) = \sigma\left((z_1,z_2)\cdot\overline{v}\right) = \sigma((z_1^{-p}z_2^{-q}\overline{v}_{p,q})_{p,q}) = \sigma((z_1^{-p}z_2^{-q}\overline{v_{q,p}})_{p,q}) = (\overline{z}_1^{-q}\overline{z}_2^{-p}v_{p,q})_{p,q}.$$

Hence h is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant, and therefore descends to $\mathbb{R}$. $\square$

Thus from any R-Hodge structure V of weight n, we have constructed a morphism $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$. Conversely given any $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$, we can define $V^{p,q}$ to be the eigenspace of the character $(p,q) \in X^*(\mathbb{S})$ of $\mathbb{S}_{\mathbb{C}}$. Then $V = \bigoplus V^{p,q}$, and $\overline{V^{q,p}} = V^{p,q}$ because h is defined over $\mathbb{R}$. Hence we have the following proposition.

Proposition 1.2.4. *Let $R = \mathbb{Z},\mathbb{Q}$ and let V be a torsion-free R-module of finite type.*

Then there are bijections between the following sets of:

(i) Hodge structures of weight n on V;

(ii) morphisms $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that the eigenspace of $(p,q) \in X^*(\mathbb{S})$ is 0 unless $p+q = n$.

(iii) morphisms $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that the composite $h \circ w\colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ sends each $z \in \mathbb{R}^{\times}$ to the multiplication by z^{-n}.

If a Hodge structure on V corresponds to $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$, by abuse of notation we use (V, h) to denote this Hodge structure. In this terminology, the Weil operator C of the Hodge structure (V, h) in the definition of polarizations (1.1.3) is simply $h(\sqrt{-1})$.

Proposition 1.2.5. *Let (V, h) and (W, h') be two R-Hodge structures of weight n, and let $\varphi\colon V \to W$ be an R-linear map.*

Then φ is a morphism of Hodge structures if and only if $\varphi(h(z)v) = h'(z)\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$ and $v \in V_{\mathbb{R}}$.

The proof of Lemma 1.1.7.(i) can be much simplified by this proposition: $\psi(y, x) = \psi(Cy, Cx) = (2\pi\sqrt{-1})^{-2n}\psi_C(Cy, x) = (2\pi\sqrt{-1})^{-2n}\psi_C(x, Cy) = \psi(x, C^2y) = (-1)^n\psi(x, y)$, and hence ψ is $(-1)^n$-symmetric.

Proof. Write $v = (v_{p,q})_{p,q} \in V_{\mathbb{C}} = \bigoplus V^{p,q}$. Then $h(z)v = (z^{-p}\overline{z}^{-q}v_{p,q})_{p,q}$. So $\varphi(h(z)v) = (z^{-p}\overline{z}^{-q}\varphi(v_{p,q}))_{p,q}$ by linearity of φ.

Assume φ is a morphism of Hodge structures. Then $\varphi(V^{p,q}) \subseteq W^{p,q}$ for all p, q, and hence $\varphi(v_{p,q}) = \varphi(v)_{p,q}$ for all p, q. So $\varphi(h(z)v) = (z^{-p}\overline{z}^{-q}\varphi(v)_{p,q})_{p,q} = h'(z)\varphi(v)$.

Conversely assume $\varphi(h(z)v) = h'(z)\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$ and $v \in V_{\mathbb{R}}$. Let $v \in V^{p,q}$. By considering $v + \overline{v}$ and $(v - \overline{v})/\sqrt{-1}$, we have $\varphi(h(z)v) = h'(z)\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$. So $h'(z)\varphi(v) = \varphi(h(z)v) = \varphi(z^{-p}\overline{z}^{-q}v) = z^{-p}\overline{z}^{-q}\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$. Therefore $\varphi(v) \in W^{p,q}$. $\square$

This proposition has the following immediate corollary.

Corollary 1.2.6. *Let (V, h) be an R-Hodge structure of weight n, and let W be a torsion-free R-submodule of V.*

Then $h|_W$ is an R-Hodge structure if and only if $W_{\mathbb{R}}$ is an $h(\mathbb{S})$-submodule of V.

In this case, we call the Hodge structure $(W, h|_W)$ a *sub-R-Hodge structure* of (V, h).

Another corollary is:

Corollary 1.2.7. *Let $Q\colon V \times V \to R$ induce a polarization on (V, h). Then $h(\mathbb{S}) \subseteq \mathrm{Aut}(V, Q)$.*

Proof. By definition, Q induces a morphism of Hodge structures between $V \otimes V$ and $R(-n)$. Thus the conclusion follows immediately from Proposition 1.2.5. $\square$

1.2.4 Mumford–Tate group

In this subsection, assume $R = \mathbb{Z}$ or $\mathbb{Q}$. Let (V, h) be an R-Hodge structure.

Definition 1.2.8. *The* **Mumford–Tate group** *of (V, h) is the smallest $\mathbb{Q}$-subgroup $\mathrm{MT}(h)$ of $\mathrm{GL}(V_{\mathbb{Q}})$ such that $h(\mathbb{S}) \subseteq \mathrm{MT}(h)(\mathbb{R})$.*

It is easy to check that $\mathrm{MT}(h)$ is connected since $\mathbb{S}$ is, and $\mathrm{MT}(h)(\mathbb{C})$ is the subgroup of $\mathrm{GL}(V(\mathbb{C}))$ generated by $\sigma(h(\mathbb{S}(\mathbb{C})))$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We also have the following characterization of $\mathrm{MT}(h)$ using the principal cocharacter μ defined above (1.2.5).

Lemma 1.2.9. *$\mathrm{MT}(h)$ is the smallest $\mathbb{Q}$-subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ such that $\mu_h := h \circ \mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ factors through $\mathrm{MT}(h)_{\mathbb{C}}$.*

Proof. By definition $\mu_h(\mathbb{G}_{\mathrm{m},\mathbb{C}}) \subseteq \mathrm{MT}(h)_{\mathbb{C}}$. Conversely let M be a $\mathbb{Q}$-subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ which contains $\mu_h(\mathbb{G}_{\mathrm{m},\mathbb{C}}) = h(\mu(\mathbb{G}_{\mathrm{m},\mathbb{C}}))$. Then $M(\mathbb{C})$ contains $h(z, 1) \in \mathrm{GL}(V(\mathbb{C}))$ for all $z \in \mathbb{C}^{\times}$. Since M is defined over $\mathbb{Q}$ and h is defined over $\mathbb{R}$, we have that $M(\mathbb{C})$ contains $\sigma(h(z, 1)) = h(\sigma(z, 1)) = h(1, \overline{z})$ for all $z \in \mathbb{C}^{\times}$, where $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$. Hence $M(\mathbb{C})$, as a group, contains $h(z, 1)h(1, \overline{z}') = h(z, \overline{z}')$ for all $z, z' \in \mathbb{C}^{\times}$. Hence $h(\mathbb{S}_{\mathbb{C}}) \subseteq M_{\mathbb{C}}$, so $\mathrm{MT}(h) \subseteq M$. $\square$

It is not hard to check that the Mumford–Tate of the dual Hodge structure of (V, h) is still $\mathrm{MT}(h)$.

Now assume $R = \mathbb{Q}$. For $m, n \in \mathbb{Z}_{\geq 0}$, we have a Hodge structure $T^{m,n}V := V^{\otimes m} \otimes (V^\vee)^{\otimes n}$, and $\mathrm{MT}(h)$ acts on $T^{m,n}V$ componentwise. The following proposition is an immediate consequence of Corollary 1.2.6 (applied to $T^{m,n}V$).

Proposition 1.2.10. *Let W be a $\mathbb{Q}$-subspace of $T^{m,n}V$. Then W is a sub-$\mathbb{Q}$-Hodge structure of $T^{m,n}V$ if and only if W is a $\mathrm{MT}(h)$-submodule of $T^{m,n}V$.*

This proposition gives rise to another useful characterization of $\mathrm{MT}(h)$, which is important in the study of (sub-)Shimura varieties. We make the following definition.

Definition 1.2.11. *The elements of $(T^{m,n}V_{\mathbb{C}})^{0,0} \cap T^{m,n}V$, with m and n running over all non-negative integers, are called the* **Hodge tensor** *for (V, h).*

Denote by Hdg_h the set of all Hodge tensors for (V, h).

Proposition 1.2.12. *We have $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathrm{Hdg}_h)$.*

In particular by dimension reasons, $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathfrak{I})$ for some finite set $\mathfrak{I} \subseteq \mathrm{Hdg}_h$.

Proof. Take $t \in \mathrm{Hdg}_h$. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we have $\sigma(t) = t$ since t is a $\mathbb{Q}$-element. By (1.2.6) we have $h(z_1, z_2)t = z_1^0 \overline{z_2}^0 t = t$ for any $(z_1, z_2) \in \mathbb{S}(\mathbb{C})$. Applying the action of any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and recalling that $\mathrm{MT}(h)(\mathbb{C})$ is generated by the $\sigma(h(\mathbb{S}(\mathbb{C})))$'s, we have that t is fixed by $\mathrm{MT}(h)(\mathbb{Q})$. This establishes "$\subseteq$".

To get $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathrm{Hdg}_h)$, notice that $\mathrm{MT}(h)$ is a closed subgroup of $\mathrm{GL}(V)$. By theory of algebraic groups, $\mathrm{MT}(h)$ is thus the stabilizer of some 1-dimensional $\mathbb{Q}$-subspace L in $\bigoplus_{(m,n)\in I} T^{m,n}V$ for some finite subset $I \subseteq \mathbb{Z}^2_{\geq 0}$. Now that L is a 1-dimensional $\mathrm{MT}(h)$-submodule of $\bigoplus_{(m,n)\in I} T^{m,n}V$, Proposition 1.2.10 implies that L is a 1-dimensional $\mathbb{Q}$-Hodge structure, and hence $L_{\mathbb{C}} = L^{p,q}$ for some p and q. But then $p = q$ since $L^{p,q} = \overline{L^{q,p}}$.[2] In other words, $L \simeq \mathbb{Q}(-p)$ has weight $2p$.

If $p = 0$, take a $\mathbb{Q}$-generator ℓ of L. Then $\mathrm{MT}(h)(\mathbb{Q})$ fixes ℓ by the same argument on proving "$\subseteq$". So $\mathrm{MT}(h)$, being the stabilizer of $\mathbb{Q}\ell$, equals $Z_{\mathrm{GL}(V)}(\ell)$. If $p \neq 0$, then the weight of (V, h) is not zero, and hence the weight r of the Hodge structure $\det V := \bigwedge^{\dim V} V$ is non-zero (since $\det V$ can be realized as a $\mathrm{MT}(h)$-submodule of $V^{\otimes \dim V}$). We may assume $r > 0$ up to replacing V by $V^\vee$. The 1-dimensional $\mathbb{Q}$-space $L^{\otimes r} \otimes (\det V)^{\otimes -2p}$ is a Hodge structure of weight 0 and hence equals its $(0, 0)$-piece. Let ℓ be a generator of $L^{\otimes r} \otimes (\det V)^{\otimes -2p}$. Then ℓ is fixed by $\mathrm{MT}(h)(\mathbb{Q})$ by the same argument on proving "$\subseteq$". Hence $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\ell)$ as in the case of $p = 0$.

To summarize, there exists a finite sum of Hodge tensors $t_1 + \cdots + t_N$ such that $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(t_1 + \cdots + t_N)$. So $\mathrm{MT}(h) \subseteq \bigcap_{i=1}^N Z_{\mathrm{GL}(V)}(t_i) \subseteq Z_{\mathrm{GL}(V)}(t_1 + \cdots + t_N)$ becomes an equality. We are done. $\square$

Finally, we point out that the Mumford–Tate group of any *polarized* $\mathbb{Q}$-Hodge structure of weight n is a reductive group. A detailed discussion on this will be given in the next chapter (Corollary 2.2.5).

1.3 Passing to families

In practice it is important for us to work with families. We discuss two aspects, and end up with a question to relate them.

[2] To make the argument in this paragraph vigorous, we need to argue with *mixed* Hodge structures because $\bigoplus_{(m,n)\in I} T^{m,n}V$ may have more than one weight. However, since $\bigoplus_{(m,n)\in I} T^{m,n}V$ is a direct sum of (pure) Hodge structures and $\dim L = 1$, we are essentially working with a pure Hodge structure.

1.3.1 Variation of Hodge structures

Let S be a complex manifold.

Definition 1.3.1. *A* $\mathbb{Z}$**-variation of Hodge structures** *($\mathbb{Z}$-VHS) of weight n on S is* $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ *where*

- $\mathbb{V}_{\mathbb{Z}}$ *is a local system of free* $\mathbb{Z}$*-modules on S of finite rank,*
- $\mathcal{F}^\bullet$ *is a finite decreasing filtration (called the* **Hodge filtration***) of the holomorphic vector bundle* $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathcal{O}_S$ *by holomorphic subbundles,*

such that

(i) $(\mathbb{V}_{\mathbb{Z},s}, \mathcal{F}^\bullet_s)$ *is a* $\mathbb{Z}$*-Hodge structure of weight n for each $s \in S$,*

(ii) the connection $\nabla\colon \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega^1_S$ *whose sheaf of horizontal sections is* $\mathbb{V}_{\mathbb{C}} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ *satisfies the* **Griffiths' transversality** *condition*

$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S \qquad \text{for all } p. \tag{1.3.1}$$

A **polarization** *on* $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ *is a morphism of local systems*

$$\mathbb{V}_{\mathbb{Q}} \otimes \mathbb{V}_{\mathbb{Q}} \to \mathbb{Q}_S$$

inducing on each fiber a polarization of the corresponding $\mathbb{Q}$*-Hodge structure.*

Example 1.3.2. *Let $f\colon X \to S$ be a smooth projective morphism. Then $\mathbb{V} := R^n f_* \mathbb{Z}_X$ is a local system of $\mathbb{Z}$-modules on S with fiber $\mathbb{V}_s = H^n(X_s, \mathbb{Z})$. Replace $\mathbb{V}$ by its quotient by torsion. Under the isomorphism $\mathcal{V} \simeq R^n f_* \Omega^\bullet_{X/S}$, the Hodge filtration is $\mathcal{F}^p\mathcal{V} = R^n f_* \Omega^{\geq p}_{X/S}$. Notice that the subbundle of (p,q)-forms is not holomorphic if $q \neq 0$, but $\mathcal{F}^p\mathcal{V}$ is holomorphic. The fiberwise polarization from Example 1.1.9 gives a polarization on $\mathbb{V}$.*

And this example is the geometric origin of the Griffiths' transversality.

1.3.2 Parametrizing space

Next we turn to the following question. Let V be a finite-dimensional $\mathbb{R}$-vector space, and let $n \in \mathbb{Z}$.

Fix a partition $\{h^{p,q}\}_{p,q\in\mathbb{Z}}$ of $\dim V_{\mathbb{C}}$ into non-negative integers with $p+q=n$ such that $h^{p,q} = h^{q,p}$. Consider the set of all Hodge structures on V such that in the Hodge decomposition, we have $\dim V^{p,q} = h^{p,q}$ for all p,q. Equivalently by Proposition 1.2.4, we are considering the subset $\mathcal{M}_0$ of $\mathrm{Hom}(\mathbb{S}, \mathrm{GL}_V)$ such that the eigenspace of $(p,q) \in X^*(\mathbb{S})$ has dimension $h^{p,q}$. Notice that GL_V acts on $\mathrm{Hom}(\mathbb{S}, \mathrm{GL}_V)$, by sending $h \mapsto \mathrm{Int}(g) \circ h$.

Lemma 1.3.3. *$\mathcal{M}_0$ is a GL_V-orbit.*

Proof. Fix $h \in \mathcal{M}_0$. Then $V_h^{p,q} = \{v \in V_{\mathbb{C}} : h(z)v = z^{-p}\overline{z}^{-q}v \text{ for all } z \in \mathbb{S}(\mathbb{R})\}$.

For any $g \in \mathrm{GL}_V$, it is easy to check that $\{v \in V_{\mathbb{C}} : (g\cdot h)(z)v = z^{-p}\overline{z}^{-q}v \text{ for all } z \in \mathbb{S}(\mathbb{R})\}$ equals $gV_h^{p,q}$, and hence has dimension $h^{p,q}$. Hence the Hodge structure on V determined by $g \cdot h$ is in $\mathcal{M}_0$. Namely $\mathrm{GL}_V \cdot h \subseteq \mathcal{M}_0$.

Conversely let $h' \in \mathcal{M}_0$. By assumption $\dim V_{h'}^{p,q} = \dim V_h^{p,q}$ for all p,q. Assume $h^{p,q} = 0$ unless $r \leq p \leq s$. Such r and s exist since $\dim V_{\mathbb{C}} < \infty$. Now there exists a $g_1 \in \mathrm{GL}_V$ such that $V_{h'}^{r,n-r} = g_1 V_h^{r,n-r}$ by dimension reasons. Now we work with h' and $g_1 \cdot h$, and there exists $g_2 \in \mathrm{GL}_V$ such that $g_2 V_{h'}^{r,n-r} = V_{h'}^{r,n-r}$ and $V_{h'}^{r+1,n-r-1} = g_2 V_{g_1\cdot h}^{r+1,n-r-1}$. We continue to work with h' and $g_2g_1 \cdot h$ and repeat this process which stops after finitely many steps. Hence we find a $g \in \mathrm{GL}_V$ such that $V_{h'}^{p,q} = V_{g\cdot h}^{p,q}$ for all (p,q). So $h' = g \cdot h$. Thus $\mathcal{M}_0 \subseteq \mathrm{GL}_V \cdot h$. $\square$

Next we fix furthermore a non-degenerate $(-1)^n$-symmetric pairing $Q\colon V \times V \to \mathbb{R}$. We furthermore consider the subset $\mathcal{M}$ of $\mathcal{M}_0$ consisting of Hodge structures on V for which Q is a polarization. Then by Corollary 1.2.7, we have $\mathcal{M} \subseteq \mathrm{Hom}(\mathbb{S}, \mathrm{Aut}(V,Q))$. Moreover using (the proof of) Lemma 1.3.3, we see that $\mathcal{M}$ is an $\mathrm{Aut}(V,Q)$-orbit.

Example 1.3.4. *Assume* $\dim V = 2g$ *and let* $Q\colon V \times V \to \mathbb{R}$ *be the standard symplectic pairing. Then* $\mathrm{Aut}(V,Q) = \mathrm{GSp}_{2g}$. *If* $g = 1$, *then* $\mathrm{Aut}(V,Q) = \mathrm{GL}_2$.

Finally fix a collection of tensors $\{s_\alpha\}$ on $T^{m,n} = V^{\otimes m} \otimes (V^\vee)^{\otimes n}$ with m,n running over all non-negative integers. Set

$$G := \mathrm{Aut}(V,Q) \cap \bigcap\nolimits_\alpha \mathrm{Stab}_{\mathrm{GL}_V}(s_\alpha). \tag{1.3.2}$$

Let G^+ be the identity component of the real Lie group $G(\mathbb{R})$. Then $[G(\mathbb{R}) : G^+] < \infty$.

Fix $h\colon \mathbb{S} \to \mathrm{Aut}(V,Q)$ such that each s_α is a Hodge tensor for the Hodge structure (V,h). Then the same holds true for the Hodge structure $(V, g\cdot h)$ for all $g \in G^+$. Let $X^+ := G^+ \cdot h \subseteq \mathrm{Hom}(\mathbb{S}, G)$.[3]

Now we have a family of Hodge structures on X^+ as follows: $X^+ \times V \to X^+$, with the Hodge structure on V over each $h \in X^+$ being precisely the one given by h. Now $X^+ \times V$ can be seen as a smooth vector bundle on X^+, and for each p there is a subbundle F^p whose fiber over each $h \in X^+$ is the Hodge filtration F^p_h.

In view of the definition of VHS (Definition 1.3.1), we wish the investigate the following questions:

(i) Is there a complex structure on X^+ for which each subbundle F^p is a holomorphic?

(ii) When does Griffiths' transversality hold true, *i.e.* $\nabla(F^p) \subseteq F^{p-1} \otimes \Omega^1_{X^+}$ for all p?

1.3.3 Constraint on the Hodge type

Continue to use the notation above. Let $h \in X^+ \subseteq \mathrm{Hom}(\mathbb{S}, G)$. Composing with the adjoint representation $G \to \mathrm{GL}(\mathrm{Lie}G)$, we have a Hodge structure on $\mathrm{Lie}G$ by Proposition 1.2.4. By abuse of notation, we use $(\mathrm{Lie}G, h)$ to denote this Hodge structure. Since X^+ is a G^+-orbit, the Hodge type of $(\mathrm{Lie}G, h)$ is independent of the choice of $h \in X^+$.

Moreover h induces a Hodge structure on $\mathrm{End}(V) = V^\vee \otimes V$, which must be of weight 0 and by abuse of notation we denote by $(\mathrm{End}(V), h)$. The inclusion $G \subseteq \mathrm{GL}(V)$ induces $\mathrm{Lie}G \subseteq \mathrm{End}(V) = V^\vee \otimes V$. Hence the weight of $(\mathrm{Lie}G, h)$ is 0.

In what follows, we use $\mathfrak{g}$ to denote $\mathrm{Lie}G$.

Proposition 1.3.5. *There exists a unique complex structure on* X^+ *such that* F^p *is holomorphic for each* p. *Griffiths' transversality holds true if and only if the Hodge structure* $(\mathrm{Lie}G, h)$ *has type* $(-1,1) + (0,0) + (1,-1)$ *for one (and hence all)* $h \in X^+$.

Proof. For each $h \in X^+$, let $F^\bullet_h$ be the Hodge filtration of the Hodge structure (V,h). For each p, write $d_p := \dim F^p_h = \sum_{r \geq p} h^{r,n-r}$ which does not depend on h. We have a flag variety $\mathcal{F}\ell$ parametrizing sequences (called *flags*) $\cdots \supseteq V_p \supseteq V_{p+1} \supseteq \cdots$ of subspaces of $V_\mathbb{C}$ with $\dim V_p = d_p$ for each p. By general theory, $\mathcal{F}\ell$ is a complex algebraic variety which is a $\mathrm{GL}(V_\mathbb{C})$-orbit. Moreover, the tangent space of $\mathcal{F}\ell$ at the flag $\cdots \supseteq V_p \supseteq V_{p+1} \supseteq \cdots$ is a subspace of

$$\bigoplus_p \mathrm{Hom}(V_p, V_\mathbb{C}/V_p). \tag{1.3.3}$$

[3]In fact, X^+ is known to be a connected component of $X \subseteq \mathcal{M}$ which parametrizes all Hodge structures on V for which each s_α is a Hodge tensor.

There is a natural map

$$\varphi\colon X^+ \to \mathcal{F}\ell, \quad h \mapsto F_h^\bullet,$$

which is injective since a Hodge structure is uniquely determined by its Hodge filtration. The group $\mathrm{GL}(V_{\mathbb{C}})$ naturally acts on $\mathcal{F}\ell$, and it is not hard to check that the stabilizer of $F_h^\bullet$ is $\exp F_h^0\mathrm{End}(V_{\mathbb{C}})$.

Let us show that φ makes X^+ into a complex subvariety of $\mathcal{F}\ell$. Fix $h_0 \in X^+$ and let $K_\infty := \mathrm{Stab}_{G^+}(h_0)$. Then $X^+ = G^+ \cdot h_0 \simeq G^+/K_\infty$, and $\mathrm{Lie}K_\infty = \mathfrak{g} \cap F_{h_0}^0\mathfrak{g}_{\mathbb{C}}$ which is the $(0,0)$-constituent of the Hodge structure $(\mathfrak{g}, h_0)$. So φ factors through

$$X^+ = G^+/K_\infty \to X^\vee := G(\mathbb{C})/\exp F_{h_0}^0\mathfrak{g}_{\mathbb{C}} \to \mathcal{F}\ell \simeq \mathrm{GL}(V_{\mathbb{C}})/\exp F_{h_0}^0\mathrm{End}(V). \tag{1.3.4}$$

The first map makes X^+ into an open submanifold of $X^\vee$, and the second map is a closed immersion as complex algebraic varieties. So X^+ has a natural complex structure induced from $X^\vee$.

Next we turn to the Griffiths' transversality. The tangent map of φ at h_0 is

$$\mathrm{d}\varphi\colon T_{h_0}X^+ \to T_{h_0}\mathcal{F}\ell \simeq \mathrm{End}(V_{\mathbb{C}})/F_{h_0}^0\mathrm{End}(V_{\mathbb{C}}) \subseteq \bigoplus_p \mathrm{Hom}(F_{h_0}^p, V_{\mathbb{C}}/F_{h_0}^p).$$

Griffiths' transversality holds true if and only if

$$\mathrm{im}(\mathrm{d}\varphi) \subseteq \bigoplus_p \mathrm{Hom}(F_{h_0}^p, F_{h_0}^{p-1}/F_{h_0}^p),$$

and hence if and only if

$$\mathrm{im}(\mathrm{d}\varphi) \subseteq F_{h_0}^{-1}\mathrm{End}(V_{\mathbb{C}})/F_{h_0}^0\mathrm{End}(V_{\mathbb{C}}).$$

But $\mathrm{im}(\mathrm{d}\varphi) = \mathrm{Lie}G_{\mathbb{C}}/F_{h_0}^0\mathfrak{g}_{\mathbb{C}}$. So Griffiths' transversality holds true if and only if $\mathfrak{g}_{\mathbb{C}} = F_{h_0}^{-1}\mathfrak{g}_{\mathbb{C}}$. Therefore we can conclude. □

We yet to understand the polarization attached to this family, for which we need to recall some background knowledge on reductive groups. The full discussion will be carried out in §2.2.

Chapter 2

From Hodge theory to Hermitian symmetric domains

2.1 Basic background knowledge on reductive groups

Let k be a field. Let G be a connected linear group defined over k. Let $\overline{k}$ be an algebraic closure of k.

Denote by $\mathbb{G}_{\mathrm{a},k}$ the group defined by: for any k-algebra R, we have $\mathbb{G}_{\mathrm{a}}(R) = R$. When k is clear in the context, we simply write $\mathbb{G}_{\mathrm{a}}$.

Definition 2.1.1. *G is called a* **reductive group** *if $G_{\overline{k}}$ does not contain a normal subgroup isomorphic to $\mathbb{G}_{\mathrm{a}}$.*

A notion closely related to reductive groups is the *unipotent radical*. Let us briefly recall the definition. Recall that G can be embedded as a closed subgroup scheme of GL_N for some N. An element $g \in G$ is said to be *unipotent* if $(I_N - g)^N = 0$ (as matrix). A subgroup of G is said to be *unipotent* if all its elements are unipotent. As an example, U_N (consisting of upper triangular matrices whose diagonal entries are 1) is a unipotent subgroup of GL_N. Moreover, it is known that any unipotent subgroup of GL_N is a subgroup of gU_Ng^{-1} for some $g \in \mathrm{GL}_N$.

Definition 2.1.2. *The* **unipotent radical** *of G, denoted by $R_u(G)$, is the identity component of its maximal normal unipotent subgroup.*

As an example, $R_u(\mathrm{GL}_N) = 1$. Moreover, any algebraic torus has trivial unipotent radical.

Since $\mathbb{G}_{\mathrm{a}}$ is a unipotent subgroup of GL_N via $x \mapsto \begin{bmatrix} 1 & 0 & \cdots & 0 & x \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$, we have:

Lemma 2.1.3. *G is a reductive group if and only if $R_u(G_{\overline{k}}) = 1$.*

For any reductive group G, its connected center $Z(G)^\circ$ is an algebraic torus. Among reductive groups, those with trivial connected center are of particular importance.

Definition 2.1.4. *A reductive group G is called* **semi-simple** *if its connected center $Z(G)^\circ$ is trivial. It is called* **simple** *if its only connected normal subgroups are 1 and G.*

Clearly, simple groups are semi-simple because $Z(G)$ is a normal subgroup of G. Given a reductive group G, one can naturally construct two semi-simple subgroups:

(i) the *derived subgroup* $G^{\mathrm{der}} := [G,G]$ which is a normal subgroup of G,

(ii) the *adjoint* $G^{\mathrm{ad}} := G/Z(G)$ which is a quotient of G.

The composite $G^{\mathrm{der}} \to G \to G^{\mathrm{ad}}$ is a central isogeny, *i.e.* it is surjective and has finite kernel contained in $Z(G)$. As an example, $\mathrm{GL}_N^{\mathrm{der}} = \mathrm{SL}_N$ and $\mathrm{GL}_N^{\mathrm{ad}} = \mathrm{PGL}_N$, and the kernel of $\mathrm{SL}_N \to \mathrm{PGL}_N$ is $\{\pm I_N\}$.

Next we recall the following structural theorem of reductive groups.

Theorem 2.1.5 (Structural theorem of reductive groups)**.** *Let G be a reductive group. Then there are only finitely many non-trivial simple normal subgroups $G_1, \ldots, G_n$ of G, and*

$$G = Z(G)G_1 \cdots G_n$$

with the intersections $G_i \cap G_j < Z(G)$.

We end this revision by a characterization of a $\mathbb{C}$-group to be reductive.

Proposition 2.1.6. *Assume* $\mathrm{char} k = 0$*. Then the followings are equivalent:*

(i) G is a reductive group;

(ii) Any representation V of G can be decomposed into the direct sum of irreducible ones.

Corollary 2.1.7. *Let G be a connected linear algebraic group defined over $\mathbb{C}$. Then G is reductive if and only if G has a real form $G_{\mathbb{R}}$ (i.e. $G_{\mathbb{R}} \otimes \mathbb{C} \simeq G$) such that $G_{\mathbb{R}}(\mathbb{R})$ is compact.*

Proof. We only sketch for $\Leftarrow$. By definition it is enough to prove that $G_{\mathbb{R}}$ is reductive. For any representation V of $G_{\mathbb{R}}$, define an inner product on V induced by $\|v\| := \int_{G_{\mathbb{R}}(\mathbb{R})} gv$ with respect to a Haar measure on $G_{\mathbb{R}}(\mathbb{R})$. Then this inner product is $G_{\mathbb{R}}$-invariant. Thus V can be decomposed into the direct sum of irreducible sub-representations of $G_{\mathbb{R}}$. $\square$

Example 2.1.8. *Let $G = \mathrm{GL}_{N,\mathbb{C}}$. Then $\mathrm{GL}_{N,\mathbb{R}}$ and (write $J_{p,q} = \mathrm{diag}\{I_p, -I_q\}$ and denote for simplicity by $J = J_{p,q}$)*

$$U(p,q) := \left\{ g \in \mathrm{GL}_{N,\mathbb{C}} : \overline{g}^{\mathrm{t}} J g = J \right\}$$

are $\mathbb{R}$-forms of G, with all $p + q = N$. The associated complex conjugation for $U(p,q)$ is $\sigma \colon g \mapsto J(\overline{g}^{\mathrm{t}})^{-1} J$. A compact $\mathbb{R}$-form is $U(N)$.

2.2 Polarization on families and reductive groups

Recall the setting of §1.3.2: V is a finite-dimensional $\mathbb{R}$-vector space, $n \in \mathbb{Z}$, $G < \mathrm{GL}(V)$ and $X^+ \subseteq \mathrm{Hom}(\mathbb{S}, G)$ is a G^+-orbit. We know that X^+ parametrizes certain Hodge structures on V of weight n, and hence has carries a family of Hodge structures. By Proposition 1.3.5, X^+ has a unique complex structure such that this family of Hodge structures varies holomorphically.

Better, we have fixed a $(-1)^n$-symmetric pairing $Q \colon V \times V \to \mathbb{R}$ which induces a polarization for the Hodge structure on V given by each $h \in X^+$. In this section, we prove that this extra information forces G to be a reductive group.

2.2.1 Cartan involution

We need some background knowledge on Cartan involutions.

Let $G_{\mathbb{R}}$ be a linear algebraic group defined over $\mathbb{R}$. Let $\sigma\colon G_{\mathbb{C}} \to G_{\mathbb{C}}$ be the associated conjugation.

Definition 2.2.1. *A* **Cartan involution** *is a morphism $\theta\colon G_{\mathbb{R}} \to G_{\mathbb{R}}$ such that $\theta^2 = 1$ and that $(G_{\mathbb{C}})^{\tau} := \{g \in G_{\mathbb{C}} : \tau(g) = g\}$ is a compact real form of $G_{\mathbb{C}}$, where $\tau = \theta_{\mathbb{C}} \circ \sigma = \sigma \circ \theta_{\mathbb{C}}$.*

Example 2.2.2. *Let us look at the following examples with $G_{\mathbb{C}} = \mathrm{GL}_{N,\mathbb{C}}$.*

(a) $G_{\mathbb{R}} = U(N)$, *with* $\theta = 1$.

(b) $G_{\mathbb{R}} = U(p,q)$, *with* $\theta(g) = JgJ$ *where* $J = J_{p,q}$.

(c) $G_{\mathbb{R}} = \mathrm{GL}_{N,\mathbb{R}}$, *with* $\theta(g) = (g^{\mathrm{t}})^{-1}$.

Proposition 2.2.3. *$G_{\mathbb{R}}$ is reductive if and only if $G_{\mathbb{R}}$ admits a Cartan involution. And any two Cartan involutions of $G_{\mathbb{R}}$ are conjugate.*

In Example 2.2.2, the Cartan involutions in (a) and (b) are induced by an element of $G(\mathbb{R})$, while in (c) it is not. The first kind is called *inner Cartan involution* and is of particular importance because of its relation with polarizations explained by the following lemma.

Lemma 2.2.4 (Deligne)**.** *Let $C \in G(\mathbb{R})$ with $C^2 = 1$. Then the followings are equivalent:*

(i) $\mathrm{Int}(C)$ *is a Cartan involution of* $G_{\mathbb{R}}$*,*

(ii) any $G_{\mathbb{R}}$-representation V is C-polarizable, i.e. there exists a $G_{\mathbb{R}}$-invariant bi-linear map $\phi\colon V \times V \to \mathbb{R}$ such that $(x,y) \mapsto \phi_{\mathbb{C}}(x, C\overline{y})$ is Hermitian and positive-definite (equivalently, $(x,y) \mapsto \phi(x, Cy)$ is symmetric and positive-definite),

(iii) $G_{\mathbb{R}}$ admits one faithful representation which is C-polarizable.

Proof. Let ϕ be a bi-linear map. Observe that the followings are equivalent:

- ϕ is G-invariant;
- $\phi_{\mathbb{C}}(gx, \sigma(g)\overline{y}) = \phi_{\mathbb{C}}(x, \overline{y})$ for all $g \in G_{\mathbb{C}}$ and $x, y \in V_{\mathbb{C}}$;
- $\phi_{\mathbb{C}}(gx, \sigma(g)C\overline{y}) = \phi_{\mathbb{C}}(x, C\overline{y})$ for all $g \in G_{\mathbb{C}}$ and $x, y \in V_{\mathbb{C}}$;
- $(x,y) \mapsto \phi_{\mathbb{C}}(x, C\overline{y})$ is U-invariant, where $U = (G_{\mathbb{C}})^{\tau}$ with $\tau = \mathrm{Int}(C) \circ \sigma$.

The last equivalence follows from $\phi_{\mathbb{C}}(gx, \sigma(g)C\overline{y}) = \phi_{\mathbb{C}}(gx, C\tau(g)\overline{y})$.

Now let us go back to the proof of the lemma. (ii) implying (iii) is trivial. (iii) implies that U is compact, and hence implies (i). It remains to show that (i) implies (ii).

Assume (i). Then $G_{\mathbb{C}}$ has a compact real form U, which is the set of fixed points of $\tau = \mathrm{Int}(C) \circ \sigma$. There exists a U-invariant positive-definite symmetric bi-linear map $\phi\colon V \times V \to \mathbb{R}$ since U is compact. Hence $\phi_{\mathbb{C}}$ is $G_{\mathbb{C}}$-invariant, and so $\phi_{\mathbb{C}}(gx, \tau(g)\overline{y}) = \phi_{\mathbb{C}}(x, \overline{y})$ for all $g \in G_{\mathbb{C}}$. But $\tau(g) = C\sigma(g)C^{-1} = C\sigma(g)C$, hence $\phi_{\mathbb{C}}(gx, \sigma(g)C\overline{y}) = \phi_{\mathbb{C}}(x, C\overline{y})$ for all $g \in G_{\mathbb{C}}$. Thus ϕ is also $G_{\mathbb{R}}$-invariant. This establishes (ii). □

Here is a corollary on the Mumford–Tate group.

Corollary 2.2.5. *Let (V,h) be a $\mathbb{Q}$-Hodge structure of weight n with a polarization ψ. Then* $\mathrm{MT}(h)$ *is a reductive group.*

Proof. Let $G_{\mathbb{R}} := \mathrm{MT}(h)_{\mathbb{R}}$ and $C := h(\sqrt{-1})$. Then $C^2 = 1$, and $V_{\mathbb{R}}$ is a faithful representation of $G_{\mathbb{R}}$ which is C-polarization. Hence $\mathrm{Int}(C)$ is a Cartan involution of $G_{\mathbb{R}}$ by Lemma 2.2.4. So $G_{\mathbb{R}}$ is reductive by Proposition 2.2.3. Hence $\mathrm{MT}(h)$ is a reductive group. □

2.2.2 Polarization on parametrizing space

Now let us go back to our setting at the beginning of this section.

Let $h \in X^+$. Let G_1 be the subgroup of G generated by $h(\mathbb{S})$ for all $h \in X^+$. In other words, G_1 is the smallest subgroup of G which contains $h(\mathbb{S})$ for all $h \in X^+$. It is easy to check that G_1 is a normal subgroup of G, and that X^+ is a G_1^+-orbit, where G_1^+ is the identity component of the real Lie group $G_1(\mathbb{R})$.

Recall the weight cocharacter $w\colon \mathbb{G}_\mathrm{m} \to \mathbb{S}$ induced by $\mathbb{R}^\times \subseteq \mathbb{C}^\times$.

Proposition 2.2.6. *Assume $h \circ w$ factors through $Z(G)$ for one (and hence all) $h \in X^+$. Then the followings are equivalent:*

(1) There exists $\psi\colon V \otimes V \to \mathbb{R}(-n)$ which is a polarization for the Hodge structure determined by each $h \in X^+$;

(2) G_1 is a reductive group for one (and hence all) $h \in X^+$, and $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of G_1^{ad}.

In (2), $\mathrm{Int}(h(\sqrt{-1}))$ is an automorphism of G_1 which acts trivially on $Z(G_1)$, and so can be seen as an automorphism of G_1^{ad}.

In our setting, ψ is induced by Q. But this proposition also gives an abstract way of showing the existence of a polarization on a family of Hodge structures, which will be useful in §2.3.

Proof. By assumption, the subgroup $(h \circ w)(\mathbb{G}_\mathrm{m})$ of G_1 is independent of the choice of $h \in X^+$, and we denote it by W. Then $W < Z(G_1)$.

Recall the short exact sequence of group over $\mathbb{R}$

$$1 \to U(1) \to \mathbb{S} \xrightarrow{\mathrm{Nm}} \mathbb{G}_\mathrm{m} \to 1.$$

Let G_2 be the subgroup of G_1 generated by $h(U(1))$ for all $h \in X^+$. Then $G_1 = W \cdot G_2$. Moreover since $W < Z(G_1)$, the inclusion $G_2 < G_1$ induces $G_2^{\mathrm{ad}} \simeq G_1^{\mathrm{ad}}$. So (2) is equivalent to:

(*) G_2 is a reductive group for $h \in X^+$, and $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of G_2^{ad}.

Take a map $\psi\colon V \otimes V \to \mathbb{R}$. Then

$$\begin{aligned} &\psi\colon V \otimes V \to \mathbb{R}(-n) \text{ is a morphism of Hodge structures for all } h \in X^+ \\ \Leftrightarrow &\psi \text{ is } h(\mathbb{S})\text{-equivariant for all } h \in X^+ \\ \Leftrightarrow &\psi \text{ is } h(U(1))\text{-invariant for all } h \in X^+ \quad \text{because } \mathbb{S} = w(\mathbb{G}_\mathrm{m}) \cdot U(1) \\ \Leftrightarrow &\psi \text{ is } G_2\text{-invariant.} \end{aligned}$$

Thus $\psi\colon V \otimes V \to \mathbb{R}(-n)$ is a polarization for all $h \in X^+$ if and only if the G_2-equivariant map $(x, y) \mapsto \psi(x, h(\sqrt{-1})\overline{y})$ is Hermitian and positive-definite. Hence by Lemma 2.2.4, (1) is equivalent to $\mathrm{Int}(h(\sqrt{-1}))$ being a Cartan involution of G_2. Hence by (*), it suffices to prove that $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of G_2 if and only if it is a Cartan involution of G_2^{ad}. So it remains to prove that $Z(G_2)$ is compact. This is true because G_2 is generated by compact subgroups (since $U(1)$ is compact). $\square$

2.3 Hermitian symmetric domains

Motivated by Proposition 1.3.5 and 2.2.6, we shall study pairs (G, X^+) where

- G is a reductive group defined over $\mathbb{R}$,

- X^+ is a G^+-orbit contained in $\mathrm{Hom}(\mathbb{S}, G)$, with G acting on $\mathrm{Hom}(\mathbb{S}, G)$ via conjugation (with G^+ the identity component of the real Lie group $G(\mathbb{R})$)

satisfying the following properties:

(i) For any $h \in X^+$, the Hodge structure $(\mathrm{Lie}G, h)$ has type $(-1,1)+(0,0)+(1,-1)$,

(ii) For any $h \in X^+$, $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution for G^{ad}.

In fact, it is enough to require (i) and (ii) for one $h \in X^+$. And condition (i) implies that $h \circ w \colon \mathbb{G}_{\mathrm{m}} \to G$ factors through $Z(G)$. Indeed by (i), $\mathrm{Ad} \circ h \circ w \colon \mathbb{G}_{\mathrm{m}} \to \mathrm{GL}(\mathrm{Lie}G)$ sends z to the multiplication by $z^0 = 1$, and hence is trivial. So $\mathrm{im}(h \circ w) \subseteq \mathrm{Ker}(\mathrm{Ad}) = Z(G)$.

Now take any representation V of G. Then $X^+ \times V \to X^+$ is a family of $\mathbb{R}$-Hodge structures, with the Hodge structure on $h \in X^+$ determined by $\mathbb{S} \xrightarrow{h} G \to \mathrm{GL}(V)$. By Proposition 1.3.5 and 2.2.6, this family is an $\mathbb{R}$-variation of Hodge structures endowed with a polarization.

Theorem 2.3.1. *X^+ is a Hermitian symmetric domain. More precisely, this means:*

(1) $X^+ \simeq X_1^+ \times \cdots \times X_k^+$;

(2) Each X_i^+ is a Riemannian symmetric space of non-compact type, i.e. $X_i^+ \simeq G_i^+/K_{i,\infty}$ where G_i is a simple group defined over $\mathbb{R}$ and $K_{i,\infty}$ is a maximal compact subgroup of G_i^+;

(3) For each $i \in \{1, \ldots, k\}$, X_i^+ has a G_i-invariant complex structure.

Conversely, any Hermitian symmetric domain can be obtained as X^+ for a pair (G, X^+) as above. But we will not prove this in this course.

2.3.1 The example of Siegel case

Let $V = \mathbb{R}^{2d}$. Let $\psi \colon V \times V \to \mathbb{R}$ be $(x, y) \mapsto x^{\mathrm{t}} J y$ with $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$.

Define the $\mathbb{R}$-group

$$G_{\mathbb{R}} = \mathrm{GSp}(\psi) = \mathrm{GSp}_{2d} := \left\{ g \in \mathrm{GL}(V) : \psi(gx, gy) = c\psi(x,y) \text{ for some } c \in \mathbb{R}^{\times} \right\}$$
$$= \left\{ g \in \mathrm{GL}_{2d,\mathbb{R}} : gJg^{\mathrm{t}} = cJ \text{ for some } c \in \mathbb{R}^{\times} \right\}.$$

The derived subgroup $G_{\mathbb{R}}^{\mathrm{der}} = \mathrm{Sp}_{2d} = \{g \in \mathrm{GL}(V) : \psi(gx, gy) = \psi(x,y)\} = \left\{ g \in \mathrm{GL}_{2d,\mathbb{R}} : gJg^{\mathrm{t}} = J \right\}$.

Define

$$h_0 \colon \mathbb{S} \to \mathrm{GSp}_{2d}, \qquad a + b\sqrt{-1} \mapsto aI_{2d} + bJ.$$

Indeed, this map is well-defined since $(aI_{2d} + bJ)J(aI_{2d} + bJ)^{\mathrm{t}} = (a^2 + b^2)J$. Notice that $h_0 \circ w \colon \mathbb{G}_{\mathrm{m}} \to \mathrm{GSp}_{2d}$ sends $r \in \mathbb{R}^{\times}$ to multiplication on V by r. Hence the Hodge structure (V, h_0) has weight -1.

The eigenvalues for J are $\pm\sqrt{-1}$. Let $V^{-1,0}$ (resp. $V^{0,-1}$) be the eigenspace of $\sqrt{-1}$ (resp. of $-\sqrt{-1}$). Then one can check that each $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ acts on $V^{-1,0}$ as multiplication by z and on $V^{0,-1}$ as multiplication by $\overline{z}$. Thus (V, h_0) is a Hodge structure of type $(-1,0)+(0,-1)$, and ψ is a polarization.

Now that $\mathrm{Lie}G_{\mathbb{R}} \subseteq \mathrm{End}(V) = V \otimes V^{\vee}$, we know that the Hodge structure $(\mathrm{Lie}G, h)$ has type $(-1,1)+(0,0)+(1,-1)$. So condition (i) holds true.

For condition (ii), apply Lemma 2.2.4 to the group $(\mathrm{GSp}_{2d})^{\mathrm{ad}} = \mathrm{GSp}_{2d}/Z$, where Z is the subgroup of scalar matrices, and the element $C \in (\mathrm{GSp}_{2d})^{\mathrm{ad}}(\mathbb{R})$ being the image of $h_0(\sqrt{-1}) =$

$J \in \mathrm{GSp}_{2d}(\mathbb{R})$. Since ψ is a J-polarization of the Hodge structure (V, h_0), by Lemma 2.2.4 $\mathrm{Int}(h_0(\sqrt{-1}))$ is a Cartan involution for $(\mathrm{GSp}_{2d})^{\mathrm{ad}}$.

Let $X^+ \subseteq \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ be the GSp_{2d}^+-orbit of h_0. Then Sp_{2d} acts transitively on X^+, and $\mathrm{Stab}_{\mathrm{Sp}_{2d}}(h_0) = U(d) = O(2d) \cap \mathrm{Sp}_{2d}$ is a maximal compact subgroup of Sp_{2d}. So

$$X^+ \simeq \mathrm{Sp}_{2d}/(O(2d) \cap \mathrm{Sp}_{2d})$$

with Sp_{2d} a simple group defined over $\mathbb{R}$ which is not compact. To see the complex structure in a more concrete way, let us make the identification

$$X^+ = \mathrm{Sp}_{2n}/(O(2d) \cap \mathrm{Sp}_{2d}) \xrightarrow{\sim} \mathfrak{H}_d := \left\{\tau \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}} \text{ and } \mathrm{Im}\tau > 0\right\} \tag{2.3.1}$$

which sends

$$g \cdot h_0 \mapsto g \cdot \sqrt{-1} I_d := (\sqrt{-1}A + B)(\sqrt{-1}C + D)^{-1} \qquad \text{with } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The Sp_{2d}-invariant complex structure on X^+ is the same as the complex structure on $\mathfrak{H}_d$ inherited from the open inclusion $\mathfrak{H}_d \subseteq \left\{\tau \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}}\right\} \simeq \mathbb{C}^{d(d+1)/2}$.

2.3.2 Cartan decomposition of semi-simple groups

In this subsection, we review background knowledge (without proof) on the Cartan decomposition of semi-simple groups G defined over $\mathbb{R}$. This is closely related to the Cartan involution from §2.2.1.

Let θ be a Cartan involution of a semi-simple group G defined over $\mathbb{R}$. Composing with the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathrm{Lie}G)$, we get an involution on $\mathfrak{g} := \mathrm{Lie}G$ which we still call a *Cartan involution* and denote by θ. Then θ has eigenvalues ± 1, and let $\mathfrak{k}$ (resp. $\mathfrak{m}$) be the eigenspace for 1 (resp. for -1). Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}. \tag{2.3.2}$$

Moreover, $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$ by looking at the eigenvalues. So $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, while any Lie subalgebra contained in $\mathfrak{m}$ is commutative.

Lemma 2.3.2. $K_\infty := \exp(\mathfrak{k})$ *is a maximal compact subgroup of* G^+.

We can also recover the compact real form of G as follows. The Cartan involution θ extends to $\mathfrak{g}_{\mathbb{C}}$ and we have a corresponding $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}$. Let $\mathfrak{g}_c := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}$. Then $G_c := \exp(\mathfrak{g}_c)$ is a compact real Lie group and which is a real form of G. Notice that $K_\infty = G \cap G_c$.

2.3.3 Proof of Theorem 2.3.1

By definition of X^+, the center $Z(G)$ acts trivially on X^+. Hence the action of G^+ factors through $G^{\mathrm{ad}}(\mathbb{R})^+$. By Theorem 4.3.2, G^{ad} can be decomposed into a direct product $G^{\mathrm{ad}} = G_1 \times \cdots \times G_k$ with each G_i a simple group. Fix $h \in X^+$, and let $X_i^+ := G_i^+ \cdot h$. Then the decomposition of the group induces

$$X^+ \simeq X_1^+ \times \cdots \times X_k^+.$$

This establishes (1).

In the rest of proof, to ease notation, use G to denote G_i and X^+ to denote X_i^+. Then G is a simple group with trivial center.

Denote by $\mathfrak{g} := \mathrm{Lie}G$. Consider the action of $h(\sqrt{-1})$ on $\mathfrak{g}$ via the adjoint representation. Then $h(\sqrt{-1})$ acts on $\mathfrak{g}^{0,0}$ as identity and on $\mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}$ as multiplication by -1. Thus $X^+ \simeq G^+/K_\infty$ for the subgroup $K_\infty := \exp(\mathfrak{g}^{0,0})$ of G^+. Condition (ii) says that the action of $h(\sqrt{-1})$ on $\mathfrak{g}$ is a Cartan involution, and hence we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as in (2.3.2). Then condition (i) says that $\mathfrak{k} = \mathfrak{g}^{0,0}$ (and $\mathfrak{m}_\mathbb{C} = \mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}$). Hence $K_\infty := \exp(\mathfrak{g}^{0,0})$ is maximal compact in G^+ by Lemma 2.3.2. This establishes (2).

Remark 2.3.3. *Assume G is simple with trivial center. If G is compact, we claim that $X^+ = \{\text{trivial map}\}$. Indeed, $\mathrm{Int}(h(\sqrt{-1}))$ is identity because it is a Cartan involution for G. Thus $\mathrm{Ad} \circ h \colon \mathbb{S} \to \mathrm{GL}(\mathfrak{g})$ sends $\sqrt{-1}$ to identity, and hence $(\mathfrak{g}, h)$ has Hodge type $(0,0)$ by assumption (i) (since $\sqrt{-1}$ acts on the complement of $\mathfrak{g}^{0,0}$ by multiplication by -1). But then $\mathrm{Ad} \circ h$ is trivial since $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $\mathfrak{g}$ as multiplication by $z^0\bar{z}^0 = 1$. Thus $h(\mathbb{S}) \subseteq \mathrm{Ker}(\mathrm{Ad}) = Z(G) = \{1\}$.*

For part (3), notice that $[\mathfrak{g}^{1,-1}, \mathfrak{g}^{1,-1}] \subseteq \mathfrak{g}^{2,-2} = 0$. Hence $\mathfrak{g}^{1,-1}$ is an abelian Lie subalgebra of $\mathfrak{g}_\mathbb{C}$. Same is true for $\mathfrak{g}^{-1,1}$. Thus $F^0\mathfrak{g}_\mathbb{C} = \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$ is a Lie subalgebra of $\mathfrak{g}_\mathbb{C}$. Therefore $P_\mathbb{C} := \exp(F^0\mathfrak{g}_\mathbb{C})$ is a subgroup of $G(\mathbb{C})$, with $P_\mathbb{C} \cap G = K_\infty$. Thus the inclusion $G \subseteq G(\mathbb{C})$ induces an injective morphism of real smooth manifolds

$$X^+ = G^+/K_\infty \to X^\vee := G(\mathbb{C})/P_\mathbb{C}. \tag{2.3.3}$$

The tangent of this map is an isomorphism as real vector spaces. Hence this map realizes X^+ as an open subset of $X^\vee$. This establishes (3). We are done. □

2.3.4 Borel embedding theorem and Harish–Chandra realization

Replacing G by G^{der} does not change X^+. Hence we may assume that G is semi-simple. Fix $h \in X^+$, and take the inner Cartan involution θ obtained from $h(\sqrt{-1})$. Use the notation from §2.3.2. The real tangent space of X^+ at h, denoted by $T_\mathbb{R}(X^+)$, can be identified as $\mathfrak{m}$.

The element $J := h(e^{\pi\sqrt{-1}/4})$ satisfies $J^2 = 1$. Its action on X^+ induces a decomposition

$$T_\mathbb{R}(X^+) \otimes_\mathbb{R} \mathbb{C} = T^{1,0}(X^+) \oplus T^{0,1}(X^+)$$

where J acts by multiplication by $\sqrt{-1}$ on $T^{1,0}(X^+)$ and by $-\sqrt{-1}$ on $T^{0,1}(X^+)$. Then $T^{1,0}(X^+)$ is the holomorphic tangent space at h. On the other hand, we have $\mathfrak{m}_\mathbb{C} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ where J acts by multiplication by $\sqrt{-1}$ on $\mathfrak{m}^+$ and by $-\sqrt{-1}$ on $\mathfrak{m}^-$; in fact $\mathfrak{m}^+ = \mathfrak{g}^{-1,1}$ and $\mathfrak{m}^- = \mathfrak{g}^{1,-1}$. Then as we have seen above, both $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are abelian Lie subalgebras of $\mathfrak{g}_\mathbb{C}$.

Let $M^+ := \exp(\mathfrak{m}^+)$, $M^- := \exp(\mathfrak{m}^-)$; both are abelian subgroups of $G_\mathbb{C}$. Let $K_\mathbb{C} := \exp(\mathfrak{k}_\mathbb{C})$ and $P_\mathbb{C} := \exp(\mathfrak{k}_\mathbb{C} + \mathfrak{m}^-) = K_\mathbb{C}M^-$. Then $P_\mathbb{C}$ is a subgroup of $G_\mathbb{C}$.

Here is a more precise version of (2.3.3), with G_c the real form of G from the end of §2.3.2.

Theorem 2.3.4 (Borel Embedding Theorem)**.** *The embedding $G_c < G(\mathbb{C})$ induces an isomorphism of real manifolds $G_c/K_\infty \simeq G(\mathbb{C})/P_\mathbb{C} = X^\vee$. The embedding $G < G(\mathbb{C})$ induces an open embedding*

$$X^+ = G^+/K_\infty \to X^\vee = G(\mathbb{C})/P_\mathbb{C},$$

realizing X^+ as an open subset (in the usual topology) of $X^\vee$.

We call $X^\vee$ the *compact dual* of X^+.

Theorem 2.3.5 (Harish–Chandra)**.** *The map*

$$F \colon M^+ \times K_{\infty,\mathbb{C}} \times M^- \to G_\mathbb{C}, \qquad (m^+, k, m^-) \mapsto m^+km^-$$

is a biholomorphism of of the left hand side onto an open subset of $G(\mathbb{C})$ containing G. As a consequence, the map

$$\eta \colon \mathfrak{m}^+ \to X^\vee = G(\mathbb{C})/P_\mathbb{C}, \qquad m^+ \mapsto \exp(m^+)P_\mathbb{C}$$

is a biholomorphism onto a dense open subset of $X^\vee$ containing X^+. Futhermore, $\mathcal{D} := \eta^{-1}(X^+)$ is a bounded symmetric domain in $\mathfrak{m}^+ \simeq \mathbb{C}^N$ and $\eta^{-1}(h) = 0$.

Example 2.3.6. *Let us continue with Example 2.3.1. The Harish–Chandra realization of Siegel upper-half space $\mathfrak{H}_d$, based at $\sqrt{-1}I_d$, is*

$$\mathcal{D}_d := \{Z \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : Z = Z^{\mathrm{t}} \text{ and } I_d - Z\overline{Z} > 0\},$$

and we have the Cayley transformation in this case

$$\mathfrak{H}_d \xrightarrow{\sim} \mathcal{D}_d, \quad \tau \mapsto (\tau - \sqrt{-1}I_d)(\tau + \sqrt{-1}I_d)^{-1} \tag{2.3.4}$$

with the inverse being $Z \mapsto \sqrt{-1}(Z + I_d)(-Z + I_d)^{-1}$.

Chapter 3

Shimura data and Shimura varieties

3.1 Basic definitions

3.1.1 Shimura data

Definition 3.1.1. *A* **Shimura datum** *is a pair* $(\mathbf{G}, X)$ *where*

- $\mathbf{G}$ *is a reductive group defined over* $\mathbb{Q}$*,*
- X *is a* $\mathbf{G}(\mathbb{R})$*-orbit in* $\mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$

such that for one (and hence all) $h \in X$*, we have*

(SV1) the Hodge structure $\mathrm{Ad} \circ h$ *on* $\mathrm{Lie}\mathbf{G}$ *has type* $(-1, 1) + (0, 0) + (1, -1)$*,*

(SV2) $\mathrm{Int}(h(\sqrt{-1}))$ *is a Cartan involution of* $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$*,*

(SV3) for every $\mathbb{Q}$*-simple factor* $\mathbf{H}$ *of* $\mathbf{G}^{\mathrm{ad}}$*, the morphism* $\mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \to \mathbf{H}_{\mathbb{R}}$ *is non-trivial.*

A **(Shimura) morphism** *between two Shimura data* $\rho\colon (\mathbf{G}', X') \to (\mathbf{G}, X)$ *is a morphism* ρ *on the underlying groups such that* $\rho \circ h \in X$ *for all* $h \in X'$*. In particular, we call the image of such a Shimura morphism to be a* **sub-Shimura datum** *of* $(\mathbf{G}, X)$*.*

The main difference of a Shimura datum and the pair (G, X^+) from §2.3 is the definition field of the group (over $\mathbb{Q}$ or over $\mathbb{R}$). A similar assumption to (SV3) for (G, X^+) has been discussed in Remark 2.3.3. By Theorem 2.3.1, each connected component X^+ of X is a Hermitian symmetric domain (and the complex structure on X is $\mathbf{G}(\mathbb{R})$-invariant). By Proposition 1.3.5, each representation V of $\mathbf{G}$ gives rise to a $\mathbb{Q}$-VHS on X^+ by (SV1), which furthermore carries $\mathbb{R}$-polarization by Proposition 2.2.6 and (SV2).[1]

The following two further assumptions guarantee that this $\mathbb{Q}$-VHS carries a $\mathbb{Q}$-polarization. Notice that they may not be satisfied by an arbitrary Shimura datum.

(SV4) the morphism $w_h\colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to Z(\mathbf{G})_{\mathbb{R}}$ is defined over $\mathbb{Q}$,

(SV2') $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of $\mathbf{G}_{\mathbb{R}}/w_h(\mathbb{G}_{\mathrm{m},\mathbb{R}})$.

Example 3.1.2 (0-dimensional Shimura datum)**.** *The set* X *is a finite set if and only if* $\mathbf{G}$ *is abelian (and hence an algebraic torus). This case shows up when we study CM abelian varieties.*

[1] (SV1) implies that $w_h\colon \mathbb{G}_{\mathrm{m}} \xrightarrow{w} \mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}}$ factors through $Z(\mathbf{G})_{\mathbb{R}}$, so we can apply Proposition 2.2.6.

Example 3.1.3 (Siegel Shimura datum). *Let us take the example of the Siegel case from Example 2.3.1, except that the vector space and the groups are defined over* $\mathbb{Q}$. *More precisely,* $V = \mathbb{Q}^{2d}$ *and* $\psi\colon V \times V \to \mathbb{Q}$ *is* $(x, y) \mapsto x^{\mathrm{t}}Jy$ *with* $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. *The* $\mathbb{Q}$*-group is*

$$\begin{aligned} \mathbf{GSp}(\psi) = \mathbf{GSp}_{2d} &:= \left\{ g \in \mathrm{GL}(V) : \psi(gx, gy) = c\psi(x, y) \text{ for some } c \in \mathbb{Q}^{\times} \right\} \\ &= \left\{ g \in \mathrm{GL}_{2d,\mathbb{Q}} : gJg^{\mathrm{t}} = cJ \text{ for some } c \in \mathbb{Q}^{\times} \right\}, \end{aligned}$$

and $h_0\colon \mathbb{S} \to \mathbf{GSp}_{2d,\mathbb{R}}$ *maps* $a + b\sqrt{-1} \mapsto aI_{2d} + bJ$. *The derived subgroup is* $\mathbf{Sp}(\psi) = \mathbf{Sp}_{2d}$ *by requesting the* $c \in \mathbb{Q}^{\times}$ *in the definition to be* 1.

The $\mathbf{G}(\mathbb{R})$*-orbit is* $\mathbf{GSp}_{2d}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{GSp}_{2d,\mathbb{R}})$. *Under the identification similar to* (2.3.1), *we have*

$$\mathbf{GSp}_{2d}(\mathbb{R})h_0 \xrightarrow{\sim} \mathfrak{H}_d^{\pm} := \left\{ \tau \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}},\ \text{either } \mathrm{Im}\tau > 0 \text{ or } \mathrm{Im}\tau < 0 \right\}. \tag{3.1.1}$$

Then $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$ *is a Shimura datum by Example 2.3.1 (see Remark 2.3.3 for (SV3)). It is called the* **Siegel Shimura datum**. *Moreover, (SV4) and (SV2') are easily seen to be also satisfied. In fact, V is a representation of* $\mathbf{GSp}_{2d}$, *and* ψ *is the desired* $\mathbb{Q}$*-polarization on the induced* $\mathbb{Q}$*-VHS.*

Next we present an example where (SV4) and (SV2') are not satisfied. We also see in this example that two Shimura data may have the same underlying $\mathbb{R}$-group and the same underlying space, but the $\mathbb{Q}$-groups are different.

Example 3.1.4 (Shimura curves). *Let B be a simple quaternion algebra over a totally real number field F. Assume that B is split at exactly one real place of F, i.e. there exists a real embedding* $\sigma\colon K \to \mathbb{R}$ *such that*

$$B_\sigma \simeq \begin{cases} \mathrm{M}_2(\mathbb{R}) & \text{if } \sigma = \sigma_0 \\ \mathbb{H} & \text{otherwise} \end{cases}$$

for all read embeddings $\sigma\colon K \to \mathbb{R}$, *where* $\mathbb{H}$ *is the Hamilton quaternion algebra over* $\mathbb{R}$.

Define the $\mathbb{Q}$*-group* $\mathbf{G}$

$$\mathbf{G}(R) := (B \otimes_{\mathbb{Q}} R)^{\times} \quad \text{for all } \mathbb{Q}\text{-algebra } R,$$

and let

$$h_0\colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \simeq \mathrm{GL}_{2,\mathbb{R}} \times \mathbb{H}^{\times} \times \cdots \times \mathbb{H}^{\times}, \qquad a + b\sqrt{-1} \mapsto \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, 1, \ldots, 1 \right).$$

Thus all but the first components of $\mathbf{G}(\mathbb{R})h_0$ *are the identity map, and so* $\mathbf{G}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$ *is isomorphic to* $\mathfrak{H}_1^{\pm}$, *via an isomorphism similar to* (3.1.1) *(with* $d = 1$*). Both (SV1) and (SV2) hold true for the pair* $(\mathbf{G}, \mathfrak{H}_1^{\pm})$ *similarly to the Siegel case. To see (SV3), it suffices to observe that* $\mathbf{G}^{\mathrm{ad}}$ *is a simple group because B is a simple quaternion algebra over F.*

So $(\mathbf{G}, \mathfrak{H}_1^{\pm})$ *is a Shimura datum. However, if* $[F : \mathbb{Q}] > 1$, *then (SV4) and (SV2') do not hold true, by looking at the action of* $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.

And even in the case $F = \mathbb{Q}$, *the group* $\mathbf{G}$ *is not necessarily* $\mathbf{GL}_2$. *So* $(\mathbf{G}, \mathfrak{H}_1^{\pm})$ *needs not be the Siegel Shimura datum in this case.*

3.1.2 Shimura varieties

Denote by $\mathbb{A}_\mathrm{f} \subseteq \prod_{p\in M_{\mathbb{Q},\mathrm{f}}} \mathbb{Q}_p$ the ring of finite adèles over $\mathbb{Q}$, and by $\widehat{\mathbb{Z}} := \prod \mathbb{Z}_p$. Then $\widehat{\mathbb{Z}}$ is a (maximal) compact open subgroup of $\mathbb{A}_\mathrm{f}$, and $\mathbb{Q}$ is dense in $\mathbb{A}_\mathrm{f}$.

Let $(\mathbf{G}, X)$ be a Shimura datum. Then $\mathbf{G}(\mathbb{Q})$ acts on X by definition of Shimura data, and consider the action of $\mathbf{G}(\mathbb{Q})$ on $\mathbf{G}(\mathbb{A}_\mathrm{f})$ by multiplication on the left.

Definition 3.1.5. *Let $(\mathbf{G}, X)$ be a Shimura datum. A* **Shimura variety** *is a double coset*

$$\mathrm{Sh}_K(\mathbf{G}, X) := \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_\mathrm{f})/K$$

where $K \subseteq \mathbf{G}(\mathbb{A}_\mathrm{f})$ is a compact open subset. Here $\mathbf{G}(\mathbb{Q})$ acts on both X and $\mathbf{G}(\mathbb{A}_\mathrm{f})$ on the left as above, and K acts on $\mathbf{G}(\mathbb{A}_\mathrm{f})$ by the multiplication on the right; i.e. $q(x, g)k = (q \cdot x, qgk)$ for all $q \in \mathbf{G}(\mathbb{Q})$, $(x, g) \in X \times \mathbf{G}(\mathbb{A}_\mathrm{f})$ and $k \in K$.

We will prove in this course that the double coset $\mathrm{Sh}_K(\mathbf{G}, X)$ is the set of $\mathbb{C}$-points of an algebraic variety. This justifies the name of Shimura variety.

Example 3.1.6. *In the Siegel example above, the group $\mathbf{GSp}_{2d}$ is defined over $\mathbb{Z}$; indeed we can take V to be $\mathbb{Z}^{2d}$ and ψ maps $V \times V$ to $\mathbb{Z}$. Then $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$ is a compact open subgroup of $\mathbf{GSp}_{2d}(\mathbb{A}_\mathrm{f})$. Other compact open subgroups include gKg^{-1} for any $g \in \mathbf{GSp}_{2d}(\mathbb{A}_\mathrm{f})$ and any finite-indexed subgroup K of $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$. We will come back to this example in §3.3 and prove that the Siegel Shimura varieties are moduli spaces of abelian varieties.*

Definition 3.1.7. *A* **(Shimura) morphism** *$[\rho]\colon \mathrm{Sh}_{K'}(\mathbf{G}', X') \to \mathrm{Sh}_K(\mathbf{G}, X)$ between two Shimura varieties is a morphism of Shimura data $\rho\colon (\mathbf{G}', X') \to (\mathbf{G}, X)$ such that $\rho(K') \subseteq K$.*

Example 3.1.8. *Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety.*

Let $K' \subseteq K$ be another compact open subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f})$. Then the identity map on $(\mathbf{G}, X)$ induces a Shimura morphism $\mathrm{Sh}_{K'}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$, with finite fibers since K' must have finite index in K. In fact, this is finite morphism in the category of algebraic varieties.

Let $g \in \mathbf{G}(\mathbb{A}_\mathrm{f})$. Then gKg^{-1} is a compact open subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f})$, and we have a Shimura morphism $[g\cdot]\colon \mathrm{Sh}_{gKg^{-1}}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$, sending $[x, g'] \mapsto [x, gg']$. More generally, if K' is a compact open subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f})$ such that $K' \subseteq gKg^{-1}$, then we have a Shimura morphism $[g\cdot]\colon \mathrm{Sh}_{K'}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$ which is a finite morphism.

Example 3.1.9 (Hecke operator)**.** *Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety.*

Any $g \in \mathbf{G}(\mathbb{A}_\mathrm{f})$ induces a correspondence on $\mathrm{Sh}_K(\mathbf{G}, X)$ as follows. Write $K' := K \cap gKg^{-1}$ for simplicity; it is a compact open subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f})$ and $K' \subseteq gKg^{-1}$. We have Shimura morphisms

$$\begin{tikzcd} & \mathrm{Sh}_{K'}(\mathbf{G}, X) \arrow[dl, "{[g\cdot]}"'] \arrow[dr, "{[1]}"] & \\ \mathrm{Sh}_K(\mathbf{G}, X) & & \mathrm{Sh}_K(\mathbf{G}, X) \end{tikzcd}$$

where the right one is induces by identity on $(\mathbf{G}, X)$. Both are finite morphisms, so we have a correspondence on $\mathrm{Sh}_K(\mathbf{G}, X)$, which is called the **Hecke correspondence/operator** *and denoted by T_g.*

Definition 3.1.10. *Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety. We call any irreducible component of $(T_g \circ [\rho])(\mathrm{Sh}_{K'}(\mathbf{G}', X'))$, where $[\rho]$ is a Shimura morphism and $g \in \mathbf{G}(\mathbb{A}_\mathrm{f})$, to be a* **special subvariety** *of $\mathrm{Sh}_K(\mathbf{G}, X)$. A special subvariety of dimension 0 is called a* **special point***.*

Of course in the definition of special subvarieties, it suffices to consider the Shimura morphisms arising from sub-Shimura data of $(\mathbf{G}, X)$. Thus special points arise from sub-Shimura data $(\mathbf{T}, X_\mathbf{T})$ of $(\mathbf{G}, X)$ where $\mathbf{T}$ is an algebraic torus.

3.2 Decomposition of Shimura varieties into Hermitian locally symmetric domains

Let $(\mathbf{G}, X)$ be a Shimura datum. Then any connected component X is a Hermitian symmetric domain. Fix one such X^+.

Let K be a compact open subgroup of $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$. Then we have a Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ defined as the double coset $\mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$. We wish to prove that this double coset is the $\mathbb{C}$-points of an algebraic variety.

In this section, we start with the first step, by endowing $\mathrm{Sh}_K(\mathbf{G}, X)$ with a structure of complex varieties.

Theorem 3.2.1. *There exists a finite-indexed subgroup K' of K such that*

$$\mathrm{Sh}_{K'}(\mathbf{G}, X) \simeq \bigsqcup_{g \in \mathcal{C}} \Gamma_g \backslash X^+, \tag{3.2.1}$$

for a finite set $\mathcal{C} \subseteq \mathbf{G}(\mathbb{A}_{\mathrm{f}})$, with each Γ_g a torsion-free discrete group acting on X^+.

The actual decomposition will be given later on (3.2.3), where the definitions of $\mathcal{C}$ and Γ_g are given. At this stage, let us make the following observation: since Γ_g is torsion-free discrete, the quotient $\Gamma_g\backslash X^+$ has a natural structure of complex manifolds and even more is a Hermitian locally symmetric domain. So $\mathrm{Sh}_{K'}(\mathbf{G}, X)$ is a finite disjoint union of Hermitian locally symmetric domains. As for $\mathrm{Sh}_K(\mathbf{G}, X)$, the finite-to-1 map $\mathrm{Sh}_{K'}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$ then makes $\mathrm{Sh}_K(\mathbf{G}, X)$ into a finite union of complex orbifolds.

3.2.1 Two approximation theorems for algebraic groups

Let $\mathbf{H}$ be an algebraic group defined over $\mathbb{Q}$. We will use the following approximation theorems.

- (*Real Approximation*) $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{R})$.
- (*Strong Approximation*) If $\mathbf{H}$ is semi-simple and simply-connected, then $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{A}_{\mathrm{f}})$.

The definition of "simply-connected" will be recalled later in §3.2.5.

3.2.2 Preparation and adjoint Shimura data

Now let us introduce the *adjoint Shimura datum* $(\mathbf{G}^{\mathrm{ad}}, \overline{X})$ of $(\mathbf{G}, X)$. Take $h \in X^+$. Then h induces a morphism

$$\overline{h} \colon \mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \to \mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}.$$

Hence we obtain a $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$-orbit $\overline{X} := \mathbf{G}^{\mathrm{ad}}(\mathbb{R})\overline{h}$ in $\mathrm{Hom}(\mathbb{S}, \mathbf{G}^{\mathrm{ad}}_{\mathbb{R}})$, with a natural map $X \to \overline{X}$. The image of X^+ in $\overline{X}$ is connected, and the following lemma (applied to $G = \mathbf{G}(\mathbb{R})$)[2] easily implies that this image is again a connected component of $\overline{X}$. So by abuse of notation, we will also use X^+ to denote a connected component of $\overline{X}$.

Lemma 3.2.2. *For any algebraic group G over $\mathbb{R}$, the adjoint quotient $G^+ \to (G^{\mathrm{ad}})^+$ is surjective when restricted to the identity component.*

[2]Here is a background for this lemma. Let $\varphi \colon H \to H'$ be a morphism of algebraic groups defined over k. Assume $\mathrm{char}(k) = 0$. Then φ is called *surjective* if $\varphi(H(\overline{k})) = H'(\overline{k})$. If φ is surjective, it may happen that $\varphi(H(k)) \neq H'(k)$!

We omit the proof of this lemma. Define

$$\begin{aligned}\mathbf{G}(\mathbb{R})_+ &:= \text{inverse image of } \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+ \text{ in } \mathbf{G}(\mathbb{R}) \\ \mathbf{G}(\mathbb{Q})_+ &:= \mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})_+. \end{aligned} \tag{3.2.2}$$

Lemma 3.2.3. $\mathbf{G}(\mathbb{R})_+$ *is the stabilizer of* X^+*, i.e.* $\mathbf{G}(\mathbb{R})_+ = \{g \in \mathbf{G}(\mathbb{R}) : gX^+ = X^+\}$.

With Lemma 3.2.3, we can complete our more precise version of (3.2.1):

$$\mathrm{Sh}_K(\mathbf{G}, X) \simeq \coprod_{[g]\in \mathbf{G}(\mathbb{Q})_+\backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K} \Gamma_g\backslash X^+, \tag{3.2.3}$$

with $\Gamma_g := gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$; replacing K by a suitable finite-indexed subgroup K' guarantees that Γ_g is torsion-free, see §3.2.4. The finiteness of the double coset $\mathbf{G}(\mathbb{Q})_+\backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$ will be proved in §3.2.5; the proof uses the *Strong Approximation Theorem.*

Proof of Lemma 3.2.3. Consider the action of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ on $\overline{X}$, and recall that X^+ is a connected component of $\overline{X}$. It suffices to prove that $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+ = \{g \in \mathbf{G}^{\mathrm{ad}}(\mathbb{R}) : gX^+ = X^+\}$. This follows from general theory of Hermitian symmetric domains (and some knowledge on $\mathbb{R}$-algebraic groups *v.s.* real Lie groups) which we will not cover in this course. $\square$

3.2.3 Proof of (3.2.3)

We start by showing that there is a bijection

$$\mathbf{G}(\mathbb{Q})_+\backslash X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}}) \xrightarrow{\sim} \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}}), \qquad [x, g] \mapsto [x, g]. \tag{3.2.4}$$

Injectivity: Assume $(x, g), (x', g') \in X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ are mapped to the same point on the right hand side. Then there exists $q \in \mathbf{G}(\mathbb{Q})$ such that $(x', g') = q(x, g) = (qx, qg)$. Hence $qX^+ \cap X^+$ is non-empty as it contains $qx = x'$. So $qX^+ = X^+$. So $q \in \mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})_+$. This proves the injectivity of the map above.

Surjectivity: Assume $(x, g) \in X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$. By the *Real Approximation* in §3.2.1, $\mathbf{G}(\mathbb{Q})x$ is dense in $\mathbf{G}(\mathbb{R})x = X$. So $\mathbf{G}(\mathbb{Q})x \cap X^+ \neq \emptyset$, and hence there exists $q \in \mathbf{G}(\mathbb{Q})$ such that $qx \in X^+$. Then $(qx, qg) \in X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$, and its image under (3.2.4) is $[x, g]$. We are done for the surjectivity of (3.2.3).

Now let us prove the bijectivity of the map

$$\coprod_{[g]\in \mathbf{G}(\mathbb{Q})_+\backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K} \Gamma_g\backslash X^+ \to \mathbf{G}(\mathbb{Q})_+\backslash X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K, \qquad \Gamma_g x \mapsto [x, g]. \tag{3.2.5}$$

Injectivity: If $[x', g'] = [x, g]$, then $(qx, qgk) = (x', g')$ for some $q \in \mathbf{G}(\mathbb{Q})_+$ and $k \in K$. So $[g] = [g']$ in $\mathbf{G}(\mathbb{Q})_+\backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$. Hence it suffices to prove the injectivity of $\Gamma_g\backslash X^+ \to \mathbf{G}(\mathbb{Q})_+\backslash X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$. Now if $[x', g] = [x, g]$, then $(qx, qgk) = (x', g)$ for some $q \in \mathbf{G}(\mathbb{Q})_+$ and $k \in K$. So $q = gk^{-1}g^{-1} \in gKg^{-1}$. So $q \in \Gamma_g = gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$. Thus we have proved the injectivity of (3.2.5).

Surjectivitity: Let $[x, g]$ be an element of the right hand side. Then it is the image of $\Gamma_g x$.

We have thus proved (3.2.3). $\square$

3.2.4 Torsion-free subgroup

Here is a choice of K' so that $gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ is torsion-free for all $g \in \mathbf{G}(\mathbb{A}_{\mathrm{f}})$. Take a faithful representation V of $\mathbf{G}$. Then there exists a lattice L in V such that $\widehat{L} := L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is fixed by K. Equivalently, we are embedding $\mathbf{G}$ as a closed subgroup of $\mathbf{GL}_N$ over $\mathbb{Q}$ such that K is a subgroup of $\mathbf{GL}_N(\widehat{\mathbb{Z}})$. Let $\ell \geq 3$ be an integer. Take K' to be the subgroup of K which acts trivially on $\widehat{L}/\ell\widehat{L}$, or equivalently

$$K' := \{g \in K < \mathbf{GL}_N(\widehat{\mathbb{Z}}) : g \equiv I_N \bmod \ell\}.$$

Then any element $\gamma \in \Gamma_g := gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+ < \mathrm{GL}(V)$ acts trivially on $\widehat{gL}/\ell\widehat{gL}$, so all the eigenvalues of γ are 1 (as they are 1 modulo $\ell \geq 3$). So $\gamma = 1$ if γ is torsion. So Γ_g is torsion-free.

3.2.5 The group of connected components of a Shimura variety

In this subsection, we prove the finiteness of the double coset $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$. This finishes the proof of Theorem 3.2.1, and shows that $\pi_0(\mathrm{Sh}_K(\mathbf{G}, X)) \simeq \mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$.

Case: simply-connected derived subgroup

The result in this case is better, with a clear understanding of the group $\pi_0(\mathrm{Sh}_K(\mathbf{G}, X))$. Consider the short exact sequence of $\mathbb{Q}$-groups

$$1 \to \mathbf{G}^{\mathrm{der}} \to \mathbf{G} \to \mathbf{T} := \mathbf{G}/\mathbf{G}^{\mathrm{der}} \to 1$$

with $\mathbf{T}$ an algebraic torus defined over $\mathbb{Q}$.

Definition 3.2.4. *An algebraic group H defined over a field k of characteristic* 0 *is said to be* **simply-connected** *if any central isogeny $H' \to H$ (i.e. a surjective morphism whose kernel is finite and contained in the center of H') is an isomorphism.*

Theorem 3.2.5. *Assume $\mathbf{G}^{\mathrm{der}}$ is simply-connected. Then $\nu(\mathbf{G}(\mathbb{Q})_+)$ has finite index in $\mathbf{G}(\mathbb{Q})$, $\nu(K)$ is a compact open subgroup of $\mathbf{T}(\mathbb{A}_{\mathrm{f}})$, and $\nu(\mathbf{G}(\mathbb{Q})_+)\backslash\mathbf{T}(\mathbb{A}_{\mathrm{f}})/\nu(K)$ is a finite abelian group. Moreover, ν induces a natural isomorphism of groups*

$$\pi_0(\mathrm{Sh}_K(\mathbf{G}, X)) \simeq \nu(\mathbf{G}(\mathbb{Q})_+)\backslash\mathbf{T}(\mathbb{A}_{\mathrm{f}})/\nu(K).$$

Before proving this theorem, we point out without proof that

$$\nu(\mathbf{G}(\mathbb{Q})_+) = \mathbf{T}(\mathbb{Q}) \cap \nu(Z(\mathbf{G})(\mathbb{R})) =: \mathbf{T}(\mathbb{Q})^{\dagger}. \tag{3.2.6}$$

Proof. General theory of semi-simple simply-connected $\mathbb{Q}$-groups asserts that $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ is connected. Therefore $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ stabilizes X^+ and hence is contained in $\mathbf{G}(\mathbb{R})_+$ by Lemma 3.2.3. So $\mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \subseteq \mathbf{G}(\mathbb{Q})_+$. By the *Strong Approximation Theorem* from §3.2.1, $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ is dense in $\mathbf{G}^{\mathrm{der}}(\mathbb{A}_{\mathrm{f}})$. Hence

$$\mathbf{G}^{\mathrm{der}}(\mathbb{A}_{\mathrm{f}}) = \mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \cdot (K \cap \mathbf{G}^{\mathrm{der}}(\mathbb{A}_{\mathrm{f}})) \subseteq \mathbf{G}(\mathbb{Q})_+ \cdot (K \cap \mathbf{G}^{\mathrm{der}}(\mathbb{A}_{\mathrm{f}})). \tag{3.2.7}$$

Because $\mathbf{G}^{\mathrm{der}}$ is simply-connected, the short exact sequence of groups above Theorem 3.2.5 induces a short exact sequence

$$1 \to \mathbf{G}^{\mathrm{der}}(\mathbb{A}_{\mathrm{f}}) \to \mathbf{G}(\mathbb{A}_{\mathrm{f}}) \xrightarrow{\nu} \mathbf{T}(\mathbb{A}_{\mathrm{f}}) \to 1.$$

Here we use the knowledge on semi-simple simply-connected $\mathbb{Q}$-groups that $H^1(\mathbb{Q}_p, \mathbf{G}^{\mathrm{der}}) = 0$ for any prime p.

Now ν induces a map

$$\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K \to \nu(\mathbf{G}(\mathbb{Q})_+)\backslash\mathbf{T}(\mathbb{A}_{\mathrm{f}})/\nu(K), \tag{3.2.8}$$

which, by (3.2.7), is a bijection. The right hand side is an abelian group because $\mathbf{T}$ is an algebraic torus (hence abelian).

Now to prove the theorem, it remains to prove:

(i) $\nu(\mathbf{G}(\mathbb{Q}))$ has finite index in $\mathbf{T}(\mathbb{Q})$.

(ii) $\nu(K)$ is a compact open subgroup of $\mathbf{T}(\mathbb{A}_{\mathrm{f}})$.

(iii) The right hand side of (3.2.8) is finite.

Let us prove (i). The Hasse Principle for simply-connected $\mathbb{Q}$-groups says that the natural map $H^1(\mathbb{Q}, \mathbf{G}^{\mathrm{der}}) \to \prod_{p\le\infty} H^1(\mathbb{Q}_p, \mathbf{G}^{\mathrm{der}}) = H^1(\mathbb{R}, \mathbf{G}^{\mathrm{der}})$ is injective; here we used again the fact that $H^1(\mathbb{Q}_p, \mathbf{G}^{\mathrm{der}}) = 0$ for any prime number p (as $\mathbf{G}^{\mathrm{der}}$ is furthermore semi-simple). So by the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \arrow[r] \arrow[d] & \mathbf{G}(\mathbb{Q}) \arrow[r] \arrow[d] & \mathbf{T}(\mathbb{Q}) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, \mathbf{G}^{\mathrm{der}}) \arrow[d, hook] \\
1 \arrow[r] & \mathbf{G}^{\mathrm{der}}(\mathbb{R}) \arrow[r] & \mathbf{G}(\mathbb{R}) \arrow[r] & \mathbf{T}(\mathbb{R}) \arrow[r] & H^1(\mathbb{R}, \mathbf{G}^{\mathrm{der}})
\end{tikzcd}$$

we get that $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q})) \to \mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$ is injective. But $\nu(\mathbf{G}(\mathbb{R})^+) = \mathbf{T}(\mathbb{R})^+$. So $\mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$ is finite, and hence $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q}))$ is finite. This establishes the claim.

For (ii), we extend $\mathbf{G} \to \mathbf{T}$ to a morphism of group schemes over $\mathbb{Z}[1/N]$ for some integer N, and prove that $\mathbf{G}(\mathbb{Z}_p) \to \mathbf{T}(\mathbb{Z}_p)$ is surjective for almost all prime p. We first work on $\mathbb{F}_p$ and then list using an argument similar to Newton's Lemma. We omit this proof.

Now we prove (iii). It suffices to prove that $\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A}_{\mathrm{f}})/\nu(K)$ is finite, and up to replacing $\nu(K)$ by a smaller compact open subgroup we may assume $\nu(K) \subseteq \mathbf{T}(\widehat{\mathbb{Z}})$. As $[\mathbf{T}(\widehat{\mathbb{Z}}) : \nu(K)]$ is finite (since $\mathbf{T}(\widehat{Z})$ is compact and $\nu(K)$ is open), it suffices to prove that

$$\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A}_{\mathrm{f}})/\mathbf{T}(\widehat{\mathbb{Z}})$$

is finite. This is exactly the *class group* of the algebraic torus $\mathbf{T}$ which is known to be finite by classical theory (and this number is called the *class number* of $\mathbf{T}$). In the case where $\mathbf{T} = \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_{\mathrm{m}}$ for a number field K, this is exactly the class group of K. $\square$

General case

Let $\widetilde{\mathbf{G}}$ be the universal cover of $\mathbf{G}^{\mathrm{der}}$, *i.e.* $\widetilde{\mathbf{G}}$ is simply-connected with a central isogeny (surjective with finite kernel contained in the center) $u\colon \widetilde{\mathbf{G}} \to \mathbf{G}^{\mathrm{der}}$. Then we have a surjective morphism of $\mathbb{Q}$-groups

$$\varphi\colon \mathbf{G}' := Z(\mathbf{G}) \times \widetilde{\mathbf{G}} \to \mathbf{G}, \qquad (z, g) \mapsto zu(g)$$

which is a central isogeny. Thus to prove the finiteness of $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$, it suffices to prove the finiteness of

$$\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_{\mathrm{f}})/K'$$

for K' a compact open subgroup of $\mathbf{G}'(\mathbb{A}_{\mathrm{f}})$. But the derived subgroup of $\mathbf{G}'$ is $\widetilde{\mathbf{G}}$ which is simply-connected. So we are back to the previous case, and hence $\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_{\mathrm{f}})/K'$ is finite. So $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$ is finite.

3.2.6 An upshot of Theorem 3.2.1 on special subvarieties

Special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)$ can be better understood via Theorem 3.2.1 as follows. Let S be a special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)$, arising from the sub-Shimura datum $(\mathbf{G}', X') \subseteq (\mathbf{G}, X)$ and the Hecke operator given by $g \in \mathbf{G}(\mathbb{A}_f)$. Then under the decomposition (3.2.1), S is the image of $u((X')^+)$ under the uniformization $u\colon X^+ \to \Gamma_g \backslash X^+$ for the suitable connected component $(X')^+$ of X'. Moreover, the sub-Shimura data can be constructed as follows. Take $h \in X$, and let $\mathrm{MT}(h)$ be the smallest $\mathbb{Q}$-subgroup of $\mathbf{G}$ such that $h\colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ factors through $\mathrm{MT}(h)_{\mathbb{R}}$. Then take $\mathbf{G}' := \mathrm{MT}(h)$ and $X' := \mathbf{G}'(\mathbb{R})h$.

3.3 Siegel modular variety

Take the example of Siegel case in Example 3.1.3 and Example 3.1.6. In particular $V = \mathbb{Q}^{2d}$, $\psi\colon V \times V \to \mathbb{Q}$ is $(x, y) \mapsto x^{\mathrm{t}} J y$ with $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. Thus there is a lattice L in V such that ψ restricts to $L \times L \to \mathbb{Z}$. To simplify notation, denote by $L = V(\mathbb{Z})$.

The Siegel Shimura datum is $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$. For each N, set

$$\begin{aligned} K(N) &:= \left\{ g \in \mathbf{GSp}_{2d}(\mathbb{A}_{\mathrm{f}}) : gV(\widehat{\mathbb{Z}}) \subseteq V(\widehat{\mathbb{Z}}) \text{ and acts trivially on } V(\widehat{\mathbb{Z}})/NV(\widehat{\mathbb{Z}}) \right\} \\ &= \left\{ g \in \mathbf{GSp}_{2d}(\widehat{\mathbb{Z}}) : g \equiv I_{2d} \pmod{N} \right\}. \end{aligned}$$

Then we have the Shimura variety $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$.

Theorem 3.3.1. *Assume $N \geq 3$. Then $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$ is the fine moduli space of principally polarized abelian varieties of dimension d with a level-N-structure, i.e. there is a canonical bijection between*

- *the $\mathbb{C}$-points of* $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$,
- *and the isomorphism classes of the triples (A, λ, η_N) where A is a complex abelian variety of dimension d, λ is a principal polarization on A, and η_N is a level-N-structure on A.*

When $N = 1, 2$, the Shimura variety is a coarse moduli space.

Let us explain the meaning of this theorem. Let A be an abelian variety defined over $\mathbb{C}$.

(i) A *principal polarization* on A is a polarization on the Hodge structure $H_1(A, \mathbb{Z})$ with determinant 1, *i.e.* an alternating pairing $\lambda\colon H_1(A, \mathbb{Z}) \times H_1(A, \mathbb{Z}) \to \mathbb{Z}$, which under suitable $\mathbb{Z}$-basis of $H_1(A, \mathbb{Z})$ is $\begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. In more geometric terms, it is an isomorphism $\lambda\colon A \xrightarrow{\sim} A^\vee$.

(ii) A *(symplectic) level-N-structure* on A is a basis of $H_1(A, \mathbb{Z}/N\mathbb{Z})$ which is symplectic with respect to λ. In more geometric terms, it is a basis of the $\mathbb{Z}/N\mathbb{Z}$-module $A[N]$ which is symplectic under $e_N\colon A[N] \times A[N] \xrightarrow{(1,\lambda)} A[N] \times A^\vee[N] \to \mu_N$ where last map is the Weil pairing. Or more concretely, it is an isomorphism

$$\eta_N\colon A[N] \xrightarrow{\sim} H_1(A, \mathbb{Z}/N\mathbb{Z})$$

such that the two composites

$$\begin{aligned} A[N]\times A[N] &\xrightarrow{(\eta_N,\eta_N)} H_1(A,\mathbb{Z}/N\mathbb{Z})\times H_1(A,\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\bar{\lambda}} \mathbb{Z}/N\mathbb{Z} \\ \text{and}\quad A[N]\times A[N] &\xrightarrow{e_N} \mu_N \xrightarrow{e^{2\pi\sqrt{-1}a/N}\mapsto[a]} \mathbb{Z}/N\mathbb{Z} \end{aligned}$$

differ from the multiplication by an element in $[\ell]\in(\mathbb{Z}/N\mathbb{Z})^\times$, and we say that this level-N-structure *has twist* $[\ell]$.

Proof. Recall that each point in $\mathfrak{H}_d^\pm$ parametrizes a $\mathbb{Q}$-Hodge structure on V of type $(-1,0)+(0,-1)$; see §2.3.1.

We shall use Theorem 3.2.1 and the more precise version (3.2.3), and better, Theorem 3.2.5 because $\mathbf{Sp}_{2d}$ is simply-connected. One can compute that $\mathbf{GSp}_{2d}(\mathbb{R})_+=\mathbf{GSp}_{2d}(\mathbb{R})^+=\{g\in\mathbf{GSp}_{2d}(\mathbb{R}):\det(g)>0\}$. So $\mathbf{GSp}_{2d}(\mathbb{Q})_+=\{g\in\mathbf{GSp}_{2d}(\mathbb{Q}):\det(g)>0\}$. Thus for the quotient

$$1\to\mathbf{Sp}_{2d}\to\mathbf{GSp}_{2d}\xrightarrow{\nu}\mathbb{G}_{\mathrm{m}}\to 1,$$

we have $\nu(\mathbf{GSp}_{2d}(\mathbb{Q})_+)=\mathbb{Q}_{>0}$.[3] It is not hard to compute that $\nu(K(N))=\{z\in\widehat{\mathbb{Z}}:z\equiv 1 \pmod N\}=1+N\widehat{\mathbb{Z}}$. Thus

$$\pi_0\left(\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d},\mathfrak{H}_d^\pm)\right)\simeq\mathbb{Q}_{>0}\backslash\mathbb{A}_{\mathrm{f}}^\times/(1+N\widehat{\mathbb{Z}})\simeq(\mathbb{Z}/N\mathbb{Z})^\times.$$

Write $\Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$ for the connected component of $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d},\mathfrak{H}_d^\pm)$ indexed by $[\ell]\in(\mathbb{Z}/N\mathbb{Z})^\times$. Below we only give the constructions of the two directions, without proving that they are inverse to each other.

Given a triple (A,λ,η_N). Assume that the level-N-structure has twist $[\ell]\in(\mathbb{Z}/N\mathbb{Z})^\times$. First $H_1(A,\mathbb{Z})$ is a $\mathbb{Z}$-Hodge structure of type $(-1,0)+(0,-1)$, and hence under suitable isomorphism $(H_1(A,\mathbb{Z}),\lambda)\simeq(V(\mathbb{Z}),\psi)$ we obtain a point $\tau\in\mathfrak{H}_d^+$. Then we get a point in $\Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$ as the image of τ under $\mathfrak{H}_d^+\to\Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$.

Conversely let $x\in\Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$. Let τ be a pre-image of x under the quotient $\mathfrak{H}_d^+\to\Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$. Recall that τ parametrizes a $\mathbb{Q}$-Hodge structure on V of type $(-1,0)+(0,-1)$, and thus we can endow $V(\mathbb{R})$ with a complex structure by the bijection $V(\mathbb{R})\subseteq V(\mathbb{C})\to V(\mathbb{C})/V_\tau^{0,-1}$. This makes $A_\tau:=V(\mathbb{R})/V(\mathbb{Z})$ into a compact complex torus of dimension d, with $H_1(A_\tau,\mathbb{Z})=V(\mathbb{Z})$. Thus ψ induces a principle polarization via $H_1(A_\tau,\mathbb{Z})$. Hence A_τ is an abelian variety with a principal polarization which by abuse of notation we still use ψ to denote. The level-N-structure on A_τ is given as follows. We have $A_\tau[N]=\frac{1}{N}V(\mathbb{Z})/V(\mathbb{Z})=V(\mathbb{Z})/NV(\mathbb{Z})=V(\mathbb{Z}/N\mathbb{Z})$. Take $g\in\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$ such that $\nu(g)\in\widehat{\mathbb{Z}}^\times$ is congruent to ℓ modulo $1+N\widehat{\mathbb{Z}}$. Then g induces an isomorphism $g\colon V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})\xrightarrow{\sim}V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})$. But $V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})=V(\mathbb{Z}/N\mathbb{Z})=H_1(A_\tau,\mathbb{Z}/N\mathbb{Z})$. Thus we have $A_\tau[N]=V(\mathbb{Z}/N\mathbb{Z})=V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})\xrightarrow{g}V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})=H_1(A_\tau,\mathbb{Z}/N\mathbb{Z})$. This is the desired level-N-structure because $\psi(gx,gy)=\nu(g)\psi(x,y)$ by definition of $\mathbf{GSp}_{2d}$. ☐

More generally, we can take any symplectic pairing ψ on V, *i.e.* $\psi\colon V\times V\to\mathbb{Q}$ is non-degenerate bilinear anti-symmetric. Then we have the symplectic group $\mathbf{GSp}(\psi)$ which is the subgroup of $\mathrm{GL}(V)$ preserving ψ (up to a number in $\mathbb{Q}^\times$) and a $\mathbf{GSp}(\psi)(\mathbb{R})$-orbit in $\mathrm{Hom}(\mathbb{S},\mathbf{GSp}(\psi)_\mathbb{R})$ which can still be identified with $\mathfrak{H}_d^\pm$. This gives a Shimura datum $(\mathbf{GSp}(\psi),\mathfrak{H}_d^\pm)$. The associated Shimura varieties are then moduli spaces of abelian varieties *polarized by* ψ of dimension d with suitable level structures.

[3] In fact $\nu(g)=(\det g)^{1/d}$.

Definition 3.3.2. *A Shimura variety is called a* **Siegel modular space** *if the associated Shimura datum is isomorphic to* $(\mathbf{GSp}(\psi), \mathfrak{H}_d^{\pm})$ *for some* ψ *and* d *as above.*

A Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ *is called* **of Hodge type** *if there exists an injective Shimura morphism* $(\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}(\psi), \mathfrak{H}_d^{\pm})$.

A Shimura variety is called **of abelian type** *if it admits a finite covering (as an algebraic variety) by a Shimura variety variety of Hodge type.*

By the construction in §3.2.6 and Proposition 1.2.12, Shimura varieties of Hodge type are moduli spaces of abelian varieties A with some prescribed Hodge tensors of $H_1(A, \mathbb{Q})$.

Shimura varieties of abelian type can be detected purely on the underlying group $\mathbf{G}$, and they may not parametrize abelian varieties. As an example, all Shimura varieties associated with the Shimura data from Example 3.1.4 are of abelian type, but they do not parametrize abelian varieties unless $F = \mathbb{Q}$.

3.4 CM abelian varieties and special points

Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety. In Definition 3.1.10 we defined *special points* on $\mathrm{Sh}_K(\mathbf{G}, X)$. They are of particular importance. For example, there exists a natural number field $E(\mathbf{G}, X)$, called the *reflex field* of $(\mathbf{G}, X)$, on which $\mathrm{Sh}_K(\mathbf{G}, X)$ is "naturally" defined (or in more vigorous terms, has a canonical model), characterized by the action of the Galois group of $E(\mathbf{G}, X)$. This action is explicitly defined for special points on $\mathrm{Sh}_K(\mathbf{G}, X)$ via the class field theory, and is uniquely determined in this way by the following theorem whose proof we omit:

Theorem 3.4.1. *The set of special points is dense in* $\mathrm{Sh}_K(\mathbf{G}, X)$.

Here "dense" is true even for the usual topology. The hard part of this theorem is to prove the existence of one special point. Indeed, assume $\mathrm{Sh}_K(\mathbf{G}, X) \simeq \bigsqcup \Gamma_g \backslash X^+$ has a special point $[x]$. Then its inverse image x in X^+ gives rise to a morphism $x \colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ which factors through $\mathbf{T}_{\mathbb{R}}$ for an algebraic torus $\mathbf{T} < \mathbf{G}$. But then the morphism given by $g \cdot x$ for any $g \in \mathbf{G}(\mathbb{Q})$ factors through $(g\mathbf{T}g^{-1})_{\mathbb{R}}$, with $g\mathbf{T}g^{-1}$ clearly an algebraic torus in $\mathbf{G}$ (since it is abelian), and hence defines a Shimura datum $(g\mathbf{T}g^{-1}, g \cdot \mathbf{T}(\mathbb{R})x)$. But $\mathbf{T}(\mathbb{R})x$ is a finite set of points since $\mathbf{T}$ is abelian. So the image of $\mathbf{G}(\mathbb{Q})x$ under the quotient $X^+ \to \Gamma_g \backslash X^+$ consists of special points of $\mathrm{Sh}_K(\mathbf{G}, X)$. Notice that $X^+ = \mathbf{G}(\mathbb{R})^+ x$. Now it suffice to use the Real Approximation that $\mathbf{G}(\mathbb{Q})$ is dense in $\mathbf{G}(\mathbb{R})$ to conclude.

For the existence of special points, we shall focus on the Siegel modular variety, for which we have:

Theorem 3.4.2. *Take* $[x] \in \mathrm{Sh}_K(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})(\mathbb{C})$. *Then* $[x]$ *is a special point if and only if the abelian variety* A_x *parametrized by* $[x]$ *is CM, i.e.* $\mathrm{End}(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$ *contains a commutative* $\mathbb{Q}$*-subalgebra of dimension* $2d$. *Equivalently, an abelian variety* A *defined over* $\mathbb{C}$ *is CM if and only if the Mumford–Tate group of the* $\mathbb{Q}$*-Hodge structure* $H_1(A, \mathbb{Q})$ *is an algebraic torus.*

We will not give a full proof of this theorem, but only recall the definition of CM abelian varieties and give a brief explanation why the associated Mumford–Tate group (which we call the Mumford–Tate group of A) is an algebraic torus.

Assume A is a simple abelian variety. Then A is CM if and only if $E := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a CM field, *i.e.* there exists a totally real field F such that E/F is a totally imaginary quadratic extension. Write $\overline{(\cdot)}$ for the complex conjugation with respect to E/F. Then there exists an element $\iota \in E$ such that $\bar{\iota} = -\iota$ (totally imaginary element). Then E can be endowed with the $\mathbb{Q}$-symplectic form

$$\langle x, y \rangle := \mathrm{Tr}_{E/\mathbb{Q}}(\overline{x} \iota y).$$

This makes $(E, \langle, \rangle) \simeq (\mathbb{Q}^{2d}, \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix})$ into a symplectic space. Set $\mathbf{GU}_E$ to be the subgroup of $\mathbf{GSp}_{2d}$ generated by $\mathbb{G}_{\mathrm{m}} = Z(\mathbf{GSp}_{2d})$ and

$$\mathbf{U}_E := \{x \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{\mathrm{m}} : x\overline{x} = 1\}.$$

Then one can check that $\mathbf{GU}_E$ is an algebraic torus which contains the Mumford–Tate group of A. Thus the Mumford–Tate group of A is abelian, and hence must be an algebraic torus. In fact, one can check that $\mathbf{GU}_E$ is a maximal torus of $\mathbf{GSp}_{2d}$.

Chapter 4

Boundary components

Starting from this chapter, we will discuss compactifications of Shimura varieties $\mathrm{Sh}_K(\mathbf{G}, X)$, or locally Hermitian symmetric spaces $\Gamma\backslash X^+$. This chapter introduces boundary components of X^+.

4.1 Example: modular curves

Consider the modular curves $\mathrm{Sh}_K(\mathbf{GL}_2, \mathfrak{H}^\pm)$, *i.e.* the Siegel modular variety from §3.3 with $d = 1$. In the particular case where $K = \mathbf{GL}_2(\widehat{\mathbb{Z}})$, we are working with

$$Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}.$$

It is a well-known result that $Y(1) \simeq \mathbb{C}$ via the j-function $j : \mathfrak{H} \to \mathbb{C}$. Hence a compactification of $Y(1)$ is $\mathbb{P}^1(\mathbb{C})$. This is the *Baily–Borel compactification* or the *toroidal compactification* of $Y(1)$ (but not the *Borel–Serre compactification*). In this section, we explain how to view this compactification as the *Baily–Borel compactification* of $Y(1)$. A large part is to study the *boundary components*, which is important for other compactifications we will discuss (*toroidal compactification* and *Borel–Serre compactification*).

4.1.1 Boundary components of $\mathfrak{H}$

The *boundary* of $\mathfrak{H}$ in $\mathbb{C} \cup \{\infty\}$ is the union of the real axis and $\{\infty\}$; in other words, the boundary of $\mathfrak{H}$ in $\mathbb{P}^1(\mathbb{C})$ is $\mathbb{P}^1(\mathbb{R})$. This is better seen via the Cayley transformation (2.3.4)

$$\mathfrak{H} \xrightarrow{\sim} \mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}, \qquad \tau \mapsto (\tau - \sqrt{-1})(\tau + \sqrt{-1})^{-1},$$

and the boundary of $\mathcal{D}$ is the unit circle. Denote by $\overline{\mathcal{D}}$ the closure of $\mathcal{D}$ in $\mathbb{C}$, *i.e.* $\overline{\mathcal{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $\partial\mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D}$. Then ∞ corresponds to $1 \in \overline{\mathcal{D}}$.

Call each point in $\partial\mathcal{D}$ a *boundary component* of $\mathcal{D}$. It is justified by the following fact: Any holomorphic map $\mathcal{D} \to \overline{\mathcal{D}}$ either has image in $\mathcal{D}$ or is constant.[1]

4.1.2 Extension of the group action to $\overline{\mathcal{D}}$

The group $\mathrm{GL}_2(\mathbb{R})^+$ acts on $\mathcal{D}$, via its action on $\mathfrak{H}$ and the Cayley transformation above, by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{(a - \sqrt{-1}c)(z + 1) + (b - \sqrt{-1}d)\sqrt{-1}(z - 1)}{(a + \sqrt{-1}c)(z + 1) + (b + \sqrt{-1}d)\sqrt{-1}(z - 1)}, \qquad \forall z \in \mathcal{D}.$$

[1] This is a consequence of the Open Mapping Theorem in complex analysis, which asserts that any holomorphic function on a connected set in the complex plane is open.

Lemma 4.1.1. *The action of* $\mathrm{GL}_2(\mathbb{R})^+$ *on* $\mathcal{D}$ *extends to* $\overline{\mathcal{D}}$*. Moreover, the action of* $\mathrm{GL}_2(\mathbb{R})^+$ *on* $\partial\mathcal{D}$ *is transitive.*

Proof. Take $z \in \overline{\mathcal{D}}$, and set

$$u_\pm := (a \pm \sqrt{-1}c)(z+1) + (b \pm \sqrt{-1}d)\sqrt{-1}(z-1).$$

For the first part of the lemma, we need to show that $u_+ \neq 0$ and $u_- u_+^{-1} \in \overline{\mathcal{D}}$.

Then

$$\begin{bmatrix} u_- \\ u_+ \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix},$$

and one can compute that

$$u_+\overline{u}_+ - u_-\overline{u}_- = 4(1 - z\overline{z}).$$

So $u_+\overline{u}_+ \geq u_-\overline{u}_-$ because $z \in \overline{\mathcal{D}}$. If $u_+ = 0$, then $u_+ = u_- = 0$, contradiction to $\operatorname{rank} \begin{bmatrix} u_- \\ u_+ \end{bmatrix} = \operatorname{rank} \begin{bmatrix} z \\ 1 \end{bmatrix} = 1$. So $u_+ \neq 0$, and $(u_- u_+^{-1})\overline{u_- u_+^{-1}} = \frac{u_-\overline{u}_-}{u_+\overline{u}_+} \leq 1$. Hence $u_- u_+^{-1} \in \overline{\mathcal{D}}$. We are done.

Let us prove the "Moreover" part. We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot 1 = \frac{a^2 - c^2}{a^2 + c^2} + \frac{-2ac}{a^2 + c^2}\sqrt{-1}.$$

The right hand side is easily checked to be in $\partial\mathcal{D} = \{z \in \mathbb{C} : |z| = 1\}$. Conversely any $z \in \partial\mathcal{D}$ can be written as the right hand side for some 2×2-matrix in $\mathrm{GL}_2(\mathbb{R})^+$. Hence we are done. $\square$

4.1.3 Compactifying at each boundary component

To see how to compactify $\mathcal{D} \simeq \mathfrak{H}$ at each boundary component, we need to study the stabilizer of each $z \in \overline{\mathcal{D}}$. Since $Z(\mathrm{GL}_2)(\mathbb{R})$ acts trivially on $\overline{\mathcal{D}}$, it suffices to consider the stabilizer in $\mathrm{SL}_2(\mathbb{R})$. By Lemma 4.1.1, it suffices to study this for $1 \in \overline{\mathcal{D}}$. For this purpose, it is easier to use the upper half plan. Define

$$P := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : b \in \mathbb{R},\ a \neq 0 \right\} \tag{4.1.1}$$

Then it is easy to check that $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(g \cdot \infty) = gP(\mathbb{R})^+ g^{-1}$ for any $g \in \mathrm{SL}_2(\mathbb{R})$. Indeed, it suffices to check this with $g = I_2$, and then it suffices to notice that elements on the right hand side of (4.1.1) correspond to translations along the real axis.

Lemma 4.1.2. *The followings hold true:*

(i) $\mathrm{SL}_2(\mathbb{C})/P(\mathbb{C})$ *is a projective space.*

(ii) For any $g \in \mathrm{SL}_2(\mathbb{R})$*, the group* gPg^{-1} *is defined over* $\mathbb{Q}$ *if and only if* $g \in \mathrm{SL}_2(\mathbb{Q})$*.*

(iii) Let $\tau \in \mathbb{P}^1(\mathbb{R})$ *(the boundary of* $\mathfrak{H}$ *in* $\mathbb{P}^1(\mathbb{C})$*). Then* $\tau \in \mathbb{P}^1(\mathbb{Q}) \Leftrightarrow \tau = g \cdot \infty$ *for some* $g \in \mathrm{SL}_2(\mathbb{Q}) \Leftrightarrow \tau = g \cdot \infty$ *for some* $g \in \mathrm{SL}_2(\mathbb{Z})$*.*

Proof. (ii) and (iii) are simple computations. For (i), it suffices to notice that the homogeneous space $\mathrm{SL}_2(\mathbb{C})/P(\mathbb{C}) \simeq \mathrm{GL}_2(\mathbb{C})/\left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{C},\ ad \neq 0 \right\}$ is the Grassmannian parametrizing 1-dimensional $\mathbb{C}$-subspaces in $\mathbb{C}^2$. $\square$

Let us go further. We have:

Lemma 4.1.3. *For each $g \in \mathrm{SL}_2(\mathbb{R})$, the group $gP(\mathbb{R})^+g^{-1}$ acts transitively on $\mathfrak{H}$.*

The proof itself is important. As a preparation, the group P has the following subgroups:

- The *unipotent radical* $N_P := \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{R} \right\}$, where elements act on $\mathfrak{H}$ as $\tau \mapsto \tau + b$.
- the *split torus* $A_P := \left\{ \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} : a > 0 \right\}$,[2] where elements act on $\mathfrak{H}$ as $\tau \mapsto a^{-2}\tau$.
- $M_P := \{\pm I_2\}$, which acts trivially on $\mathfrak{H}$.

such that

$$P = N_P A_P M_P \tag{4.1.2}$$

and the map $N_P \times A_P \times M_P \to P$, $(n, a, m) \mapsto nam$, is a diffeomorphism.

Proof. We only need to prove this lemma for P. For any $\tau = x + \sqrt{-1}y \in \mathfrak{H}$, we have

$$\tau = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{bmatrix} \sqrt{-1}.$$

Hence we are done. $\square$

Now we are ready to explain how the point ∞ is added to $\mathfrak{H}$ via the group P (in other words, how compactify $\mathfrak{H}$ at ∞). The decomposition 4.1.2 induces, by Lemma 4.1.3,

$$\mathfrak{H} \simeq P/(P \cap \mathrm{SO}(2)) = P/M_P \simeq N_P \times A_P \simeq \mathbb{R} \times \mathbb{R}_{>0}, \qquad \tau = x + \sqrt{-1}y \mapsto (x, \sqrt{y}^{-1}). \tag{4.1.3}$$

The A_P-factor is isomorphic to $\mathbb{R}_{>0}$, and a natural way to add a boundary to $\mathbb{R}_{>0}$ is to add 0 and make it into $\mathbb{R}_{\geq 0}$. In doing this, we are adding the point $x + \sqrt{-1}0^{-2} = \infty$ to $\mathfrak{H}$.

This process can be carried out for $g \cdot \infty \in \mathbb{P}^1(\mathbb{R})$ for any $g \in \mathrm{SL}_2(\mathbb{R})$, by replacing N_P and A_P by gN_Pg^{-1} and gA_Pg^{-1}. In this way, the point $g \cdot \infty \in \mathbb{P}^1(\mathbb{R})$ is added to $\mathfrak{H}$ by "compactifying" $gA_Pg^{-1} \simeq \mathbb{R}_{>0}$ into $\mathbb{R}_{\geq 0}$.

4.1.4 Rational vs real boundaries, and Siegel sets

We wish to compactify the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \simeq \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{D}$. The idea is to do the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\overline{\mathcal{D}}$, for the extended action of $\mathrm{SL}_2(\mathbb{R})$ on $\overline{\mathcal{D}}$ defined in Lemma 4.1.1. However, $\partial\mathcal{D} = \overline{\mathcal{D}} \setminus \mathcal{D} \simeq \mathbb{P}^1(\mathbb{R})$ contains infinitely many $\mathrm{SL}_2(\mathbb{Z})$-orbits.

A solution to this is to consider the *rational boundary components*, which are precisely the points in $\mathbb{P}^1(\mathbb{Q}) \subseteq \mathbb{P}^1(\mathbb{R})$. Equivalently by (ii) and (iii) of Lemma 4.1.2, a boundary component $z \in \partial\mathcal{D}$ is called a *rational boundary component* if its stabilizer in $\mathrm{SL}_2(\mathbb{R})$ is defined over $\mathbb{Q}$. Now part (iii) of Lemma 4.1.2 asserts that there is only one $\mathrm{SL}_2(\mathbb{Z})$-class of rational boundary components.

Another important notion is the *Siegel sets* associated with $P = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\infty)$ defined as follows; one needs this for example to pass from (partial) compactification of $\mathfrak{H}$ to compactification of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$. For each $t > 0$ and any compact bounded set $U \subseteq N_P \simeq \mathbb{R}$, define

$$\Sigma_{P,U,t} := U \times \{a \in \mathbb{R}_{>0} : a \leq t\} \simeq \{\tau = x + \sqrt{-1}y : x \in U,\ y \geq t^{-2}\} \subseteq \mathfrak{H}.$$

[2]Notice that A_P is not an algebraic subgroup of P, but only a Lie subgroup. This is a minor issue: Indeed, if we replace GL_2 by $\mathrm{PGL}_2 = \mathrm{SL}_2/\{\pm I_2\}$, then the quotient of A_P becomes an algebraic subgroup.

Then we have the following classical result on the j-function:[3] for a suitable U and suitable $t \gg 1$, $\Sigma_{P,U,t}$ is a fundamental set for the uniformization $j\colon \mathfrak{H} \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \simeq \mathbb{C}$ (*i.e.* $j|_{\Sigma_{P,U,t}}$ is surjective and has finite fibers). Then one can define the Siegel sets associated with $gPg^{-1} = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(g\cdot\infty)$ (for any $g \in \mathrm{SL}_2(\mathbb{R})$) to be $g \cdot \Sigma_{P,U,t}$.

We can also compactify $\Gamma\backslash\mathfrak{H}$ to be, as a set, $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$ for any finite-indexed subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$, by the following lemma which is a direct consequence of the discussion above.

Lemma 4.1.4. *(i) There are finitely many rational boundary components $\alpha_1,\dots,\alpha_n$ of $\mathfrak{H}$ such that $\mathbb{P}^1(\mathbb{Q}) = \bigcup_j \Gamma\cdot\alpha_j$.*

(ii) Let $P_j := \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\alpha_j)$. Then there are suitable Siegel sets Σ_j associated with P_j for $j \in \{1,\dots,n\}$ such that $\bigcup_j \Sigma_j$ is a fundamental set for the uniformization $u\colon \mathfrak{H} \to \Gamma\backslash\mathfrak{H}$.

4.1.5 Satake topology on $\overline{\mathcal{D}}$

This subsection is for the Baily–Borel compactification of $\Gamma\backslash\mathfrak{H}$. We will revisit the materials later in more generality.

Our desired compactification is $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$. We yet to explain the topology on this set, so that it is Hausdorff and compact. Notice that we cannot take the one induced by the usual topology on $\mathbb{C}$ because $x \in \mathbb{P}^1(\mathbb{Q})$ there are infinitely many $\gamma \in \Gamma$ which fixed x, and hence the quotient $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$ is not Hausdorff under this topology.

The topology which we consider is the Satake topology, induced from the Satake topology on $\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$ defined as follows. On $\mathfrak{H}$, the Satake topology is the usual topology, induced from $\mathbb{C}$. Next, an open neighborhood basis of ∞ consists of the open sets $U_t := \{z \in \mathfrak{H} : \mathrm{Im}(z) > t\}$ for all $t \geq 2$; equivalently a sequence $\tau_j = x_j + \sqrt{-1}y_j \in \mathfrak{H}$ converges to ∞ if and only if $y_j \to \infty$. Finally, an open neighborhood basis of $g\cdot\infty \in \mathbb{P}^1(\mathbb{Q})$ (with $g \in \mathrm{SL}_2(\mathbb{Q})$) consists of $g \cdot U_t$ for all $t \geq 2$. We state without proof the following assertions (whose proof needs to use Siegel sets):

(i) For any $x \in \mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$, there exists a fundamental system of neighborhoods $\{U\}$ of x such that

$$\gamma U = U,\ \forall\gamma\in\Gamma_x;\quad \gamma U \cap U = \emptyset,\ \forall\gamma\notin\Gamma_x$$

where $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$.

(ii) If $x, x' \in \mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$ are not in one Γ-orbit, then there exist neighborhoods U of x and U' of x' such that

$$\Gamma U \cap U' = \emptyset.$$

These properties guarantee that $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$ is Hausdorff under the Satake topology. The compactness follows easily from part (ii) of Lemma 4.1.4.

4.2 Parabolic subgroups and Levi subgroups: definitions and statements

For the simplest Siegel Shimura datum $(\mathbf{GL}_2, \mathfrak{H}^{\pm})$, Lemma 4.1.2.(i) suggests that parabolic subgroups of SL_2 (*i.e.* subgroups of SL_2 such that the homogeneous space $\mathrm{SL}_2(\mathbb{C})/P(\mathbb{C})$ is a projective variety) are closely related to the boundary components of $\mathfrak{H}$. This is true for an arbitrary Shimura datum $(\mathbf{G}, X)$.

In this section, we review background knowledge on parabolic subgroups of reductive groups over algebraically closed fields. In the next section, we do it over an arbitrary field.

Let k be a field, and let G be a reductive group defined over k. Let $\overline{k}$ be an algebraic closed field containing k. For our purpose, we will take $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\overline{k} = \mathbb{C}$.

[3] A well-known fundamental domain of the j-function is $\{z \in \mathbb{C} : |z| \geq 1,\ -1 \leq \mathrm{Re}(z) < 1\}$.

Definition 4.2.1. *A subgroup P of G is called a* **parabolic subgroup** *if the homogeneous space $G(\overline{k})/P(\overline{k})$ is a projective variety.*

It is a theorem of Chevalley that *parabolic subgroups are always connected.* We are more interested in the *proper* parabolic subgroups.

Example 4.2.2. *For $G = \mathrm{GL}_N$. Let P be the subgroup of upper triangular matrices in blocks (with the length of the ℓ-th diagonal block being n_ℓ). Then if we write $G = \mathrm{GL}(V)$ with $V \simeq k^N$, then P is the stabilizer of a flag $F^\bullet = (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{m-1} \subsetneq V_m = V)$ of subspaces of V, with $\dim V_\ell - \dim V_{\ell-1} = n_\ell$ for each ℓ. Hence G/P is a flag variety and hence is projective. So P is a parabolic subgroup of GL_N.*

Let P be a parabolic subgroup of G. The unipotent radical $\mathcal{R}_u(P)$ is a closed normal subgroup of P, and hence P acts on $\mathcal{R}_u(P)$ via conjugation. This induces an action of any subgroup of H on $\mathcal{R}_u(H)$.

Definition 4.2.3. *A* **Levi subgroup** *of P is a closed subgroup L of P such that $H = \mathcal{R}_u(P) \rtimes L$.*

A Levi subgroup, if exists, is then isomorphic to $P/\mathcal{R}_u(P)$ and hence is a reductive group (in particular is connected).

Theorem 4.2.4. *P has Levi subgroups, and any two Levi subgroups of P are conjugate by a unique element in $\mathcal{R}_u(P)$.*

We are more interested in more concrete constructions of Levi subgroups of P. This will be given in combinatorial data in the next two sections.

The following construction of parabolic subgroups of G is useful. Let λ be a cocharacter of G, *i.e.* a morphism of algebraic groups $\mathbb{G}_{\mathrm{m}} \to G$.

Theorem 4.2.5. *(i) The set*

$$P(\lambda) := \{x \in G : \lim_{t\to 0} \lambda(t)x\lambda(t)^{-1} \text{ exists}\}$$

is a parabolic subgroup of G, and the centralizer of $\lambda(\mathbb{G}_{\mathrm{m}})$ is a Levi subgroup of $P(\lambda)$. Moreover $\mathcal{R}_u(P(\lambda)) = \{x \in G : \lim_{t\to 0} \lambda(t)x\lambda(t)^{-1} = 1\}$.

(ii) Any parabolic subgroup of G is $P(\lambda)$ for some λ.

If $\lambda(\mathbb{G}_{\mathrm{m}}) < Z(G)$, then $P(\lambda) = G$. In fact, this theorem will serve as a bridge from the theory over algebraically closed fields to the theory over an arbitrary field. Indeed, if $k = \overline{k}$ then Theorem 4.2.5 follows easily from Theorem 4.3.14. Then we can descend to arbitrary k.

4.3 Parabolic subgroups via root systems: over algebraically closed fields

In this section, we take $k = \overline{k}$ to be an algebraically closed field, and G a reductive group defined over k. For our purpose, it is harmless to take $k = \mathbb{C}$. We will explain the combinatorial construction of parabolic subgroups of G, and Example 4.2.2 will be revisited in this language as Example 4.3.15.

Let $\mathfrak{g} := \mathrm{Lie}G$. Then we have the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ whose kernel is $Z(G)$. Notice that $Z(G)^\circ$ is an algebraic torus since G is reductive.

4.3.1 Root system for G

Let T be a maximal torus of G, *i.e.* an algebraic torus contained in G and maximal under the inclusion. For example if $G = \mathrm{GL}_N$, we can take $T = D_N$ to be the subgroup of diagonal matrices with non-zero diagonal entries. We have the standard properties:

Lemma 4.3.1. *(i) Any maximal torus of G equals gTg^{-1} for some $g \in G(\overline{k})$.*

(ii) $T = Z_G(T) = \{g \in G(\overline{k}) : gtg^{-1} = t$ for all $t \in T(\overline{k})\}$.

(iii) $W(T,G) := N_G(T)/T$ is finite and is called the **Weyl group**.

Thus $T \supseteq Z(G)^\circ$.

Now consider the action of T on $\mathfrak{g}$ via $T < G$ and the adjoint action. Let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_\mathrm{m})$ be the group of characters of T. For each $\alpha \in X^*(T)$, define $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : t \cdot x = \alpha(t)x$ for all $t \in T\}$ to be the eigenspace for α. Then we have a decomposition as in (1.2.2)

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi(T,G)} \mathfrak{g}_\alpha \tag{4.3.1}$$

where $\mathfrak{g}^T := \{x \in \mathfrak{g} : T \cdot x = x\}$ is the eigenspace for the trivial character, and $\Phi(T,G) \subseteq X^*(T) \setminus \{\text{trivial character}\}$ is the subset of non-trivial characters α of T such that $\mathfrak{g}_\alpha \neq 0$. By Lemma 4.3.1.(ii), we have $\mathfrak{g}^T = \mathfrak{t} := \mathrm{Lie}T$.

Denote for simplicity by $\Phi = \Phi(T,G)$. Elements in Φ are called *roots of* T. The following theorem, which gives combinatorial data associated with G and T, is extremely important in the theory of reductive groups.

Theorem 4.3.2. *(1) Φ generates a subgroup of finite index in $X^*(T/Z(G)^\circ) \subseteq X^*(T)$.*

(2) Let $\alpha \in \Phi$ and $\beta \in X^*(T)$ which is a multiple of α. Then $\beta \in \Phi \Leftrightarrow \beta = \pm\alpha$.

(3) Let $\alpha \in \Phi$, and set $G_\alpha := Z_G((\mathrm{Ker}\alpha)^\circ)$. Then

(a) $\dim \mathfrak{g}_\alpha = 1$, and there is a unique connected T-stable (unipotent) subgroup U_α of G such that $\mathrm{Lie}U_\alpha = \mathfrak{g}_\alpha$,[4]

(b) G_α is a reductive group and $\mathrm{Lie}G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$,[5] *and $G_\alpha^{\mathrm{ad}} \simeq \mathrm{PGL}_2$,*[6]

(c) the subgroup $W(T, G_\alpha)$ is $W(T,G)$ is generated by a reflection r_α such that $r_\alpha(\alpha) = -\alpha$.

(4) Let $\alpha \in \Phi$ and $r_\alpha \in W(T,G)$ be as in (3.c). Then for any $\beta \in \Phi$, we have

$$r_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$$

with $n_{\beta,\alpha} \in \mathbb{Z}$. Moreover, $n_{\alpha,\alpha} = 2$.

Thus Φ is a reduced root system in the vector space $E := X^*(T/Z(G)^\circ)_\mathbb{R}$ with Weyl group $W(T,G)$ in the sense below.

Definition 4.3.3. *Let E be a finite-dimensional real vector space with a Euclidean inner product $\langle , \rangle$. A* **root system** *Φ in E is a finite set of non-zero vectors (called* **roots***) such that:*

[4]Thus U_α is isomorphic to $\mathbb{G}_\mathrm{a}$ since it is a unipotent group of dimension 1.

[5]In other words, G_α is generated by T, U_α and $U_{-\alpha}$.

[6]Indeed, we can choose a generator X_α of $\mathfrak{g}_\alpha$ for each $\alpha \in \Phi$ such that X_α, $X_{-\alpha}$, $[X_\alpha, X_{-\alpha}]$ is an $\mathfrak{sl}_2$-triple for all $\alpha \in \Phi$.

(1) Φ spans E,

(2) If $\alpha, c\alpha \in \Phi$ for some $c \neq 0$, then $c \in \{1, -1, 1/2, -1/2\}$,

(3) For any $\alpha \in \Phi$, the set Φ is closed under the reflection through the hyperplane perpendicular to α (which we denote by r_α),

(4) For any $\alpha, \beta \in \Phi$, we have $r_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$ with $n_{\beta,\alpha} \in \mathbb{Z}$.

A root system is called **reduced** *if furthermore it satisfies:*

(2') The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are $\pm\alpha$.

We call $\dim E$ *the* **rank** *of Φ.*

The **Weyl group** *of Φ, denoted by $W(\Phi)$, is the group of* $\mathrm{Aut}(\Phi)$ *generated by r_α for all $\alpha \in \Phi$.*

Conversely, given a *root datum* (root system and "coroot system") one can associate a unique reductive group. We shall not go into details for this, but restrict our discussion to root systems. In practice, we often take G to be semi-simple, so that $\Phi(T, G)$ is a reduced root system in $X^*(T)_{\mathbb{R}}$.

Example 4.3.4. *Let $G = \mathrm{GL}_N$ and $T = D_N$. The Weyl group is isomorphic to the permutation group $\mathfrak{S}_N$. For each $j \in \{1, \ldots, N\}$, define $e_j \in X^*(D_N)$ to be $\mathrm{diag}(t_1, \ldots, t_N) \mapsto t_j$. Then we have an isomorphism $X^*(D_N) \simeq \bigoplus_{j=1}^N \mathbb{Z}e_j$. One can check that $\Phi(D_N, \mathrm{GL}_N) = \{e_i - e_j : i \neq j\}$.*

Highly related to this example is $G = \mathrm{SL}_N$ and $T = D_N \cap \mathrm{SL}_N$. Then $X^*(T) \simeq \bigoplus_{j=1}^N \mathbb{Z}e_j/\mathbb{Z}(e_1 + \ldots + e_N)$. And $\Phi(T, G)$ in this case is precisely the image of $\Phi(D_N, \mathrm{GL}_N)$ under the natural projection $X^*(D_N) \to X^*(T)$.

Example 4.3.5. *Let $G = \mathrm{Sp}_{2d}$ and $T = \mathrm{Sp}_{2d} \cap D_{2d} = \{\mathrm{diag}(t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}) : t_1 \cdots t_d \neq 0\}$. The Weyl group is isomorphic to $\{\pm 1\}^d \rtimes \mathfrak{S}_d$. For each $j \in \{1, \ldots, d\}$, define $e_j \in X^*(T)$ to be $\mathrm{diag}(t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}) \mapsto t_j$. Then $X^*(T) \simeq \bigoplus_{j=1}^d \mathbb{Z}e_j$. One can check that $\Phi(T, \mathrm{Sp}_{2d}) = \{\pm 2e_i, \pm e_i \pm e_j : 1 \leq i, j \leq d,\ i \neq j\}$.*

Root systems in Example 4.3.4 are called *of type A_{N-1}*, and root systems in Example 4.3.5 are called *of type C_d*. We also have root systems of type B_n (dual to C_n; coming from SO_{2n+1}) and D_n (coming from SO_{2n}), and exceptional types E_6, E_7, E_8, F_4, G_2). We will not go into details for this, but only point out that the last 3 types do not show up in the theory of Shimura varieties and that a Shimura variety is of abelian type unless the underlying group has $\mathbb{Q}$-factors of mixed type D or of exceptional types.

4.3.2 Positive roots and Borel subgroups

We start with the abstract theory of root systems $\Phi \subseteq E$.

Definition 4.3.6. *A* **basis** *of Φ is a subset Δ of Φ which is a basis of E such that each root $\beta \in \Phi$ is a linear combination $\beta = \sum_{\alpha \in \Delta} m_\alpha \alpha$ with $m_\alpha \in \mathbb{Z}$ of the same sign.*

Given a basis Δ of Φ, a root $\beta \in \Phi$ is said to be **positive (with respect to Δ)** *if $m_\alpha \geq 0$ for the decomposition above. Denote by Φ^+ the set of positive roots, and $\Phi^- := -\Phi^+$. Then $\Phi = \Phi^+ \sqcup \Phi^-$.*

A root $\alpha \in \Phi^+$ is said to be **simple** *if it is not the sum of two other positive roots.*

Lemma 4.3.7. *Δ is precisely the set of simple roots in Φ^+.*

In practice, one can start from a subset Φ^+ of Φ such that $\Phi = \Phi^+ \sqcup (-\Phi^+)$ and that $\alpha \in \Phi^+ \Rightarrow -2\alpha \notin \Phi^+$, and call these roots *positive*. Then we get a basis Δ consisting of simple roots in Φ^+, with respect to which Φ^+ is the set of positive roots. See Lemma 4.3.7.

Back to the theory of reductive groups, choosing Φ^+ is equivalently to taking a *Borel group*.

Definition 4.3.8. *A* **Borel group** *B is G is a closed connected solvable subgroup G, which is maximal for these properties.*

Example 4.3.9. *If $G = \mathrm{GL}_N$, then the subgroup T_N of upper triangular matrices is a Borel subgroup. Notice that T_N is a parabolic subgroup; see Example 4.2.2.*

Here are some basic properties of Borel subgroups. Part (iv) asserts that Borel subgroups are precisely the minimal parabolic subgroups (as we are working over $\overline{k}$).

Theorem 4.3.10. *(i) Any two Borel subgroups of G are conjugate.*

(ii) Every element of G lies in a Borel subgroup. And the intersection of all Borel subgroups of G is $Z(G)$.

(iii) (Lie–Kolchin) Assume $G < \mathrm{GL}_N$. Then there exists $x \in \mathrm{GL}_N(\overline{k})$ such that xGx^{-1} is contained in the subgroup of upper triangular matrices.

(iv) A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.

Back to our root system $\Phi(T, G)$ constructed from a maximal torus T of G. Let B be a Borel subgroup containing T. For each $\alpha \in \Phi(T, G)$, Theorem 4.3.2.(3) constructs a reductive group G_α with $\mathrm{Lie}G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$.

Theorem 4.3.11. *For each $\alpha \in \Phi(T, G)$, the intersection $B \cap G_\alpha$ is a Borel subgroup of G_α, and $\mathrm{Lie}(B \cap G_\alpha)$ is either $\mathfrak{t} \oplus \mathfrak{g}_\alpha$ or $\mathfrak{t} \oplus \mathfrak{g}_{-\alpha}$.*

Now define

$$\Phi^+(B) := \{\alpha \in \Phi(T, G) : \mathrm{Lie}(B \cap G_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha\}. \tag{4.3.2}$$

Then $\Phi(T, G) = \Phi^+(B) \sqcup (-\Phi^+(B))$ by Theorem 4.3.11. Thus we obtain the subset of positive roots determined by B, and the basis $\Delta(B)$ of $\Phi(T, G)$ consisting of simple (positive) roots in $\Phi^+(B)$ as below Lemma 4.3.7.

Conversely given any subset of positive roots Φ^+ of Φ, we can construct a subgroup B of G such that $\mathrm{Lie}B = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ (so that B is generated by T and U_α for all $\alpha \in \Phi^+$, with U_α from Theorem 4.3.2.(3a)).

Example 4.3.12. *In Example 4.3.4 with $(G, T) = (\mathrm{GL}_N, D_N)$, a set of positive roots is $\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq N\}$, and the corresponding basis is $\Delta = \{e_i - e_{i+1} : 1 \leq i \leq N-1\}$. The corresponding Borel subgroup is the subgroup of upper triangular matrices T_N.*

Example 4.3.13. *In Example 4.3.5 with $G = \mathrm{Sp}_{2d}$, a set of positive roots is $\Phi^+ = \{2e_i, e_i \pm e_j : 1 \leq i < j \leq d\}$, and the corresponding basis is $\{e_i - e_{i+1} : 1 \leq i \leq d-1\} \cup \{2e_d\}$. The corresponding Borel subgroup consists of upper triangular matrices.*

4.3.3 Standard parabolic subgroups

Consider the root system $\Phi = \Phi(T, G) \subseteq X^*(T)$ constructed from a maximal torus T in G. Let B be a Borel subgroup of G which contains T. Then B defines the set of positive roots $\Phi^+ = \Phi^+(B)$ as in (4.3.2) and hence the basis $\Delta = \Delta(B)$ of Φ. Recall that $\mathrm{Lie}B = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.

A parabolic subgroup of G is said to be *standard* (with respect to B) if it contains B. By parts (i) and (iv) of Theorem 4.3.10, every parabolic subgroup of G is conjugate to a standard one.

For any subset $I \subseteq \Delta$, denote by $\Phi_I \subseteq \Phi$ the set of roots which are linear combinations of roots in I. Let $\Phi_I^+ := \Phi^+ \cap I$. Then Φ_I is a root system in which Φ_I^+ is the set of positive roots and I is the corresponding basis. The Weyl group of Φ_I is the subgroup W_I of $W = W(T, G) = N_G(T)/T$ generated by the reflections r_α for all $\alpha \in I$.

We will use w to denote either an element in W or its representative in $N_G(T)$, whenever it is clear from the context. Then $C(w) := BwB$ is a subset of G, which by *Bruhat decomposition* satisfies: (a) $C(w)$ is a locally closed subvariety of G, (b) $G = \bigsqcup_{w\in W} C(w)$, (c) the closure $\overline{C(w)}$ is a union of certain $C(w')$.

Theorem 4.3.14. *(i) $P_I := \bigcup_{w\in W_I} BwB$ is a parabolic subgroup of G which contains B, with $\mathrm{Lie}P_I = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+\cup\Phi_I} \mathfrak{g}_\alpha$. In other words, P_I is generated by T and U_α for all $\alpha \in \Phi^+ \cup \Phi_I$, with U_α from Theorem 4.3.2.(3a).*

(ii) If P is a parabolic subgroup of G which contains B, then $P = P_I$ for a unique subset $I \subseteq \Delta$.

(iii) $\mathrm{Lie}\mathcal{R}_u(P_I) = \bigoplus_{\alpha\in\Phi^+\setminus\Phi_I} \mathfrak{g}_\alpha$.

(iv) Let L_I be the subgroup of G such that $\mathrm{Lie}L_I = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi_I} \mathfrak{g}_\alpha$. Then L_I is a Levi subgroup of P_I, i.e. is a reductive group contained in P_I such that $P_I = \mathcal{R}_u(P_I) \rtimes L_I$.

This theorem gives a combinatorial construction of all the standard parabolic subgroups of G: we add to Φ^+ the roots in Φ_I, and there is an inclusion-preserving bijection $I \mapsto P_I$ between subsets of Δ and standard parabolic subgroups. We have $P_\emptyset = B$, $P_\Delta = G$, and the maximal proper standard parabolic subgroups $P_{\Delta\setminus\{\alpha\}}$ for all $\alpha \in \Delta$. Moreover, if we define $T_I =: \left(\bigcap_{\alpha\in I} \mathrm{Ker}\alpha\right)^\circ$, then $L_I = Z_G(T_I)$. This is a more precise version of Theorem 4.2.4 for parabolic subgroups of reductive groups, when $k = \overline{k}$.

We can say more about the pieces $C(w) := BwB$ in Theorem 4.3.14. To ease notation, for any root $\alpha \in \Phi$ we shall write $\alpha > 0$ if $\alpha \in \Phi^+$ and $\alpha < 0$ if $\alpha \notin \Phi^+$.

For any $w \in W$, we can define a subset of Φ

$$\Phi(w)' := \{\alpha > 0 : w\alpha < 0\} = \{\alpha \in \Phi^+ : -w\alpha \in \Phi^+\}.$$

and define U'_w to be the subgroup of $U := \mathcal{R}_u(B)$ such that $\mathrm{Lie}U'_w = \bigoplus_{\alpha\in\Phi(w)'} \mathfrak{g}_\alpha$. Then the map $U'_w \times B \to G$, $(u, b) \mapsto uwb$ is an isomorphism of varieties.

Example 4.3.15. *In the Example 4.3.12 with $(G, T) = (\mathrm{GL}_N, D_N)$ and the Borel group being the subgroup of upper triangular matrices, the basis is $\Delta = \{e_i - e_{i+1} : 1 \le i \le N-1\}$ which identify with $\{1, \ldots, N-1\}$ (with $e_i - e_{i+1} \leftrightarrow i$). Take a subset $I \subseteq \Delta$ and write its complement*

$$\Delta \setminus I = \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{s-1}\}$$

with $a_j > 0$. Then P_I consists of upper triangular block matrices, with diagonal blocks of lengths $a_1, \ldots, a_{s-1}, a_s := N - \sum_{j=1}^{s-1} a_j$. And $L_I \simeq \mathrm{GL}_{a_1} \times \cdots \times \mathrm{GL}_{a_s}$ consists of diagonal block matrices, and $\mathcal{R}_u(P_I)$ consists of those matrices in P_I where the diagonal blocks are identity.

This is the combinatorial construction of Example 4.2.2.

The result for the Siegel case $G = \mathrm{Sp}_{2d}$ (corresponding to Example 4.3.13) will be given in later sections.

Remark 4.3.16. *Now, Theorem 4.2.5 in the case $k = \overline{k}$ follows easily from Theorem 4.3.14.*

4.4 Parabolic subgroups via root systems: over arbitrary fields

In this section, we take k to be a field, and G a reductive group defined over k. Then $Z(G)^\circ$ is an algebraic torus defined over k. Let $\mathfrak{g} := \mathrm{Lie}G$.

Let $\overline{k}$ be an algebraically closed field which contains k. For our purpose, it is harmless to take $k = \mathbb{Q}, \mathbb{R}$ and $\overline{k} = \mathbb{C}$.

By a *subgroup of* G, we mean a closed algebraic subgroup of G defined over k. In this section, we will discuss the combinatorial construction of parabolic subgroups of G, similar to the case $k = \overline{k}$.

4.4.1 Relative root systems

The first thing to do is to take a maximal torus T of $G_{\overline{k}}$ which is defined over k. It is known that such maximal tori always exist. But this is not enough, since characters of T may not be defined over k. We need:

Definition 4.4.1. *Let k'/k be an extension of fields. An algebraic torus A defined over k is said to be* k'**-split** *if $A_{k'} \simeq \mathbb{G}^r_{\mathrm{m},k'}$. Equivalently, A is k'-split if all characters of A are defined over k.*

Theorem 4.4.2. *(i) G contains a proper parabolic subgroup if and only if G contains a k-split torus which is not contained in $Z(G)$.*

(ii) Two maximal k-split tori contained in G are conjugate by an element of $G(k)$.

Here is a brief explanation to (i). Indeed, all parabolic subgroups of $G_{\overline{k}}$ are described by Theorem 4.2.5 using cocharacters, and having a parabolic subgroup of G (which by our convention means a parabolic subgroup defined over k) amounts to having a cocharacter of G which is defined over k.

Now take A to be a *maximal k-split torus* contained in G. Then $A_{\overline{k}}$ is contained in some maximal torus T of $G_{\overline{k}}$ defined over k. For each $\alpha' \in X^*(A)$, define $\mathfrak{g}_{\alpha'} := \{x \in \mathfrak{g} : a \cdot x = \alpha'(a)x \text{ for all } s \in A\}$ to be the eigenspace for α'. Then the adjoint action of $A < G$ on $\mathfrak{g}$ induces a decomposition of $\mathfrak{g}$ similar to (4.3.1)

$$\mathfrak{g} = \mathfrak{g}^A \oplus \sum_{\alpha' \in \Phi(A,G)} \mathfrak{g}_{\alpha'} \tag{4.4.1}$$

where $\Phi(A, G) \subseteq X^*(A) \setminus \{\text{trivial character}\}$ is the subset of non-trivial characters α' of A such that $\mathfrak{g}_{\alpha'} \neq 0$. The decomposition (4.4.1) is defined over k since all characters of A are defined over k.

Denote by ${}_k\Phi := \Phi(A, G)$.

Theorem 4.4.3. *${}_k\Phi$ is a root system, whose Weyl group is isomorphic to*

$${}_kW = W(A, G) := N_G(A)/Z_G(A).$$

Unlike the case $k = \overline{k}$, this root system may not be reduced. We call ${}_k\Phi$ the *relative root system* and ${}_kW$ the *relative Weyl group.*

Let us explain the analogue of G_α from Theorem 4.3.2.(3) in this relative setting. For any $\alpha' \in \Phi(A, G)$, the torus $S_{\alpha'} := (\mathrm{Ker}\alpha')^\circ$ is defined over k, and denote by $(\alpha') \subseteq \Phi(A, G)$ the subset consisting of rational multiples of α'. Then

Proposition 4.4.4. *There exists a unique closed connected unipotent k-subgroup $U_{(\alpha')}$ normalized by $Z_G(A)$ such that* $\mathrm{Lie}U_{(\alpha')} = \mathfrak{g}_{(\alpha')} := \sum_{\beta\in(\alpha')} \mathfrak{g}_\beta$.

The subgroup $G_{\alpha'} := Z_G(A_{\alpha'})$ is a reductive group defined over k, has S as a maximal k-split torus, and is generated by $Z_G(A)$ and $U_{(\alpha')}$.

4.4.2 Standard parabolic subgroups

Over $\overline{k}$, we have seen in §4.3.2 that choosing a basis of a root system (equivalently assigning the positive roots) amounts to fixing a Borel subgroup, and that Borel subgroups are precisely the minimal parabolic subgroups (Theorem 4.3.10.(iv)). Now over arbitrary k, we shall work with *minimal parabolic subgroups.*

Assign a subset ${}_k\Phi^+ = \Phi^+(A, G)$ of positive roots in ${}_k\Phi = \Phi(A, G)$, as below Lemma 4.3.7. Define

$$\mathfrak{n} := \sum_{\alpha'\in{}_k\Phi^+} \mathfrak{g}_{(\alpha')}. \tag{4.4.2}$$

It is a Lie subalgebra of $\mathfrak{g}$, and the corresponding subgroup N is unipotent and normalized by $Z_G(A)$. It is known that $P_0 := NZ_G(A)$ is a minimal parabolic subgroup of G, and every minimal parabolic subgroup of G which contains A is obtained in this way.

Now fix a minimal parabolic subgroup P_0 which contains A. A parabolic subgroup of G is said to be *standard* (with respect to P_0) if it contains P_0. As in the case $k = \overline{k}$, we have:

Theorem 4.4.5. *Every parabolic subgroup of G is conjugate, by an element in $G(k)$, to a unique standard parabolic subgroup.*

Let us construct the standard parabolic subgroups in combinatorial terms. Let ${}_k\Phi^+$ be the set of positive roots determined by P_0. Then we obtain a basis ${}_k\Delta$ of ${}_k\Phi$ as below Lemma 4.3.7.

For any subset $I \subseteq {}_k\Delta$, denote by ${}_k\Phi_I \subseteq {}_k\Phi$ the set of roots which are linear combinations of roots in I.

Let $A_I := \left(\bigcap_{\alpha'\in I} \mathrm{Ker}\alpha'\right)^\circ < A$. Then the group $L_I := Z_G(A_I)$ satisfies

$$\mathrm{Lie}L_I = \mathfrak{g}^A + \sum_{\alpha'\in{}_k\Phi_I} \mathfrak{g}_{(\alpha')}.$$

The Lie subalgebra of $\mathfrak{g}$

$$\mathfrak{n}_I := \sum_{\alpha'\in{}_k\Phi^+\setminus{}_k\Phi_I} \mathfrak{g}_{(\alpha')}$$

defines a unipotent subgroup N_I of G which is normalized by L_I, and we have:

Theorem 4.4.6. *The product $P_I := N_I \cdot L_I$ is a standard parabolic subgroup, with $N_I = \mathcal{R}_u(P_I)$ and L_I a Levi subgroup of P_I.*

Any standard parabolic subgroup of G equals P_I for some $I \subseteq {}_k\Delta$.

Moreover, observe that A_I a k-split torus, which is not contained in $Z(P_I)$. But A_I is the maximal k-split torus in $Z(L_I)$.

We close this subsection by the following immediate consequence of the construction above.

Lemma 4.4.7. *Assume $I \subseteq I' \subseteq {}_k\Delta$. Then $A_I > A_{I'}$ and $P_I < P_{I'}$.*

4.5 Horospherical decompositions and Siegel sets

Let $(\mathbf{G}, X)$ be a Shimura datum, and X^+ a connected component of X. We will use the following notation:

$$G = \mathbf{G}^{\mathrm{der}}_{\mathbb{R}}, \quad \mathfrak{g} := \mathrm{Lie}\mathbf{G}^{\mathrm{der}}, \quad \mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}G. \tag{4.5.1}$$

To ease notation, we will also *use X to denote X^+*.

We need to use maximal $\mathbb{Q}$-split (resp. $\mathbb{R}$-split) torus contained in $\mathbf{G}^{\mathrm{der}}$, for which we make the following definition.

Definition 4.5.1. *The* $\mathbb{Q}$**-rank** *(resp.* $\mathbb{R}$**-rank***) of an algebraic group H defined over $\mathbb{Q}$ is the dimension of the maximal $\mathbb{Q}$-split torus in H (resp. of the maximal $\mathbb{R}$-split torus in H), and is denoted by* $\mathrm{rk}_{\mathbb{Q}}H$ *(resp. by* $\mathrm{rk}_{\mathbb{R}}H$*).*

Theorem 4.5.2. *The followings are equivalent:*

(i) $\Gamma\backslash X$ *is compact for any arithmetic subgroup* Γ *of* $\mathbf{G}^{\mathrm{der}}$*;*

(ii) $\mathrm{rk}_{\mathbb{Q}}\mathbf{G}^{\mathrm{der}} = 0$*;*

(iii) $\mathbf{G}^{\mathrm{der}}$ *does not contain proper parabolic subgroups.*

The equivalence of (ii) and (iii) follows immediately from Theorem 4.2.5, and can be read off from the relative root system construction of parabolic subgroups.

Thus to discuss on compactifications of $\Gamma\backslash X$, we may assume $\mathrm{rk}_{\mathbb{Q}}\mathbf{G}^{\mathrm{der}} \geq 1$ and that $\mathbf{G}^{\mathrm{der}}$ contains proper parabolic subgroups. In this section, we discuss about the horospherical decomposition and Siegel sets associated with each proper parabolic subgroup $\mathbf{P}$.

4.5.1 Horospherical decompositions over $\mathbb{R}$

Let P be a parabolic subgroup of G. We start with the discussion for standard parabolic subgroups, for which we need to fix a maximal $\mathbb{R}$-split torus and a minimal parabolic subgroup of G. The general case will be reduced to the standard case by Theorem 4.4.5.

Fix $x_0 \in X$. Then (SV3) gives a Cartan involution θ of G which induces the Cartan decomposition (4.6.1)

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{m}.$$

Let $K_\infty := \exp(\mathfrak{k})$ which is a maximal compact subgroup of $G(\mathbb{R})^+$; see Lemma 2.3.2. Let $\mathfrak{a}$ to be a maximal Lie subalgebra contained in $\mathfrak{m}$.

Theorem 4.5.3. *There exists a maximal $\mathbb{R}$-split torus A in G such that* $\mathrm{Lie}A = \mathfrak{a}$*.*

Proof. First $\mathfrak{a}$ is abelian since $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a} \cap [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m} \cap \mathfrak{k} = 0$. Hence $\exp\colon \mathfrak{a} \to \exp(\mathfrak{a})$ is an isomorphism as Lie groups, and thus $\exp(\mathfrak{a}) \simeq (\mathbb{R}_{>0})^r \times \mathbb{R}^s$ (as Lie groups) for some $r, s \geq 0$. This gives rise to an $\mathbb{R}$-algebraic subgroup A_0 of G with $A_0(\mathbb{R})^+ = \exp(\mathfrak{a})$; indeed, $A_0(\mathbb{R}) \simeq (\mathbb{R}^\times)^r \times \mathbb{R}^s$.

We claim that $s = 0$. Indeed, for $\mathfrak{g}_{\mathbb{R},c} := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}$, we know that $\exp(\mathfrak{g}_{\mathbb{R},c})$ is a compact Lie group, and hence $\exp(\sqrt{-1}\mathfrak{a}) \simeq \mathbb{T}^r \times \mathbb{R}^s$ (with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) is compact, and hence $s = 0$.

Thus A_0 is an $\mathbb{R}$-split torus in G. It is contained in a maximal torus T of G defined over $\mathbb{R}$, and hence $T = A \cdot A'$ for some algebraic torus A' defined over $\mathbb{R}$. Then $\mathrm{Lie}A' \cap \mathfrak{m} = 0$ by the maximality of $\mathfrak{a}$ in $\mathfrak{m}$. One can choose A' such that $\mathrm{Lie}A' \subseteq \mathfrak{k}$, and then $A'(\mathbb{R}) < K_\infty$ which is compact. Hence A' has no $\mathbb{R}$-split factor; otherwise $\mathbb{R}^\times$ is a closed subset in $A'(\mathbb{R})$, contradiction to $A'(\mathbb{R})$ being compact. This finishes the proof. ∎

Thus we have the relative root system ${}_{\mathbb{R}}\Phi := \Phi(A, G)$ as below (4.4.1). Assign a subset ${}_{\mathbb{R}}\Phi^+$ of positive roots in ${}_{\mathbb{R}}\Phi$ as below Lemma 4.3.7. It defines a basis ${}_{\mathbb{R}}\Delta$ of ${}_{\mathbb{R}}\Phi$ (as below Lemma 4.3.7) and a minimal parabolic subgroup P_0 of G (as below (4.4.2)).

Remark 4.5.4. *An alternative approach to study the theory over $\mathbb{R}$ is to use Cartan's theory of symmetric spaces and the restricted root system (to $\mathbb{R}$). We shall not take this point of view in our course to have a uniform treatment over $\mathbb{R}$ and over $\mathbb{Q}$, but only point out that these two points of view are equivalent for our study by the the following easy observation: $\mathfrak{g}^A \cap \mathfrak{m} = \mathfrak{a}$.*

Standard parabolic subgroups. Any parabolic subgroup P of G which contains P_0 is of the form P_I for some subset $I \subseteq {}_{\mathbb{R}}\Delta$, where P_I is defined in Theorem 4.4.6. Now P_I has unipotent radical N_I and a Levi subgroup $L_I = Z_G(A_I)$. Moreover, $A < Z(L_I)$ since $A_I < A$. It is not hard to construct a θ-stable subgroup M_I of L_I such that $L_I = A \times M_I$ (inner direct product).[7] Then we have the following *real Langlands decomposition based at $x_0 \in X^+$*

$$P_I(\mathbb{R})^+ = N_I(\mathbb{R})A_I(\mathbb{R})^+M_I(\mathbb{R}) \simeq N_I(\mathbb{R}) \times A_I(\mathbb{R})^+ \times M_I(\mathbb{R}) \tag{4.5.2}$$

where the first equality is as groups, and the second isomorphism is in the category of real algebraic manifolds (the inverse map is $(n, a, m) \mapsto nam$).

We have more. The reductive subgroup M_I is θ-stable, and thus $K_{I,\infty} := M_I \cap K_\infty$ is maximal compact in $M_I(\mathbb{R})^+$. So

$$X_I := M_I(\mathbb{R})^+/K_{I,\infty} = P_I(\mathbb{R})^+/K_{I,\infty}A_I(\mathbb{R})^+N_I(\mathbb{R}) \tag{4.5.3}$$

is a symmetric space, called the *boundary symmetric space* associated with P_I. Notice however X_I may not admit an $M_I(\mathbb{R})^+$-invariant complex structure.

Lemma 4.5.5. *$P_I(\mathbb{R})^+$ acts transitively on X.*

Proof. It is not hard to check that $\mathfrak{g}_{\mathbb{R}} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ with $\mathfrak{n}$ from (4.4.2). Thus $G = NAK_\infty$, which is called the *Iwasawa decomposition* of G. On the other hand, $\mathfrak{n} \oplus \mathfrak{a} \subseteq \mathrm{Lie}P_I$ by construction of P_I. Hence $NA \subseteq P_I$ and we are done. $\square$

Thus $X = P_I(\mathbb{R})^+x_0$, and by (4.5.2) and (4.5.3) (and $\mathfrak{n}_I \cap \mathfrak{k} = 0$) we then have the following *real horospherical decomposition based at $x_0 \in X$*

$$X \simeq N_I(\mathbb{R}) \times A_I(\mathbb{R})^+ \times X_I \tag{4.5.4}$$

where the isomorphism is in the category of real algebraic manifolds.

General parabolic subgroups. Now let P be an arbitrary parabolic subgroup of G. By Theorem 4.4.5, P is conjugate to a unique standard parabolic subgroup P_I for some $I \subseteq {}_{\mathbb{R}}\Delta$. But $G = NAK_\infty$ and $NA \subseteq P_I$. So there exists $k \in K_\infty$ such that $P = kP_Ik^{-1}$. Define

$$N_P := kN_Ik^{-1} = \mathcal{R}_u(P), \quad A_P := kA_Ik^{-1}, \quad M_P := kM_Ik^{-1}.$$

Then both A_P and M_P are θ-stable, and $L_P := A_PM_P$ is a Levi subgroup of P, and A_P is an $\mathbb{R}$-split torus in P. We have the *real Langlands decomposition (based at $x_0 \in X$)*

$$P(\mathbb{R})^+ = N_P(\mathbb{R})A_P(\mathbb{R})^+M_P(\mathbb{R}) \simeq N_P(\mathbb{R}) \times A_P(\mathbb{R})^+ \times M_P(\mathbb{R}) \tag{4.5.5}$$

which induces *real horospherical decomposition (based at $x_0 \in X$)*

$$X \simeq N_P(\mathbb{R}) \times A_P(\mathbb{R})^+ \times X_P \tag{4.5.6}$$

with $X_P := M_P(\mathbb{R})/(M_P \cap K_\infty)$ called the *boundary symmetric space associated with P*.

[7]One can construct using Lie algebras: $\mathrm{Lie}M_I$ is the direct sum of $\mathrm{Lie}Z_G(A) \cap \mathfrak{k}$, $\sum_{\alpha' \in {}_k\Phi_I} \mathfrak{g}_{(\alpha')}$ and the (orthogonal) complement of $\mathrm{Lie}A_I$ in $\mathfrak{a}$.

4.5.2 Horospherical decompositions over $\mathbb{Q}$

Before the discussion over $\mathbb{Q}$, let us state the following result.

Let $\mathbf{A}$ be a maximal $\mathbb{Q}$-split torus in $\mathbf{G}^{\mathrm{der}}$. Then we obtain a relative root system ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{A}, \mathbf{G}^{\mathrm{der}})$ as below (4.4.1). Assign a subset ${}_{\mathbb{Q}}\Phi^+$ of positive roots and thus get a basis ${}_{\mathbb{Q}}\Delta$ of ${}_{\mathbb{Q}}\Phi$ as below Lemma 4.3.7. Then we get a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$ as below (4.4.2). All standard parabolic subgroups (*i.e.* those containing $\mathbf{P}_0$) are of the form $\mathbf{P}_I$ for some $I \subseteq {}_{\mathbb{Q}}\Delta$, and every parabolic subgroup $\mathbf{P}$ of $\mathbf{G}^{\mathrm{der}}$ is conjugate to a unique $\mathbf{P}_I$ by some element in $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$; see Theorem 4.4.6. We have the unipotent radical $\mathbf{N}_I$ of $\mathbf{P}_I$ and a Levi subgroup $\mathbf{L}_I = Z_{\mathbf{G}^{\mathrm{der}}}(\mathbf{A}_I)$ with $\mathbf{A}_I := \left(\bigcap_{\alpha' \in {}_{\mathbb{Q}}\Phi_I} \mathrm{Ker}\alpha'\right)^{\circ} < \mathbf{A}$. Moreover, $\mathbf{A}_I$ is the maximal $\mathbb{Q}$-split torus in $Z(\mathbf{L}_I)$. Notice that for $P_I = \mathbf{P}_{I,\mathbb{R}}$, our $\mathbf{A}_{I,\mathbb{R}}$ is a subgroup of the A_I constructed in the real case (which is the maximal $\mathbb{R}$-split torus in $Z(\mathbf{L}_{I,\mathbb{R}})$) and is *proper* if $\mathrm{rk}_{\mathbb{Q}}\mathbf{P}_I < \mathrm{rk}_{\mathbb{R}}\mathbf{P}_I$. So we need to further decompose A_I into the product of $\mathbf{A}_{I,\mathbb{R}}$ and an $\mathbb{R}$-algebraic torus $A_I^{\perp}$ whose $\mathbb{Q}$-rank is 0.[8] For this purpose, define $\mathbf{M}_I := \bigcap_{\chi} \mathrm{Ker}\chi^2$ where χ runs over all non-trivial $\mathbf{L}_I \to \mathbb{G}_{\mathrm{m}}$. Then $\mathbf{M}_I$ is a reductive group with $\mathrm{rk}_{\mathbb{Q}} Z(\mathbf{M}_I) = 0$. Then we have $\mathbf{L}_I = \mathbf{A}_I \mathbf{M}_I$ and $A_I = \mathbf{A}_{I,\mathbb{R}} A_I^{\perp}$. Denote by $\Delta(\mathbf{A}_I, \mathbf{P}_I) := {}_{\mathbb{Q}}\Delta \setminus I$.

For an arbitrary parabolic subgroup $\mathbf{P}$ of $\mathbf{G}^{\mathrm{der}}$, we can conjugate $\mathbf{P}$ to be a unique standard parabolic subgroup $\mathbf{P}_I$. Then we obtain the unipotent radical $\mathbf{N}_{\mathbf{P}}$ of $\mathbf{P}$, the Levi subgroup $\mathbf{L}_{\mathbf{P}}$ of $\mathbf{P}$, the maximal $\mathbb{Q}$-split torus $\mathbf{A}_{\mathbf{P}}$ in $Z(\mathbf{L}_{\mathbf{P}})$, and the subgroup $\mathbf{M}_{\mathbf{P}} = \bigcap_{\chi} \mathrm{Ker}\chi^2$ of $\mathbf{L}_{\mathbf{P}}$. Denote by $P := \mathbf{P}_{\mathbb{R}}$, $N_P := \mathbf{N}_{\mathbf{P},\mathbb{R}}$, $L_P := \mathbf{L}_{\mathbf{P},\mathbb{R}}$,

$$A_{\mathbf{P}} := \mathbf{A}_{\mathbf{P},\mathbb{R}}, \qquad M_{\mathbf{P}} := \mathbf{M}_{\mathbf{P},\mathbb{R}}. \tag{4.5.7}$$

Then we are in conformity with the notation in the real case, while $A_{\mathbf{P}}$ is a subgroup of A_P which is proper if $\mathrm{rk}_{\mathbb{Q}}\mathbf{P} < \mathrm{rk}_{\mathbb{R}}\mathbf{P}$. Denote by

$$\Delta(A_{\mathbf{P}}, P) \subseteq X^*(\mathbf{A}_{\mathbf{P}}) \tag{4.5.8}$$

to be the conjugate of ${}_{\mathbb{Q}}\Delta \setminus I$.

Now we have the *rational Langlands decomposition* of $\mathbf{P}$

$$P(\mathbb{R})^+ = N_P(\mathbb{R}) A_{\mathbf{P}}(\mathbb{R})^+ M_{\mathbf{P}}(\mathbb{R}) \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times M_{\mathbf{P}}(\mathbb{R}) \tag{4.5.9}$$

where the second isomorphism is in the category of real algebraic manifolds.

To get the rational horospherical decomposition, we need to fix a point $x_0 \in X$ and the associated Cartan involution θ on G, and require $A_{\mathbf{P}}$ and $M_{\mathbf{P}}$ to be θ-stable. To achieve this, we can work with the Levi quotient $\mathbf{P}/\mathbf{N}_{\mathbf{P}}$ instead of working with the Levi subgroup $\mathbf{L}_{\mathbf{P}}$ of $\mathbf{P}$, and then lift the resulting $\mathbf{A}_{\mathbf{P}}$ and $\mathbf{M}_{\mathbf{P}}$ to the $\mathbb{R}$-Levi subgroup L_P of P (the one constructed in the real case) which is θ-stable. The resulting groups may not be $\mathbb{Q}$-subgroups of $\mathbf{P}$, but this is enough for our purpose.

Remark 4.5.6. *In fact, it is known that for any $\mathbf{P}$, there exists a base point $x_1 \in X$ such that they are still defined over $\mathbb{Q}$.*

Let $K_\infty := \mathrm{Stab}_{G(\mathbb{R})^+}(x_0)$. Then $M_{\mathbf{P}} \cap K_\infty$ is maximal compact in $M_{\mathbf{P}}(\mathbb{R})^+$ by the θ-stability of $M_{\mathbf{P}}$. Now (4.5.5) induces the *rational horospherical decomposition* of $X = P(\mathbb{R})^+ x_0$

$$X \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}} \tag{4.5.10}$$

[8] That is, there is no non-trivial subgroup of $A_I^{\perp} < G = \mathbf{G}_{\mathbb{R}}^{\mathrm{der}}$ which is defined over $\mathbb{Q}$.

with $X_{\mathbf{P}} = M_{\mathbf{P}}(\mathbb{R})^+/(M_{\mathbf{P}} \cap K_\infty)$ called the *boundary symmetric space associated with* $\mathbf{P}$. Moreover, let $A_{\mathbf{P}}^{\perp}$ be the orthogonal complement of $A_{\mathbf{P},\mathbb{R}}$ in A_P, *i.e.* $A_{\mathbf{P}}^{\perp}$ is θ-stable with $A_P(\mathbb{R})^+ = \mathbf{A}_{\mathbf{P}}(\mathbb{R})^+ \times A_{\mathbf{P}}^{\perp}(\mathbb{R})^+$. Then

$$X_{\mathbf{P}} \simeq X_P \times A_{\mathbf{P}}^{\perp}(\mathbb{R})^+, \qquad A_P(\mathbb{R})^+ = A_{\mathbf{P}}(\mathbb{R})^+ \times A_{\mathbf{P}}^{\perp}(\mathbb{R})^+. \tag{4.5.11}$$

While $A_{\mathbf{P}}^{\perp}$ has $\mathbb{Q}$-rank 0, taking the quotient by Γ will roll up the fact $A_{\mathbf{P}}^{\perp}(\mathbb{R})^+$ into circles and hence does not interfere with the compactification of $\Gamma\backslash X$.

4.5.3 Siegel sets

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$. Continue to use the notation in the previous subsections. For $t > 0$, define

$$A_{\mathbf{P},t} := \{a \in A_{\mathbf{P}}(\mathbb{R})^+ : \alpha'(a) \geq t^{-1} \text{ for all } \alpha' \in \Delta(A_{\mathbf{P}}, P)\} \tag{4.5.12}$$

with $\Delta(A_{\mathbf{P}}, P)$ defined by (4.5.8).

Definition 4.5.7. *For any bounded sets $U \subseteq N_P(\mathbb{R})$ and $V \subseteq X_{\mathbf{P}}$, the subset*

$$S_{\mathbf{P},U,V,t} := U \times A_{\mathbf{P},t} \times V \subseteq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}} \simeq X$$

is called a **Siegel set** *in X associated with* $\mathbf{P}$.

4.6 Analytic boundary components

In this section focus on our discussion over $\mathbb{R}$ instead of $\mathbb{Q}$. The discussion over $\mathbb{Q}$ will be executed in Chapter 6.

Let (G, X^+) be a pair as in §2.3, *i.e.* G is a reductive group defined over $\mathbb{R}$ and X^+ is a $G(\mathbb{R})^+$-orbit contained in $\mathrm{Hom}(\mathbb{S}, G)$ satisfying conditions (i) and (ii) at the beginning of §2.3.[9]

To ease notation, we will replace G by G^{der}, so that G is from now on a semi-simple algebraic group defined over $\mathbb{R}$. We will also use X to denote X^+. We have shown that X is a Hermitian symmetric domain; see Theorem 2.3.1.

Denote by $\mathfrak{g} = \mathrm{Lie}G$.

It is known that under holomorphic isometry, X is isomorphic to an open bounded subset $\mathcal{D}$ in the affine space $\mathbb{C}^N$ where $N = \dim_{\mathbb{C}} X$; we shall review this *Harish–Chandra realization* later on at the end of §4.6.1. Let $\overline{\mathcal{D}}$ be the closure of $\mathcal{D}$ in $\mathbb{C}^N$ under the usual topology (we sometimes denote it by $\overline{X}^{\mathrm{BB}}$).

Definition 4.6.1. *An* **analytic boundary component** *of X is an equivalence class in $\overline{\mathcal{D}}$ under the equivalence relation generated by $x \sim y$ if there exists a holomorphic map $\rho\colon \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}^N$ such that $x, y \in \mathrm{Im}(\rho) \subseteq \overline{\mathcal{D}}$.*

Notice that $\mathcal{D}$ is a boundary component of X by definition. This definition generalizes the case where X is the upper half plane, in view of the last sentence of §4.1.1.

It is clear that $\overline{\mathcal{D}}$ is the disjoint union of its analytic boundary components. We shall prove:

[9]For our purpose, (G, X^+) are obtained as follows. Let $(\mathbf{G}, X)$ be a Shimura datum. Decompose $\mathbf{G}_{\mathbb{R}} = Z(\mathbf{G})_{\mathbb{R}} G_1 \cdots G_k$ for simple $\mathbb{R}$-groups $G_1, \ldots, G_k$, which induces a decomposition $X^+ \simeq X_1^+ \times \cdots \times X_k^+$. Then we can take our G to be $Z(\mathbf{G})_{\mathbb{R}} \prod_{j \in J} G_j$ and our new X^+ to be $\prod_{j \in J} X_j^+$, for any subset $J \subseteq \{1, \ldots, k\}$.

Theorem 4.6.2. *The action of $G(\mathbb{R})^+$ on $X \simeq \mathcal{D}$ extends to $\overline{\mathcal{D}}$. For any analytic boundary component $F \neq X$ of X, its normalizer*

$$N(F) := \{g \in G(\mathbb{R})^+ : gF = F\}$$

is a proper parabolic subgroup of the Lie group $G(\mathbb{R})^+$, which means that it equals $P_F(\mathbb{R}) \cap G(\mathbb{R})^+$ for a parabolic subgroup P_F of G.

Morever if X is irreducible as a Hermitian symmetric domain, then P_F is maximal proper parabolic in G. And the association $F \mapsto P_F$ defines a bijection between the set of analytic boundary components of X and the set of proper maximal parabolic subgroups of G.

In the statement, X is irreducible as a Hermitian symmetric domain if and only if X cannot be written as the product of two non-trivial Hermitian symmetric domains, or equivalently G is a simple group. [10]

In fact, we will prove a more precise version of Theorem 4.6.2, describing how P_F is constructed in terms of the root systems. Moreover, we will prove that the analytic boundary component F can be identified with the boundary symmetric space (defined below the real horospherical decomposition (4.5.6)) associated with some parabolic subgroup P'_F, and explain the relation of P_F and P'_F.

Before moving on, let us make the following very brief discussion over $\mathbb{Q}$; more details will be given in Chapter 6. Let us temporarily go back to our pair (G, X^+) at the beginning of this section (*i.e.* the pair from §2.3).

Definition 4.6.3. *Assume there exists a Shimura datum $(\mathbf{G}, X)$ such that $G = \mathbf{G}_{\mathbb{R}}$ and that X^+ is a connected component of X. Then an analytic boundary component F of X^+ is called* **rational** *if the parabolic subgroup P_F is defined over $\mathbb{Q}$.*

As hinted by Theorem 4.5.2, only the rational analytic boundary components should account for the compactification of $\Gamma\backslash X$. We will focus on the discussion of any analytic boundary component in this section, while in the end give a characterization of which ones are rational.

4.6.1 Complex structure on X and the Harish–Chandra realization

Take $x_0 \in X$ which corresponds to $h_0 \colon \mathbb{S} \to G$, and let $\theta = h_0(\sqrt{-1})$ be the Cartan involution on G given by condition (ii) at the beginning of §2.3. We thus have the Cartan decomposition (2.3.2)

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{4.6.1}$$

with $\mathfrak{k}$ (resp. $\mathfrak{m}$) be the eigenspace for 1 (resp. for -1). Notice that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$ by looking at the eigenvalues.

Then $K_\infty := \exp(\mathfrak{k})$ is a maximal compact subgroup of $G(\mathbb{R})^+$ by Lemma 2.3.2, and the real tangent space of X at x_0, denoted by $T_{\mathbb{R}}X$, can be identified as $\mathfrak{m}$.

The element $J := h_0(e^{\pi\sqrt{-1}/4})$ satisfies $J^2 = 1$. The action of J on X induces a decomposition of $\mathfrak{m}_{\mathbb{C}} = T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$

$$\mathfrak{m}_{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^- \tag{4.6.2}$$

where J acts by multiplication by $\sqrt{-1}$ on $\mathfrak{m}^+$ and by $-\sqrt{-1}$ on $\mathfrak{m}^-$. Thus the holomorphic tangent space of X at x_0 can be identified with $\mathfrak{m}^+$. Moreover, as J acts on $T_{\mathbb{R}}X = \mathfrak{m}$, we have $J \in \exp(\mathfrak{k}) = K_\infty$, and thus $J \in Z(K_\infty)$.

[10] In general, $X \simeq X_1 \times \cdots \times X_n$ decomposes into the product of irreducible Hermitian symmetric domains, and analytic boundary components of X are precisely the products of the analytic boundary components of the X_j's. Then one can also obtain a description of the bijective association $F \mapsto P_F$.

Let us recall the *Harish–Chandra realization/embedding* $\mathcal{D}$ of X in Theorem 2.3.5. We only need a brief version: We can identify X with an open bounded subset $\mathcal{D}$ of $\mathfrak{m}^+$. The identification $X \simeq \mathcal{D}$ is called the *Harish–Chandra realization* and the inclusion $\mathcal{D} \subseteq \mathfrak{m}^+$ is called the *Harish–Chandra embedding*. Moreover, it is known that there exists an open holomorphic map $\mathfrak{m}^+ \to X^\vee$ which embeds $\mathfrak{m}^+$ as an open subset (in the usual topology) of the complex algebraic variety $X^\vee$. So we can summarize into:

$$X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \subseteq X^\vee. \tag{4.6.3}$$

Example 4.6.4. *In the Siegel case* $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ *and the base point* $x_0 = \sqrt{-1}I_{2d}$, *we have* $K_\infty = U(d) = O(2d) \cap \mathrm{Sp}_{2d}$ *(and* $G = \mathrm{Sp}_{2d}$*). In this case,* $\mathfrak{m}^+ \simeq \{\tau \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}}\}$, *and the Harish–Chandra realization is* $\mathcal{D}_d := \{Z \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : Z = Z^{\mathrm{t}}$ *and* $I_d - Z\overline{Z} > 0\}$ *as in Example 2.3.6.*

4.6.2 Complex roots and the Polydisc Theorem

Let T be a maximal torus of G contained in K_∞. Consider the root system $\Phi := \Phi(T, G_{\mathbb{C}})$. We have the root decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

with each $\mathfrak{g}_\alpha$ having dimension 1.

We say that a root α is *compact* (resp. *non-compact*) if $\mathfrak{g}_\alpha \subseteq \mathfrak{k}_{\mathbb{C}}$ (resp. if $\mathfrak{g}_\alpha \subseteq \mathfrak{m}_{\mathbb{C}}$). Let Φ_K be the set of compact roots and Φ_M be the set of non-compact roots. One can check that $\Phi = \Phi_K \cup \Phi_M$.

Lemma 4.6.5. *There exists a choice of positive roots* Φ^+ *such that*

$$\mathfrak{m}^+ = \bigoplus\nolimits_{\alpha\in\Phi^+\cap\Phi_M} \mathfrak{g}_\alpha.$$

The proof uses the complex structure on X, or more precisely the action of J on $\mathfrak{m}_{\mathbb{C}}$. One can show that $J\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$ for any non-compact root α.

Definition 4.6.6. *Two roots* $\alpha, \beta \in \Phi$ *are called* **strongly orthogonal** *if* $\alpha \pm \beta$ *are not roots.*

From now on, we fix a maximal subset of strongly orthogonal roots in $\Phi^+ \cap \Phi_M$, maximal under inclusions

$$\Psi = \{\alpha_1, \ldots, \alpha_r\}. \tag{4.6.4}$$

This can be done be choosing successively the lowest positive root.

For each $\alpha \in \Psi$, choose a non-zero $e_\alpha \in \mathfrak{g}_\alpha$ and set $e_{-\alpha}$ to be the complex conjugation on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of e_α. Then $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$, and $h_\alpha := \sqrt{-1}[e_\alpha, e_{-\alpha}] \in \mathfrak{t} \subseteq \mathfrak{k}$ and is non-zero. Set

$$\mathfrak{g}_{\mathbb{C}}[\alpha] := \mathbb{C}h_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} = \mathbb{C}h_\alpha + \mathbb{C}e_\alpha + \mathbb{C}e_{-\alpha}. \tag{4.6.5}$$

Then $\mathfrak{g}_{\mathbb{C}}[\alpha] \simeq \mathfrak{sl}_{2,\mathbb{C}}$, since $[h_\alpha, e_\alpha] = \alpha(\exp(h_\alpha))e_\alpha$ and $[h_\alpha, e_{-\alpha}] = -\alpha(\exp(h_\alpha))e_{-\alpha}$ are both non-zero. Hence

$$\mathfrak{g}[\alpha] := \mathfrak{g}_{\mathbb{C}}[\alpha] \cap \mathfrak{g} = \mathbb{R}h_\alpha + \mathbb{R}x_\alpha + \mathbb{R}y_\alpha \simeq \mathfrak{sl}_{2,\mathbb{R}}$$

where $x_\alpha := e_\alpha + e_{-\alpha}$ and $y_\alpha := \sqrt{-1}(e_\alpha - e_{-\alpha})$. Notice that $Jx_\alpha = y_\alpha$ and $Jy_\alpha = -x_\alpha$ by Lemma 4.6.5, and $\{x_\alpha, y_\alpha : \alpha \in \Phi^+ \cap \Phi_M\}$ is an $\mathbb{R}$-basis of $\mathfrak{m}$.

For each $\alpha \in \Psi$, let $G[\alpha]$ be the subgroup of G such that $\mathrm{Lie}G[\alpha] = \mathfrak{g}[\alpha]$. Let $G[\Psi]$ be the subgroup of G with $\mathrm{Lie}G[\Psi] = \sum_{\alpha\in\Psi} \mathfrak{g}[\alpha]$.

Theorem 4.6.7 (Polydisc Theorem). *The orbit $G[\Psi](\mathbb{R})^+x_0 \subseteq X$ is a totally geodesic submanifold which is isomorphic to a Poincaré polydisc D^r, and $X = \bigcup_{k\in K_\infty} k \cdot D^r$.*[11]

Recall that $\mathfrak{g}[\alpha] \simeq \mathfrak{sl}_{2,\mathbb{R}}$ for all $\alpha \in \Psi$. Hence the inclusion $G[\Psi](\mathbb{R})^+x_0 \subseteq X$ is induced by a morphism

$$\varphi\colon \mathrm{SL}_2(\mathbb{R})^r \to G(\mathbb{R}). \tag{4.6.6}$$

By general theory on holomorphic maps between bounded symmetric domains, φ is the second factor of a morphism $\widetilde{\varphi}\colon U(1)\times\mathrm{SL}_2(\mathbb{R})^r \to G(\mathbb{R})$ satisfying: $\left(e^{\sqrt{-1}\theta}, \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \dots, \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) \mapsto h_0(e^{\sqrt{-1}\theta})$.

The Polydisc Theorem is a key step in the proof of the Harish–Chandra embedding. To study boundary components, we need to have a finer statement. Let $S \subseteq \{1, \dots, r\}$ be a subset, and let $G[S]$ be the subgroup of G with $\mathrm{Lie}G[S] = \sum_{j\in S} \mathfrak{g}[\alpha_j]$; in particular $G[S] = G[\Psi]$ for $S = \{1, \dots, r\}$. Then the orbit $G[S](\mathbb{R})^+x_0 \subseteq X$ is still totally geodesic in X and is isomorphic to $D^{|S|}$. We have the following compatibility:

Theorem 4.6.8. *For each $j \in \{1, \dots, r\}$, the image of $G[j](\mathbb{R})^+x_0 \subseteq X$ under the Harish–Chandra embedding is the open unit disc D_j in $\mathbb{C}e_{\alpha_j} \subseteq \mathfrak{m}^+$ (with $1 \in D_j$ corresponding to e_{α_j}). The image of $G[S](\mathbb{R})^+x_0 \subseteq X$ under the Harish–Chandra embedding is the open unit polydisc $\prod_{j\in S} D_j$ in $\prod_{j\in S} \mathbb{C}e_{\alpha_j} \subseteq \mathfrak{m}^+$.*

We finish this subsection by the example of the Siegel case.

Example 4.6.9. *In the Siegel case $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ and the base point $x_0 = \sqrt{-1}I_{2d}$, we have $K_\infty = U(d) = O(2d) \cap \mathrm{Sp}_{2d}$ (and $G = \mathrm{Sp}_{2d}$). Our maximal torus is not the usual one, but is*

$$T = \left\{ \mathrm{bdiag}(t_1, \dots, t_d) := \begin{bmatrix} \cos t_1 & & & \sin t_1 & & \\ & \ddots & & & \ddots & \\ & & \cos t_d & & & \sin t_d \\ -\sin t_1 & & & \cos t_1 & & \\ & \ddots & & & \ddots & \\ & & -\sin t_d & & & \cos t_d \end{bmatrix} : t_1, \dots, t_d \in \mathbb{R} \right\}.$$

Let $\lambda_j \in X^*(T)$ be $\mathrm{bdiag}(t_1, \dots, t_d) \mapsto e^{\sqrt{-1}t_j}$. Then $\Phi = \{\pm\sqrt{-1}(\lambda_i + \lambda_j) : 1 \le i \le j \le d\} \cup \{\pm\sqrt{-1}(\lambda_i - \lambda_j) : 1 \le i < j \le d\}$ and $\Phi_M \cap \Phi^+ = \{\sqrt{-1}(\lambda_i + \lambda_j) : 1 \le i \le j \le d\}$. The basis for this choice of Φ^+ is $\{\sqrt{-1}(\lambda_i - \lambda_{i+1}) : 1 \le i \le d-1\} \cup \{2\sqrt{-1}\lambda_d\}$.

The set Ψ is $\{\alpha_j := 2\sqrt{-1}\lambda_j : 1 \le j \le d\}$ (so $r = d$). Then the corresponding normalized e_{α_j}, h_{α_j}, x_{α_j}, y_{α_j} are:

$$\frac{1}{2}\begin{bmatrix} 1_{j,j} & \sqrt{-1}_{d+j,j} \\ \sqrt{-1}_{j,d+j} & -1_{g+j,g+j} \end{bmatrix}, \begin{bmatrix} 0 & -\sqrt{-1}_{d+j,j} \\ \sqrt{-1}_{j,d+j} & 0 \end{bmatrix}, \begin{bmatrix} 1_{j,j} & 0 \\ 0 & -1_{g+j,g+j} \end{bmatrix}, \begin{bmatrix} 0 & -1_{d+j,j} \\ -1_{j,d+j} & 0 \end{bmatrix}.$$

Here for a number s, we use $s_{i,j}$ to denote the matrix with the (i,j)-entry being s and all the rest being 0.

[11]The Poincaré unit disc D is $\{z \in \mathbb{C} : |z| < 1\}$ endowed with the hyperbolic metric, and D^r is the r-copy of D.

The extension $U(1) \times \mathrm{SL}_2(\mathbb{R})^d \to G(\mathbb{R})$ of the morphism φ from (4.6.6) is:

$$\left(u, \begin{bmatrix} a_1 & b_1 \\ c_1 & s_1 \end{bmatrix}, \dots, \begin{bmatrix} a_d & b_d \\ c_d & s_d \end{bmatrix}\right) \mapsto \begin{bmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_d & & & b_d \\ c_1 & & & s_1 & & \\ & \ddots & & & \ddots & \\ & & c_d & & & s_d \end{bmatrix}.$$

4.6.3 Real roots and Cayley transformation

Next we need to study a relative root system over $\mathbb{R}$, for which we need to take a maximal $\mathbb{R}$-split torus A in G. Our construction is as follows. Recall $\Psi = \{\alpha_1, \dots, \alpha_r\}$ from (4.6.4) is the maximal subset of strongly orthogonal roots in $\Phi^+ \cap \Phi_M$. By definition of strong orthogonality, the sum

$$\mathfrak{a} := \sum_{\alpha \in \Psi} \mathbb{R} x_\alpha, \tag{4.6.7}$$

with x_α as below (4.6.5), is commutative, and hence is a Lie subalgebra. In fact we have more:

Proposition 4.6.10. *$\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{m}$.*

Thus by Theorem 4.5.3, there exists a maximal $\mathbb{R}$-split tours A in G with $\mathrm{Lie}A = \mathfrak{a}$, and hence we have the relative root system ${}_{\mathbb{R}}\Phi := \Phi(A, G)$.

Example 4.6.11. *In the Siegel case, A is the standard torus $\{\mathrm{diag}(t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}) : t_1, \dots, t_d \in \mathbb{R}^\times\}$.*

We wish to use the root system Φ constructed in §4.6.2 to study ${}_{\mathbb{R}}\Phi$. For this purpose, we need to conjugate the maximal torus T in §4.6.2, which is contained in K_∞, to a maximal torus which contains A. For this purpose, it suffices to find an abelian Lie subalgebra $\mathfrak{a}'$ in $\mathfrak{t} \subseteq \mathfrak{k}$ which is a conjugate of $\mathfrak{a}$. This is the *Cayley transformation* which we introduce now.

Let $h_\alpha \in \mathfrak{t} \subseteq \mathfrak{k}$ be as above (4.6.5). Define

$$\mathfrak{a}' := \sum_{\alpha \in \Psi} \mathbb{R} h_\alpha \subseteq \mathfrak{t}. \tag{4.6.8}$$

For each $\alpha \in \Psi$, set $C_\alpha := \exp(\pi\sqrt{-1} y_\alpha / 4) \in G(\mathbb{C})$.[12] Then $\mathrm{Ad}(C_\alpha) h_\alpha = [\pi\sqrt{-1} y_\alpha/4, h_\alpha] \in \mathbb{R} x_\alpha \subseteq \mathfrak{a}$. The *Cayley transformation* is defined to be:

$$\mathrm{Ad}(C_\Psi) \colon \mathfrak{a}' \xrightarrow{\sim} \mathfrak{a}, \qquad \text{with } C_\Psi = \prod\nolimits_{\alpha \in \Psi} C_\alpha. \tag{4.6.9}$$

In terms of the morphism $\varphi\colon \mathrm{SL}_2(\mathbb{R})^r \to G(\mathbb{R})$ from (4.6.6) based changed to $\mathbb{C}$,

$$C_\Psi = \varphi\left(\cdots, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix}, \cdots\right). \tag{4.6.10}$$

Now, $\mathfrak{a}'$ gives rise to an $\mathbb{R}$-split torus A' with $\mathrm{Lie}A' = \mathfrak{a}'$, and the relative root system $\Phi(A', G)$ is exactly $\mathrm{Int}(C_\Psi)^* {}_{\mathbb{R}}\Phi$. Since $A' < T$ for the maximal torus T in §4.6.2, we can directly compare

[12]Notice that our y_α is well-defined up to scalar. We usually take a normalization in the definition of e_α and h_α, and hence x_α and y_α. Then the resulting C_Ψ will be as in (4.6.10).

$\Phi = \Phi(T, G)$ and $\mathrm{Int}(C_\Psi)^*{}_{\mathbb{R}}\Phi = \Phi'(A, G)$. More precisely, we can regroup the eigenspace decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ to be:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{A'} \oplus \sum_{\alpha'\in\Phi'(A,G)} \mathfrak{g}_{\alpha'} \tag{4.6.11}$$

with $\mathfrak{g}_{\mathbb{C}}^{A'} = \mathfrak{a}'_{\mathbb{C}} \oplus \sum_{\beta\sim 0} \mathfrak{g}_\beta$ and $\mathfrak{g}_{\alpha'} = \sum_{\beta\in\Phi,\ \beta\sim\alpha'} \mathfrak{g}_\beta$. Here, the equivalence $\sim$ on Φ is defined by: $\beta_1 \sim \beta_2$ if and only if $\beta_1|_{A'} = \beta_2|_{A'}$. This decomposition is defined over $\mathbb{R}$ because A' is $\mathbb{R}$-split. Applying the Cayley transformation to (4.6.11), we get the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}^A \oplus \sum_{\varphi\in{}_{\mathbb{R}}\Phi} \mathfrak{g}_\varphi \tag{4.6.12}$$

with each $\mathfrak{g}_\varphi$ being a suitable Ad-conjugate of a suitable $\mathfrak{g}_{\alpha'}$ above.

Finally each $\alpha_j \in \Psi$ defines a character $\alpha_j|_{A'} \in X^*(A')$, and hence a character $\gamma_j \in X^*(A)$ via the Cayley transformation. We thus have the following set

$${}_{\mathbb{R}}\Psi := \{\gamma_1, \ldots, \gamma_r\}. \tag{4.6.13}$$

Since $\Psi \subseteq \Phi$, we have ${}_{\mathbb{R}}\Psi \subseteq {}_{\mathbb{R}}\Phi$. In general, we have the following proposition, which is a consequence of the classification of (real) representations of $U(1) \times \mathrm{SL}_2(\mathbb{R})^r$ by analyzing the action of Weyl groups.

Proposition 4.6.12. *Assume X is irreducible as a Hermitian symmetric domain, i.e. X cannot be written as the product of two non-trivial Hermitian symmetric domains (equivalently, G is a simple group). Then one of the following possibilities occurs:*

- (Type C_r) ${}_{\mathbb{R}}\Phi = \{\pm\frac{1}{2}(\gamma_i + \gamma_j)$ *for* $i \geq j,\ \pm\frac{1}{2}(\gamma_i - \gamma_j)$ *for* $i > j\}$.
- (Type BC_r) ${}_{\mathbb{R}}\Phi = \{\pm\frac{1}{2}(\gamma_i + \gamma_j)$ *for* $i \geq j,\ \pm\frac{1}{2}(\gamma_i - \gamma_j)$ *for* $i > j,\ \pm\frac{1}{2}\gamma_i\}$.

If we order the roots such that $\gamma_1 > \ldots > \gamma_r$, then the set of simple roots is:

- (Type C_r) ${}_{\mathbb{R}}\Delta = \{\mu_1 := \frac{1}{2}(\gamma_1 - \gamma_2), \ldots, \mu_{r-1} := \frac{1}{2}(\gamma_{r-1} - \gamma_r), \mu_r := \gamma_r\}$.
- (Type BC_r) ${}_{\mathbb{R}}\Delta = \{\mu_1 := \frac{1}{2}(\gamma_1 - \gamma_2), \ldots, \mu_{r-1} := \frac{1}{2}(\gamma_{r-1} - \gamma_r), \mu_r := \frac{1}{2}\gamma_r\}$.

In each case, the simple root μ_r is called the *distinguished root*, and is the longest (resp. shortest) simple root in Type C_r (resp. Type BC_r).

Example 4.6.13. *In the Siegel case, $\gamma_j : A \to \mathbb{R}^\times$ is $\mathrm{diag}(t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}) \mapsto t_j^2$, and we are of Type C_d.*

4.6.4 Standard form of analytic boundary components

Recall the maximal subset of strongly orthogonal roots $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ of Φ, and the induced subset ${}_{\mathbb{R}}\Psi = \{\gamma_1, \ldots, \gamma_r\}$ of ${}_{\mathbb{R}}\Phi$.

For any $S \subseteq \{1, \ldots, r\}$, define the Lie subalgebra

$$\mathfrak{l}_S := \sum_{\substack{\varphi\in{}_{\mathbb{R}}\Phi \\ \varphi \text{ is a linear combination of } \gamma_j \text{ with } j\notin S}} (\mathfrak{g}_\varphi + [\mathfrak{g}_\varphi, \mathfrak{g}_{-\varphi}]) \tag{4.6.14}$$

of $\mathfrak{g}$, with each $\mathfrak{g}_\varphi$ the eigenspace of φ for the adjoint action of A on $\mathfrak{g}$; see (4.6.12).

Let L_S be the subgroup of G with $\mathrm{Lie}L_S = \mathfrak{l}_S$. Denote by $\mathfrak{m}_S^+ := \mathfrak{m}^+ \cap \mathfrak{l}_S$.

Proposition 4.6.14. *L_S is a semi-simple subgroup of G without compact factors, and*

$$X_S := L_S(\mathbb{R})^+ x_0 \simeq L_S(\mathbb{R})^+/(L_S(\mathbb{R}) \cap K_\infty)$$

is a sub-Hermitian symmetric domain in X.

Moreover, for the Harish–Chandra realizations $\mathcal{D}$ of X and $\mathcal{D}_S$ of X_S (see (4.6.3)*), we have the following equivariant diagram of symmetric holomorphic maps*

$$\begin{tikzcd} D^{|S|} \times \mathcal{D}_S \arrow[r, "f_1"] \arrow[d, "\subseteq"'] & \mathcal{D} \arrow[d, "\subseteq"] \\ \mathbb{C}^{|S|} \times \mathfrak{m}_S^+ \arrow[r, "f_2"] \arrow[d, "\subseteq"'] & \mathfrak{m}^+ \arrow[d, "\subseteq"] \\ (\mathbb{P}^1)^{|S|} \times X_S^\vee \arrow[r, "f_3"] & X^\vee \end{tikzcd} \tag{4.6.15}$$

where $D = \{z \in \mathbb{C} : |z| < 1\}$ is the Poincaré unit disc.

Here the map f_1 arises from the Polydisc Theorem as follows. We have a geodesic embedding $D^r \to \mathcal{D}$, which can be written as $D^{|S|} \times D^{|\{1,\dots,r\}\setminus S|} \to \mathcal{D}$. Now $\mathcal{D} = K_\infty \cdot D^r$ and $\mathcal{D}_S = (L_S(\mathbb{R}) \cap K_\infty) \cdot D^{|\{1,\dots,r\}\setminus S|}$, and thus we obtain f_1.

Proof. We have the following decomposition of $\mathfrak{l}_{S,\mathbb{C}}$ in terms of the complex roots in Φ by (4.6.11) and (4.6.12):

$$\mathfrak{l}_{S,\mathbb{C}} := \sum_{\substack{\alpha\in\Phi,\ \alpha\not\sim 0 \\ \alpha\sim\sum_{j\notin S} a_j\alpha_j}} (\mathfrak{g}_\alpha + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]);$$

see below (4.6.11) for the definition of $\sim$. Hence $\mathfrak{l}_{S,\mathbb{C}}$ is stable under $\mathrm{Ad}h_0(e^{\sqrt{-1}\theta})$, and so

(a) $\mathfrak{l}_S = (\mathfrak{k} \cap \mathfrak{l}_S) \oplus (\mathfrak{m} \cap \mathfrak{l}_S)$,

(b) $\mathfrak{m}_\mathbb{C} \cap \mathfrak{l}_{S,\mathbb{C}} = \mathfrak{m}_S^+ \oplus \mathfrak{m}_S^-$ with $\mathfrak{m}_S^- := \mathfrak{m}^- \cap \mathfrak{l}_{S,\mathbb{C}}$.

Hence L_S is a reductive group and X_S is a sub-Hermitian symmetric domain of X. Better, L_S is semi-simple without compact factors because it is generated by unipotent elements; see Theorem 4.3.2.(3.a).

For the "Moreover" part, notice that L_S commutes with (modulo center) the subgroup $\varphi(\prod_{j\in S} \mathrm{SL}_2(\mathbb{R}))$ for the morphism φ from (4.6.6); this is an immediate consequence of the construction of $\mathfrak{l}_S$. Hence we are done. □

We state the following theorem without proof. The proof needs the Hermann convexity theorem.

Theorem 4.6.15. *The analytic boundary components of X defined in Definition 4.6.1 are precisely the sets the form $k \cdot F_S$, where $k \in K_\infty$, $S \subseteq \{1, \dots, r\}$, and*

$$F_S := f_2\left((1, \dots, 1) \times \mathcal{D}_S\right) \subseteq \mathfrak{m}^+$$

with f_2 from (4.6.15)*.*

Two other ways to write F_S are given by (4.6.16) and, in terms of partial Cayley transformations, by (4.6.17) respectively.

Example 4.6.16. *In the Siegel case, $r = d$. Take the subset $S = \{d'+1, \ldots, d\} \subseteq \{1, \ldots, d\}$; then $|S| = d - d'$. In this case we have*

$$L_S = \left\{ \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & I_{d-d'} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d',\mathbb{R}} \right\} \simeq \mathrm{Sp}_{2d',\mathbb{R}},$$

and $X_S \simeq \mathfrak{H}_{d'}$ with Harish–Chandra realization being $\mathcal{D}_{d'}$. Under the natural identifications $\mathfrak{m}^+ \simeq \{Z \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : Z = Z^{\mathrm{t}}\}$ and $\mathfrak{m}_S^+ \simeq \{Z' \in \mathrm{Mat}_{d'\times d'}(\mathbb{C}) : Z' = (Z')^{\mathrm{t}}\}$, the holomorphic map f_2 is

$$\big((a_{d'+1}, \ldots, a_d), Z'\big) \mapsto \mathrm{diag}(Z', a_{d'+1}, \ldots, a_d).$$

Hence in this case, we have

$$F_S = \left\{ \begin{bmatrix} Z' & 0 \\ 0 & I_{d-d'} \end{bmatrix} : Z' \in \mathcal{D}_{d'} \right\}.$$

Before moving on, let us see a corollary of Theorem 4.6.15. The proof presents an application of the construction of F_S in Theorem 4.6.15, and given another way (4.6.16) to write F_S.

Corollary 4.6.17. *An analytic component of an analytic component of X is an analytic component of X.*

Proof. Let $\mathcal{D}_1$ be an analytic component of X. Theorem 4.6.15 implies that $\mathcal{D}_1 = k \cdot F_S$ for some $k \in K_\infty$ and $S \subseteq \{1, \ldots, r\}$. By Theorem 4.6.8, we have

$$F_S = \sum_{j\in S} e_{\alpha_j} + \mathcal{D}_S. \tag{4.6.16}$$

Let $\mathcal{D}_2$ be an analytic component of $\mathcal{D}_1$. Then $\mathcal{D}_2 = k \cdot \left(\sum_{j\in S} e_{\alpha_j} + \mathcal{D}_2'\right)$ for some analytic component $\mathcal{D}_2'$ of $\mathcal{D}_S$. Theorem 4.6.15 implies that $\mathcal{D}_2' = k' \cdot \left(\sum_{i\in S'} e_{\alpha_j} + \mathcal{D}_{S\cup S'}\right)$ for some $k' \in L_S \cap K_\infty$ and $S' \subseteq \{1, \ldots, r\} \setminus S$. So

$$\mathcal{D}_2 = kk' \left(\sum_{j\in S\cup S'} e_{\alpha_j} + \mathcal{D}_{S\cup S'} \right) = kk' \cdot f_2\left((1, \ldots, 1) \times \mathcal{D}_{S\cup S'}\right) = kk' \cdot F_{S\cup S'}$$

which by Theorem 4.6.15 is an analytic component of X. □

4.6.5 Analytic boundary components and maximal parabolic subgroups

From now on, assume that X is irreducible as a Hermitian symmetric domain, *i.e.* X cannot be written as the product of two non-trivial Hermitian symmetric domains. Equivalently, G is a simple group.

Now let $S \subseteq \{1, \ldots, r\}$ be a subset. Recall the set of simple roots ${}_{\mathbb{R}}\Delta = \{\mu_1, \ldots, \mu_r\}$ from Proposition 4.6.12, with μ_r being the distinguished root. Then ${}_{\mathbb{R}}\Delta$ gives rise to a minimal parabolic subgroup P_0 of G as below (4.4.2). Denote by $I_S := \{\mu_j : j \notin S\}$. Then we have a standard parabolic subgroup P_{I_S} of G (*i.e.* a parabolic subgroup containing P_0) defined in Theorem 4.4.6. It has unipotent radical N_{I_S}, maximal $\mathbb{R}$-split torus $A_{I_S} := \left(\bigcap_{j\notin S} \mathrm{Ker}\mu_j\right)^\circ$, and Levi subgroup $L_{I_S} = Z_G(A_{I_S})$.

Proposition 4.6.18. *The analytic boundary component F_S can be identified with the boundary symmetric space $X_{P_{I_S}}$ associated with the parabolic subgroup P_{I_S}.*

See (4.5.3) and below (4.5.6) for the definition of $X_{P_{I_S}}$.

Proof. We shall use the Polydisc Theorem (Theorem 4.6.7) and its refinement Theorem 4.6.8, for which we need to go back to the complex roots.

Recall the maximal subset of strongly orthogonal roots $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ from (4.6.4). Then the polydisc is $G[\Psi](\mathbb{R})^+x_0 \subseteq X$, with $\mathrm{Lie}G[\Psi] = \sum_{j=1}^r \mathfrak{g}[\alpha_j] \simeq \mathfrak{sl}_{2,\mathbb{R}}^{\oplus r}$. For the maximal abelian Lie subalgebra $\mathfrak{a} = \sum_{j=1}^r \mathbb{R}x_{\alpha_j} \subseteq \mathfrak{m}$ from (4.6.7) and the corresponding maximal $\mathbb{R}$-split torus A, we have $A < G[\Psi]$ since $\mathfrak{a} \subseteq \mathrm{Lie}G[\Psi]$. Hence $A(\mathbb{R})^+x_0 \subseteq G[\Psi](\mathbb{R})^+x_0$. A refinement of the Cartan decomposition says that $G = K_\infty A K_\infty$ (more precisely, $\mathfrak{m} = \bigcup_{k\in K_\infty} \mathrm{Ad}(k)(\mathfrak{a})$). Hence $X = \bigcup_{k\in K_\infty} k \cdot A(\mathbb{R})^+x_0$, which in fact improves the last sentence of Theorem 4.6.7. In view of $G[\Psi](\mathbb{R})^+x_0 \simeq D^r$, the subset $A(\mathbb{R})^+x_0$ can be identified with $(-1,1)^r$ and the multiplication by elements in $Z(K_\infty)$ correspond to rotations.

Denote by $S^{\mathrm{c}} := \{1, \ldots, r\} \setminus S$. The discussion generalizes to $S^{\mathrm{c}} \subseteq \{1, \ldots, r\}$. We have the partial polydisc $G[S^{\mathrm{c}}](\mathbb{R})^+x_0 \subseteq X$, with $\mathrm{Lie}G[S^{\mathrm{c}}] = \sum_{j\notin S} \mathfrak{g}[\alpha_j]$. For the sub-Hermitian symmetric domain X_S defined in Proposition 4.6.14, it is not hard to see that $G[S^{\mathrm{c}}](\mathbb{R})^+x_0$ is a maximal polydisc in X_S by Theorem 4.6.8. We also have an abelian Lie subalgebra $\mathfrak{a}_{S^{\mathrm{c}}} := \sum_{j\notin S} \mathbb{R}x_{\alpha_j} \subseteq \mathfrak{m}$, which defines an $\mathbb{R}$-split torus $A_{S^{\mathrm{c}}}$ in $G[S]$. Moreover similar to Proposition 4.6.10, $A_{S^{\mathrm{c}}}$ is a maximal $\mathbb{R}$-split torus in L_S. So $X_S = \bigcup_{k\in L_S\cap K_\infty} k \cdot A_{S^{\mathrm{c}}}(\mathbb{R})^+x_0$ as in the last paragraph.

Next we turn to the Langlands decomposition $P_{I_S}(\mathbb{R})^+ \simeq N_{I_S}(\mathbb{R}) \times A_{I_S}(\mathbb{R})^+ \times M_{I_S}(\mathbb{R})$ (4.5.2). Both $A_{S^{\mathrm{c}}}$ and A_{I_S} are subgroups of A, so $A_{S^{\mathrm{c}}} < Z_G(A_{I_S}) = L_{I_S} = A_{I_S}M_{I_S}$. But $A_{S^{\mathrm{c}}} \cap A_{I_S} = \{1\}$ by definition, so $A_{S^{\mathrm{c}}} < M_{I_S}$. Thus $A_{S^{\mathrm{c}}}(\mathbb{R})^+x_0$ is a subset of $X_{I_S} = M_{I_S}(\mathbb{R})^+/K_{I_S,\infty}$ with $K_{I_S,\infty} := M_{I_S} \cap K_\infty$. Moreover, $A_{S^{\mathrm{c}}}$ is a maximal $\mathbb{R}$-split torus in M_{I_S} by reason of rank, and $A_{S^{\mathrm{c}}}(\mathbb{R}) \cap K_{I_S,\infty} = \{1\}$. So the refined Cartan decomposition for M_{I_S} (recall that $M_{I_S} < L_{I_S}$ is chosen to be invariant under the Cartan involution) implies that $M_{I_S} = K_{I_S,\infty}A_{S^{\mathrm{c}}}K_{I_S,\infty}$ as before. Hence $X_{I_S} = \bigcup_{k\in K_{I_S,\infty}} k \cdot A_{S^{\mathrm{c}}}(\mathbb{R})^+x_0$.

Now we are done by the conclusions of the previous two paragraphs, since F_S is a suitable translate of X_S. $\square$

In fact we can be more precise on this translate. More precisely we can write the analytic boundary component F_S in terms of X_S and the *partial Cayley transformation* defined as follows. Let $C_{S^{\mathrm{c}}} := \prod_{j\notin S} C_{\alpha_j}$ with C_{α_j} defined above (4.6.9). Recall that C_{α_j} is the image of $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix}$ under $\mathrm{SL}_2(\mathbb{C}) \xrightarrow{\iota_j} \mathrm{SL}_2(\mathbb{C})^r \xrightarrow{\varphi} G(\mathbb{C})$, where ι_j is the embedding as the j-th component; see (4.6.10) for the notation and explanation. Moreover on the closed unit disc $\overline{D} = \{z \in \mathbb{C} : |z| \le 1\}$, the matrix $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix}$ sends 0 to 1 by the formula above Lemma 4.1.1. So

$$F_S = C_{S^{\mathrm{c}}} \cdot X_S. \tag{4.6.17}$$

Theorem 4.6.19. *Assume $S = \{1, \ldots, \ell\}$. Then $N(F_S) = P_{_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+$.*

Say $S = \{1, \ldots, \ell\}$. Let $I_S = \{\mu_{\ell+1}, \ldots, \mu_r\}$ and $I'_S = \{\mu_1, \ldots, \mu_{\ell-1}\}$. Then the boundary symmetric space $X_{P_{_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}}$ can be decomposed as $X_{P_{I_S}} \times X_{P_{I'_S}}$, and $X_{P_{I_S}}$ is isomorphic to F_S (hence is complex) by Proposition 4.6.18. This decomposition is also related to (4.6.19).

Proof of Theorem 4.6.2 from Theorem 4.6.19. It is easy to reduce Theorem 4.6.2 to the case where X is irreducible. Next by Theorem 4.6.15, we may assume that the analytic boundary component is F_S for some subset $S \subseteq \{1, \ldots, r\}$.

Now S defines a subset $J_S := \{\gamma_j : j \in S\}$ of ${}_{\mathbb{R}}\Psi = \{\gamma_1, \ldots, \gamma_r\}$ defined in (4.6.13). By Proposition 4.6.12, the Weyl group $W({}_{\mathbb{R}}\Phi)$ contains all the signed permutations of the roots in ${}_{\mathbb{R}}\Psi$. So there exists an element $w \in W({}_{\mathbb{R}}\Phi)$ such that $w(J_S) = \{\gamma_1, \ldots, \gamma_\ell\}$ for some ℓ. Thus we may and do assume $S = \{1, \ldots, \ell\}$. Now the conclusion follows immediately from Theorem 4.6.19, because $P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}$ is a maximal proper parabolic subgroup of G. □

Proof of Theorem 4.6.19. We start by showing that

$$P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+ \subseteq N(F_S). \tag{4.6.18}$$

Let $g \in P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+$. Then g sends F_S to an analytic boundary component, which must be either F_S or disjoint from F_S. By (4.6.17), it suffices to show that $g \cdot (C_{S^c} \cdot x_0) \in C_{S^c}(X_S^\vee)$, which itself is equivalent to $(C_{S^c}^{-1} g C_{S^c}) \cdot x_0 \in X_S^\vee$. Thus it suffices to show

$$C_{S^c}^{-1} g C_{S^c} \in L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-).$$

Now consider $P' := C_{S^c}^{-1}(P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}} C_{S^c} = \mathrm{Int}(C_{S^c})^{-1}(P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}}$, which is a subgroup of $G_{\mathbb{C}}$. Our goal is to prove that

$$P' < L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-).$$

For this purpose, let us compute LieP'. We start with writing Lie$P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}$ as above and in Theorem 4.4.6. Then using the Cayley transformation $\mathrm{Ad}(C_{S^c})$ and by (4.6.11) and (4.6.12), we have the following decomposition of LieP' into weight spaces of roots in $\Phi = \Phi(T, G_{\mathbb{C}})$ (together with $\mathfrak{t}_{\mathbb{C}}$)

$$\mathrm{Lie}P' = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\substack{\alpha \sim \frac{\pm\alpha_i \pm \alpha_j}{2} \text{ or } \frac{\pm\alpha_i}{2} \\ i,j \notin S}} \mathfrak{g}_\alpha \oplus \sum_{\substack{\alpha \sim \frac{-\alpha_i - \alpha_j}{2} \text{ or } \frac{-\alpha_i}{2} \\ i \in S}} \mathfrak{g}_\alpha \oplus \sum_{\substack{\alpha \sim \frac{-\alpha_i + \alpha_j}{2} \\ i \in S}} \mathfrak{g}_\alpha \oplus \sum_{\alpha \sim 0} \mathfrak{g}_\alpha.$$

Again, the equivalence relation $\sim$ on Φ is defined by: $\beta_1 \sim \beta_2$ if and only if $\beta_1|_{A'} = \beta_2|_{A'}$.

Now $\mathfrak{t} \subseteq \mathfrak{k}$ by choice of T, the second term generates $\mathfrak{l}_{S,\mathbb{C}}$, the third term is in $\mathfrak{m}^-$, and the fourth and fifth terms are in $\mathfrak{k}_{\mathbb{C}}$. Now the first, third, and fourth terms together generate a normal subgroup which is contained in $K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-)$. Hence the first four terms generate a normal subgroup contained in $L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-)$. Hence $P' < L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-) K_{\infty,\mathbb{C}} = L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-)$. This establishes (4.6.18).

Now that

$$P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+ \subseteq N(F_S) \subsetneq G(\mathbb{R})^+$$

and $P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}$ is a maximal proper parabolic subgroup of G by Theorem 4.4.6, we have $P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}(\mathbb{R})^+ = N(F_S)^+$. Hence $N(F_S)$ normalizes Lie$N(F_S) = $ Lie$P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}$. Chevalley's Theorem (parabolic subgroups are always connected) says that the normalizer of $\mathrm{Lie}(P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}}$ in $G_{\mathbb{C}}$ is precisely $(P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}}$. So $N(F_S) \subseteq G(\mathbb{R})^+ \cap (P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}} = P_{{}_{\mathbb{R}}\Delta\setminus\{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+$. Now we are done. □

Now let us return to any X (not necessarily irreducible). Let $F = k \cdot F_S$ be an analytic boundary component of X, and let $N(F) = \{g \in G(\mathbb{R})^+ : gF = F\}$ be its normalizer, which by Theorem 4.6.2 equals $P_F(\mathbb{R}) \cap G(\mathbb{R})^+$ for a parabolic subgroup P_F of G. We have the following subgroups of $P_F = k P_{F_S} k^{-1}$:

- $W(F) := \mathcal{R}_u(P_F)$.
- $L(F)$ which is the Levi subgroup of P_F obtained as follows: Theorem 4.6.19 gives the construction of P_{F_S} in terms of the relative root system, and hence a Levi subgroup $L(F_S)$ of P_{F_S} as in Theorem 4.4.6; now $L(F) = k L(F_S) k^{-1}$.
- $G_h(F) := k L_S k^{-1}$ which is a subgroup of a suitable Levi subgroup of P_F.
- $G_l(F)$ which is a reductive subgroup of $L(F)$ with no compact factors,

- $M(F)$ which is a compact reductive subgroup of $L(F)$,

such that $L(F) = G_h(F) \cdot G_l(F) \cdot M(F)$ and $L(F)^{\mathrm{ad}} = G_h(F)^{\mathrm{ad}} \times G_l(F)^{\mathrm{ad}} \times M(F)^{\mathrm{ad}}$. One can easily check that

$$\{g \in G(\mathbb{R})^+ : gx = x \text{ for all } x \in F\} = W(F) \rtimes (G_l(F) \cdot M(F)). \tag{4.6.19}$$

Example 4.6.20. *In the Siegel case, $r = d$. Take $S = \{1, \ldots, d'\} \subseteq \{1, \ldots, d\}$. Then*

$$P_F = \left\{ \begin{bmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & (u^{\mathrm{t}})^{-1} \end{bmatrix} \in G : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d',\mathbb{R}},\ u \in \mathrm{GL}_{d-d',\mathbb{R}} \right\},$$

$$W(F) = \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & n \\ m^{\mathrm{t}} & I_{d-d'} & n^{\mathrm{t}} & b \\ 0 & 0 & I_{d'} & -m \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : n^{\mathrm{t}}m + b = m^{\mathrm{t}}n + b^{\mathrm{t}} \right\},$$

$$G_h(F) = \left\{ \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & I_{d-d'} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d',\mathbb{R}} \right\} \simeq \mathrm{Sp}_{2d',\mathbb{R}},$$

$$G_l(F) = \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & I_{d'} & 0 \\ 0 & 0 & 0 & (u^{\mathrm{t}})^{-1} \end{bmatrix} : u \in \mathrm{GL}_{d-d',\mathbb{R}} \right\} \simeq \mathrm{GL}_{d-d',\mathbb{R}},$$

$$M(F) = \{\pm I_{2d}\}.$$

4.6.6 Some other remarks

The analytic boundary components are closely related to the Polydisc Theorem, since in (4.6.15) $(1, \ldots, 1) \in \mathbb{C}^{|S|} \subseteq (\mathbb{P}^1)^{|S|}$ is a point on the boundary of the polydisc $D^{|S|} \subseteq (\mathbb{P}^1)^{|S|}$. Thus one can also recover information of analytic boundary components in terms of the polydisc $\mathfrak{H}^r \simeq D^r \subseteq X$ and the corresponding homomorphism $\varphi \colon \mathrm{SL}_2(\mathbb{R})^r \to G$ in (4.6.6).

More precisely , for every analytic boundary component $F = kF_S$, there are holomorphic symmetric maps

$$\begin{array}{ccc} \mathfrak{H} & \xrightarrow{f_F} & \mathcal{D} \\ \subseteq\big\downarrow & & \big\downarrow\subseteq \\ \mathbb{P}^1 & \xrightarrow{f_F} & X^\vee \end{array}$$

such that $f_F(\sqrt{-1}) = x_0$ and $f_F(\infty) \in F$, and equivariant with respect to a homomorphism

$$\varphi_F \colon \mathrm{SL}_2(\mathbb{R}) \to G \qquad (\text{stronger, } \widetilde{\varphi}_F \colon U(1) \times \mathrm{SL}_2(\mathbb{R}) \to G).$$

This homomorphism φ_F (resp. $\widetilde{\varphi}_F$) is defined using the Polydisc Theorem, or more precisely the map φ (resp. and its extension $\widetilde{\varphi}$) from (4.6.6) and below. Indeed, when $F = F_S$, then φ_{F_S} (resp. $\widetilde{\varphi}_{F_S}$) is precisely

$$\varphi_{F_S} : g \mapsto \varphi(\underbrace{\cdots, 1, \cdots}_{j \notin S}, \underbrace{\cdots, g, \cdots}_{j \in S}), \qquad (\text{resp. } \widetilde{\varphi}_{F_S} \colon (e^{\sqrt{-1}\theta}, g) \mapsto \widetilde{\varphi}(\underbrace{\cdots, e^{\sqrt{-1}\theta}, \cdots}_{j \notin S}, \underbrace{\cdots, g, \cdots}_{j \in S})).$$

And in terms of Theorem 4.2.5, P_F is defined by the cocharacter $\lambda_F \colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to G$, $t \mapsto \varphi_F\left(\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\right)$. This is easy to check with the computation in the proof of Theorem 4.6.19. Then F is a rational analytic boundary component if and only if λ_F is defined over $\mathbb{Q}$.

Chapter 5

(Reductive) Borel–Serre compactification

Let $(\mathbf{G}, X)$ be a Shimura datum. By abuse of notation use X to denote a connected component. Let $\Gamma < \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup.

Throughout the whole chapter, $\mathbf{P}$ will denote a proper parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$.

5.1 General discussion on compactifications of $\Gamma\backslash X$

5.1.1 General philosophy of compactifying $\Gamma\backslash X$

In general, here is what we get a compactification of $\Gamma\backslash X$ in the following steps:

(i) Consider a certain set of proper rational parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$. To each such $\mathbf{P}$, we associate a boundary component $e(\mathbf{P})$. Set $X^* := X \sqcup \bigsqcup e(\mathbf{P})$.

(ii) Extend the action of Γ on X to X^*.

(iii) Endow a nice topology on X^* (often called the *Satake topology*), such that the action of Γ on X^* is continuous and proper[1] and that $\Gamma\backslash X^*$ is Hausdorff and compact.

In practice, *Siegel sets* play a crucial role in defining the topology on X^* and in showing the compactness of $\Gamma\backslash X^*$. Let us review its definition in the next subsection.

5.1.2 Revision on the rational symmetric spaces and Siegel sets

Recall the rational Langlands decomposition of $\mathbf{P}$ from (4.5.9)

$$P(\mathbb{R})^+ \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times M_{\mathbf{P}}(\mathbb{R})$$

and the induced rational horospherical decomposition (4.5.10)

$$h\colon X \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}}$$

with the rational boundary symmetric space

$$X_{\mathbf{P}} = M_{\mathbf{P}}(\mathbb{R})^+/(M_{\mathbf{P}} \cap K_\infty).$$

[1]Namely, any $x \in X^*$ has an open neighborhood W such that $\{\gamma \in \Gamma : \gamma W \cap W \neq \emptyset\}$ is a finite set.

Let $\Delta(A_{\mathbf{P}}, P) = \{\alpha_1, \ldots, \alpha_r\}$ be the subset of simple roots defined as in (4.5.8); they are characters of a maximal $\mathbb{Q}$-split torus $\mathbf{A_P}$ contained in $\mathbf{P}$. Then we have an isomorphism

$$A_{\mathbf{P}}(\mathbb{R})^+ \xrightarrow{\sim} \mathbb{R}^r_{>0}, \qquad a \mapsto (\alpha_1(a)^{-1}, \ldots, \alpha_r(a)^{-1}). \tag{5.1.1}$$

A Siegel set in X associated with $\mathbf{P}$ is of the form

$$\Sigma_{\mathbf{P},U,t,V} := h^{-1}(U \times A_{\mathbf{P},t} \times V) \subseteq X \tag{5.1.2}$$

with $U \subseteq N_P(\mathbb{R})$ and $V \subseteq X_{\mathbf{P}}$ bounded and

$$A_{\mathbf{P},t} := \{a \in A_{\mathbf{P}}(\mathbb{R})^+ : \alpha(a)^{-1} \leq t, \ \forall \alpha \in \Delta(A_{\mathbf{P}}, P)\}.$$

If $\mathbf{P} < \mathbf{Q}$ are parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$, then $A_{\mathbf{P}} > A_{\mathbf{Q}}$. Moreover, it can be shown that the followings are equivalent: (i) $X_{\mathbf{P}}$ is compact, (ii) $M_{\mathbf{P}}$ has $\mathbb{Q}$-rank 0, (iii) $\mathbf{P}$ is minimal parabolic in $\mathbf{G}^{\mathrm{der}}$. Furthermore, *reduction theory* asserts the following: If $\mathbf{P}$ is a minimal parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$, then there exist a finite subset $J \subseteq \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and a Siegel set $\mathfrak{S} := U \times A_{\mathbf{P},t} \times V$ of $\mathbf{P}$ such that

$$\Sigma := J \cdot \mathfrak{S} \tag{5.1.3}$$

is a fundamental set for the Γ-action on X.

5.2 Borel–Serre compactification

5.2.1 Borel–Serre partial compactification: definition

For any $\mathbf{P}$, define the *Borel–Serre boundary component*

$$e(\mathbf{P}) := N_P(\mathbb{R}) \times X_{\mathbf{P}}. \tag{5.2.1}$$

Since N_P is a normal subgroup of P, the boundary component $e(\mathbf{P}) \simeq P(\mathbb{R})^+/A_{\mathbf{P}}(\mathbb{R})^+(M_{\mathbf{P}} \cap K_\infty)$ is then an $N_P(\mathbb{R})$-principle bundle over the rational boundary symmetric space $X_{\mathbf{P}} \simeq P(\mathbb{R})^+/N_P(\mathbb{R})A_{\mathbf{P}}(\mathbb{R})^+(M_{\mathbf{P}} \cap K_\infty)$. Another visualization of $e(\mathbf{P})$ is given in (5.2.8), where we see that $e(\mathbf{P})$ is in some way the quotient of X by $A_{\mathbf{P}}(\mathbb{R})^+$.

The *Borel–Serre partial compactification* $\overline{X}^{\mathrm{BS}}$ is defined, as a set, to be

$$\overline{X}^{\mathrm{BS}} := X \cup \bigsqcup_{\mathbf{P}} e(\mathbf{P}). \tag{5.2.2}$$

The extension of the Γ-action, or more generally the $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action, to $\overline{X}^{\mathrm{BS}}$ will be given in (5.2.11).

Let us define the topology on $\overline{X}^{\mathrm{BS}}$, for which we only need to define the neighborhoods of the boundary points. For this purpose, we need to analyze boundary components $e(\mathbf{Q})$ and $e(\mathbf{P})$ for two parabolic subgroups $\mathbf{P} < \mathbf{Q}$ of $\mathbf{G}^{\mathrm{der}}$.

For the reductive subgroup $\mathbf{M_Q}$ of $\mathbf{Q}$, set $\mathbf{P}' := \mathbf{P} \cap \mathbf{M_Q}$. Then $\mathbf{P}'$ is a parabolic subgroup of $\mathbf{M_Q}$ such that, by looking at the root system construction,

$$\mathbf{M}_{\mathbf{P}'} = \mathbf{M_P}, \ \mathbf{A_P} = \mathbf{A_Q} \times \mathbf{A}_{\mathbf{P}'}, \ N_P = N_Q \rtimes N_{P'}. \tag{5.2.3}$$

Thus the horospherical decomposition of $X_{\mathbf{Q}}$ associated with $\mathbf{P}'$ is

$$X_{\mathbf{Q}} \simeq N_{P'}(\mathbb{R}) \times A_{\mathbf{P}'}(\mathbb{R})^+ \times X_{\mathbf{P}'} = N_{P'}(\mathbb{R}) \times A_{\mathbf{P}'}(\mathbb{R})^+ \times X_{\mathbf{P}}. \tag{5.2.4}$$

Next, we find another $\mathbb{Q}$-split torus $\mathbf{A}_{\mathbf{P},\mathbf{Q}}$ of $\mathbf{A}_{\mathbf{P}}$ which is isomorphic to $\mathbf{A}_{\mathbf{P}'}$. We start with the case where $\mathbf{P}$ is a standard parabolic subgroup. Namely, we fix a basis ${}_{\mathbb{Q}}\Delta$ of the relative root system ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{A}, \mathbf{G}^{\mathrm{der}})$ for some maximal $\mathbb{Q}$-split torus $\mathbf{A}$ in $\mathbf{G}^{\mathrm{der}}$, and then we obtain a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$ as below (4.4.2), and assume $\mathbf{P} = \mathbf{P}_I$ for some subset $I \subseteq {}_{\mathbb{Q}}\Delta$ as in Theorem 4.4.6. Since $\mathbf{Q} > \mathbf{P}$, we have $\mathbf{Q} > \mathbf{P}_0$ and hence $\mathbf{Q} = \mathbf{P}_J$ for some $J \subseteq {}_{\mathbb{Q}}\Delta$ by Theorem 4.4.6, and it is clear that $I \subseteq J$. By Lemma 4.4.7, we have then $\mathbf{A}_I > \mathbf{A}_J$. Better, using definitions of $\mathbf{A}_I$ and $\mathbf{A}_J$ we get that $\mathbf{A}_I = \mathbf{A}_{I,J} \times \mathbf{A}_J$, with $\mathbf{A}_{I,J} := \left(\bigcap_{\alpha' \in J\setminus I} \mathrm{Ker}\alpha'\right)^{\circ}$. Notice that in this case, $\Delta(A_{\mathbf{P}}, P) = {}_{\mathbb{Q}}\Delta \setminus I$, and hence $J \setminus I \subseteq \Delta(A_{\mathbf{P}}, P)$. In general, $\mathbf{P}$ is conjugate to a unique $\mathbf{P}_I$, and then the conjugation of $\mathbf{Q}$ by the same element in $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ is standard (*i.e.* contains $\mathbf{P}_0$), and hence $\mathbf{Q} = \mathbf{P}_J$ for some $J \subseteq {}_{\mathbb{Q}}\Delta$. Let $\mathbf{A}_{\mathbf{P},\mathbf{Q}} < \mathbf{P}$ be the suitable conjugation of $\mathbf{A}_{I,J}$, and let $I_{\mathbf{P},\mathbf{Q}} \subseteq \Delta(A_{\mathbf{P}}, P)$ be the suitable conjugation of $J\setminus I$. Then we have $\mathbf{A}_{\mathbf{P}} = \mathbf{A}_{\mathbf{Q}} \times \mathbf{A}_{\mathbf{P},\mathbf{Q}}$. Thus $\mathbf{A}_{\mathbf{P}'} \simeq \mathbf{A}_{\mathbf{P},\mathbf{Q}}$ by the second equality in (5.2.3). So (5.2.4) becomes

$$X_{\mathbf{Q}} \simeq N_{P'}(\mathbb{R}) \times A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ \times X_{\mathbf{P}}. \tag{5.2.5}$$

Therefore by (5.2.3) and (5.2.5), we have

$$e(\mathbf{Q}) = N_Q \times X_{\mathbf{Q}} \simeq N_P(\mathbb{R}) \times A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ \times X_{\mathbf{P}}. \tag{5.2.6}$$

Definition 5.2.1. *The topology on $\overline{X}^{\mathrm{BS}}$ is defined as follows: (i) on X it is the natural one, (ii) for each parabolic subgroup $\mathbf{P}$ of $\mathbf{G}^{\mathrm{der}}$, the neighborhoods of a point $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$ is $\bigsqcup_{\mathbf{Q}>\mathbf{P}} U \times A_{\mathbf{P},\mathbf{Q},t} \times V$ for all neighborhoods U of n in $N_P(\mathbb{R})$, all neighborhoods V of z in $X_{\mathbf{P}}$, and all $t > 0$, with*

$$A_{\mathbf{P},\mathbf{Q},t} := \{a \in A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ : \alpha(a)^{-1} < t,\ \forall \alpha \in I_{\mathbf{P},\mathbf{Q}}\}.$$

A better description is given in Corollary 5.2.4.

5.2.2 Borel–Serre partial compactification: corners and Hausdorff property

Recall the isomorphism $A_{\mathbf{P}}(\mathbb{R})^+ \simeq \mathbb{R}^r_{>0}$ from (5.1.1). Use $\overline{A_{\mathbf{P}}}$ to denote the closure of $A_{\mathbf{P}}(\mathbb{R})^+$ in $\mathbb{R}^r$ under the natural inclusion $\mathbb{R}^r_{>0} \subseteq \mathbb{R}^r$. The discussion on the topology of $\overline{X}^{\mathrm{BS}}$ in the previous subsection yields easily the following results.

Lemma 5.2.2. *We have a disjoint decomposition*

$$\overline{A_{\mathbf{P}}} = A_{\mathbf{P}}(\mathbb{R})^+ \cup \bigsqcup_{\mathbf{Q}>\mathbf{P}} (A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ \times 0_{\mathbf{Q}})$$

where $0_{\mathbf{Q}}$ is the origin of the real vector space $\mathbb{R}^{r'}$ arising from $A_{\mathbf{Q}}(\mathbb{R})^+ \simeq \mathbb{R}^{r'}_{>0} \subseteq \mathbb{R}^{r'}$.

Proposition 5.2.3. *The embedding $N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}} \simeq X \subseteq \overline{X}^{\mathrm{BS}}$ can be naturally extended to an open embedding $N_P(\mathbb{R}) \times \overline{A_{\mathbf{P}}} \times X_{\mathbf{P}} \hookrightarrow \overline{X}^{\mathrm{BS}}$. Moreover, the image of $N_P(\mathbb{R}) \times \overline{A_{\mathbf{P}}} \times X_{\mathbf{P}}$ in $\overline{X}^{\mathrm{BS}}$ is equal to the subset*

$$X \cup \bigsqcup\nolimits_{\mathbf{Q}>\mathbf{P}} e(\mathbf{Q}) \tag{5.2.7}$$

in $\overline{X}^{\mathrm{BS}}$.

We will call (5.2.7) the *corner associated with* $\mathbf{P}$ and denote it by $X(\mathbf{P})$. Then we have

$$X(\mathbf{P}) \simeq X \times_{A_{\mathbf{P}}(\mathbb{R})^+} \overline{A_{\mathbf{P}}}, \quad e(\mathbf{P}) = N_P(\mathbb{R}) \times \{(0, \cdots, 0)\} \times X_{\mathbf{P}}, \quad X(\mathbf{P}) \simeq e(\mathbf{P}) \times [0, \infty)^r. \tag{5.2.8}$$

Another corollary of Lemma 5.2.2 is the following description of neighborhood bases of points in the boundaries.

Corollary 5.2.4. *For any point $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$, a neighborhood basis in $\overline{X}^{\mathrm{BS}}$ is given by $U \times \overline{A_{\mathbf{P}}}_{,t} \times V \subseteq X(\mathbf{P})$, where $n \in U, z \in V$ are bases of neighborhoods of n and z respectively, and $t > 0$ with*

$$\overline{A_{\mathbf{P}}}_{,t} := \{a \in \overline{A_{\mathbf{P}}} : \alpha(a)^{-1} < t, \ \forall \alpha \in \Delta(A_{\mathbf{P}}, P)\}.$$

This neighborhood basis is highly related to the Siegel sets (5.1.2). This relation is the key to the proof of the following proposition.

Proposition 5.2.5. *$\overline{X}^{\mathrm{BS}}$ is a Hausdorff space.*

Proof. Take two distinct points $y_1, y_2 \in \overline{X}^{\mathrm{BS}} \setminus X$, with $y_j \in e(\mathbf{P}_j)$.

If $\mathbf{P}_1 = \mathbf{P}_2$, then $e(\mathbf{P}_1) = e(\mathbf{P}_2)$ and clearly there are open neighborhoods of y_1 and y_2 which are disjoint.

From now on assume $\mathbf{P}_1 \neq \mathbf{P}_2$. Assume that y_1 and y_2 have open neighborhoods which are non-disjoint. By Corollary 5.2.4, we may assume that the neighborhoods are $U_1 \times \overline{A_{\mathbf{P}_1}}_{,t} \times V_1$ and $U_2 \times \overline{A_{\mathbf{P}_2}}_{,t} \times V_2$ with $t > 0$. We may furthermore assume that U_1, V_1, U_2, V_2 are bounded. Call the intersection W. Then W is open in $U_j \times \overline{A_{\mathbf{P}_j}}_{,t} \times V_j$.

Because $U_j \times A_{\mathbf{P}_j,t} \times V_j$ is open and dense in $U_j \times \overline{A_{\mathbf{P}_j}}_{,t} \times V_j$, we have that $W \cap (U_j \times A_{\mathbf{P}_j,t} \times V_j)$ is open and dense in W. So $W \cap (U_1 \times A_{\mathbf{P}_1,t} \times V_1) \cap (U_2 \times A_{\mathbf{P}_2,t} \times V_2)$ is non-empty.

But $\mathbf{P}_1 \neq \mathbf{P}_2$, so general theory of Siegel sets says that $(U_1 \times A_{\mathbf{P}_1,t} \times V_1) \cap (U_2 \times A_{\mathbf{P}_2,t} \times V_2) = \emptyset$ for $t \gg 1$ (say $t \geq t_0$ for some fixed $t_0 \in \mathbb{R}$). Therefore by the previous paragraph, $t < t_0$. Hence we find open neighborhoods $U_1 \times \overline{A_{\mathbf{P}_1}}_{,t_0} \times V_1$ of y_1 and $U_2 \times \overline{A_{\mathbf{P}_2}}_{,t_0} \times V_2$ of y_2 which are disjoint. We are done. $\square$

5.2.3 Extension of $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action

For any element $\gamma \in \mathbf{G}^{\mathrm{der}}(\mathbb{C})$, write ${}^{\gamma}(\cdot)$ for the conjugate $\gamma(\cdot)\gamma^{-1}$.

We start by explaining the action of $P(\mathbb{R})^+$ on the boundary component $e(\mathbf{P})$. Recall $P = N_P \rtimes (A_{\mathbf{P}} M_{\mathbf{P}})$. Let $p \in P(\mathbb{R})^+$, which decomposes into $p = n_0 a_0 m_0$ with $n_0 \in N_P(\mathbb{R})$, $a_0 \in A_{\mathbf{P}}(\mathbb{R})^+$ and $m_0 \in M_{\mathbf{P}}(\mathbb{R})$. Then for $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$, set

$$p \cdot (n, z) := (n_0 \cdot {}^{a_0 m_0} n, m_0 z) \in N_P(\mathbb{R}) \times X_{\mathbf{P}} = e(\mathbf{P}). \tag{5.2.9}$$

We can rewrite this action in the following way. Instead of decomposing $p = n_0 a_0 m_0$, we can also decompose it into $p = m' a' n'$ with $m' \in M_{\mathbf{P}}(\mathbb{R})$, $a' \in A_{\mathbf{P}}(\mathbb{R})^+$ and $n' \in N_P(\mathbb{R})$. Indeed (since $A_{\mathbf{P}}$ and $M_{\mathbf{P}}$ commute), we can take $m' = m_0$, $a' = a_0$, and $n' = {}^{(a_0 m_0)^{-1}} n_0$. Then

$$p \cdot (n, z) = ({}^{m'a'}(n'n), m'z). \tag{5.2.10}$$

Next we extend this action to the action of $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ on $\overline{X}^{\mathrm{BS}}$ as follows. Let $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$. Then we can decompose $g = kp$ for $k \in K_\infty$ and $p \in P(\mathbb{R})$, and moreover $p = m' a' n'$ with $m' \in M_{\mathbf{P}}(\mathbb{R})$, $a' \in A_{\mathbf{P}}(\mathbb{R})^+$ and $n' \in N_P(\mathbb{R})$. Notice that both k and m' are not uniquely determined by g, but determined up to an element in $K_\infty \cap P = K_\infty \cap M_{\mathbf{P}}$. In particular, the product km' is uniquely determined by g. Notice that ${}^k\mathbf{P} = {}^g\mathbf{P}$ is a $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$. Set

$$g \cdot (n, z) := ({}^{km'a'}(n'n), k \cdot m'z) \in N_{{}^kP}(\mathbb{R}) \times X_{{}^k\mathbf{P}} = e({}^k\mathbf{P}) = e({}^g\mathbf{P}). \tag{5.2.11}$$

We need to explain the notation $k \cdot m'z$. Denoting by $K_{\mathbf{P}} := K_\infty \cap M_{\mathbf{P}}$, the point $m'z \in X_{\mathbf{P}} \simeq M_{\mathbf{P}}(\mathbb{R})^+/K_{\mathbf{P}}$ can be written as $mK_{\mathbf{P}}$ for some $m \in M_{\mathbf{P}}(\mathbb{R})^+$. Then $k \cdot m'z \in X_{{}^k\mathbf{P}} = X_{{}^g\mathbf{P}}$ is ${}^k m K_{{}^k\mathbf{P}} = {}^k m K_{{}^g\mathbf{P}}$.

Proposition 5.2.6. *The action of* $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ *on* $\overline{X}^{\mathrm{BS}}$ *defined above is continuous.*

Proof. It suffices to prove the following: Let $\{y_j\}$ be a sequence of points in $\overline{X}^{\mathrm{BS}}$ which converges to y_∞, then $\{g \cdot y_j\}$ converges to $g \cdot y_\infty$ for any $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$. This is clearly true if $y_\infty \in X$. Thus we may assume $y_\infty \in e(\mathbf{P})$ for some $\mathbf{P}$.

Now there are two cases to consider: either $\{y_j\} \subseteq X$, or $\{y_j\} \subseteq e(\mathbf{Q})$ for some $\mathbf{Q} > \mathbf{P}$. Indeed, by passing to a subsequence we can always reduce to one of these two cases. In the first case, write each y_j under the horospherical decomposition associated with $\mathbf{P}$. In the second case, write the $X_{\mathbf{Q}}$-component of each y_j under the relative horospherical decomposition (5.2.5). We omit the details of the computation. □

Finally, let $\Gamma < \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ be an arithmetic subgroup. We have:

Corollary 5.2.7. Γ *acts properly on* $\overline{X}^{\mathrm{BS}}$*, i.e. any point* $x \in \overline{X}^{\mathrm{BS}}$ *has an open neighborhood* W *such that*

$$\{\gamma \in \Gamma : \gamma W \cap W \neq \emptyset\}$$

is a finite set.

Proof. It is known that Γ acts properly on X. So it suffices to prove the result for $x \in e(\mathbf{P})$ for any $\mathbf{P}$. By Corollary 5.2.4, we may take $W = U \times \overline{A_{\mathbf{P}t}} \times V$, with $U \times A_{\mathbf{P},t} \times V$ a Siegel set in X. Since W is open in $\overline{X}^{\mathrm{BS}}$ and that Γ acts continuously on $\overline{X}^{\mathrm{BS}}$ (Proposition 5.2.6), we have:

$$\gamma(U \times A_{\mathbf{P},t} \times V) \cap (U \times A_{\mathbf{P},t} \times V) \neq \emptyset \Rightarrow \gamma W \cap W \neq \emptyset$$

with an argument similar to Proposition 5.2.5. Hence the desired finiteness follows from general theory of Siegel sets. □

5.2.4 Quotient by Γ and conclusion

Theorem 5.2.8. *The quotient* $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ *is a compact Hausdorff space. If* Γ *is torsion-free, then* $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ *has a canonical structure of a real analytic manifold with corners.*

Moreover, there are finitely many Γ*-conjugacy classes of proper rational parabolic subgroups of* $\mathbf{G}^{\mathrm{der}}$*. Taking a set of representatives* $\{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$*, we have*

$$\Gamma\backslash\overline{X}^{\mathrm{BS}} = \Gamma\backslash X \cup \bigsqcup_{j=1}^{m} \Gamma_{\mathbf{P}_j}\backslash e(\mathbf{P}_j) \tag{5.2.12}$$

with $\Gamma_{\mathbf{P}_j} := \Gamma \cap \mathbf{P}_j(\mathbb{Q})$*.*

We also use $\overline{\Gamma\backslash X}^{\mathrm{BS}}$ to denote $\Gamma\backslash\overline{X}^{\mathrm{BS}}$.

Proof. $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ by Proposition 5.2.5 and Corollary 5.2.7.

For the proper rational parabolic subgroups $\mathbf{P}_1, \ldots, \mathbf{P}_m$ of $\mathbf{G}^{\mathrm{der}}$, reduction theory says that the images of some associated Siegel sets

$$U_1 \times A_{\mathbf{P}_1,t_1} \times V_1, \ldots, U_m \times A_{\mathbf{P}_m,t_m} \times V_m$$

under $X \to \Gamma\backslash X$ cover the whole space. Clearly we can take all the U_j, V_j's to be compact. By Proposition 5.2.3, the closure of $U_j \times A_{\mathbf{P}_j,t_j} \times V_j$ in $\overline{X}^{\mathrm{BS}}$ is $U_j \times \overline{A_{\mathbf{P}_j,t_j}} \times V_j$, which is a compact set. The Γ-translates of these compact sets cover $\overline{X}^{\mathrm{BS}}$ because X is dense in $\overline{X}^{\mathrm{BS}}$. So we prove the compactness of $\Gamma\backslash\overline{X}^{\mathrm{BS}}$.

Next we show that $\overline{X}^{\mathrm{BS}}$ has a canonical structure of real semi-algebraic manifolds with corners. Indeed, this is clearly true for $X(\mathbf{P}) \simeq N_P(\mathbb{R}) \times \overline{A_{\mathbf{P}}} \times X_{\mathbf{P}}$ for each $\mathbf{P}$, and it is not hard to check that the real semi-algebraic structures of different $X(\mathbf{P})$'s are compatible (it suffices to check for $\mathbf{Q} > \mathbf{P}$, for which we can use (5.2.5)). The $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action is easily seen to be given by real semi-algebraic diffeomorphisms.

Now if Γ is torsion-free, then the action of Γ on $\overline{X}^{\mathrm{BS}}$ has no fixed points. So $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ has a canonical structure of a real analytic manifold with corners.

Finally to get (5.2.12), it suffices to show that $\Gamma_{\mathbf{P}}$ is the stabilizer of $e(\mathbf{P})$ in Γ for each $\mathbf{P}$. This is true because: for any $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$, either $g \in P(\mathbb{R})^+$ and $ge(\mathbf{P}) = e(\mathbf{P})$, or $g \notin P(\mathbb{R})^+$ and $ge(\mathbf{P}) \cap e(\mathbf{P}) = \emptyset$; see (5.2.11). We are done. $\square$

Example 5.2.9. *For the Poincaré upper half plan $\mathfrak{H}$ and the group $\mathbf{SL}_2$, consider the parabolic subgroup*

$$\mathbf{P} = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{Q}^{\times},\ b \in \mathbb{Q} \right\}.$$

We have $N_P = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{R} \right\} \simeq \mathbb{G}_{\mathrm{a},\mathbb{R}}$. So $e(\mathbf{P}) \simeq \mathbb{R}$, by adding a "real axis" at the point ∞. Then $\Gamma_{\mathbf{P}}\backslash e(\mathbf{P}) \simeq \mathbb{Z}\backslash\mathbb{R}$ is a loop, with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

5.3 Reductive Borel–Serre compactification

It often occurs that the Borel–Serre compactification is too large. In this section we define the *reductive Borel–Serre compactification.* For each $\mathbf{P}$, define the *reductive Borel–Serre boundary component* to be

$$e(\mathbf{P}) := X_{\mathbf{P}}. \tag{5.3.1}$$

Then clearly it is obtained from the Borel–Serre boundary component (5.2.1) by collapsing $N_P(\mathbb{R})$. Define the *reductive Borel–Serre partial compactification* to be

$$\overline{X}^{\mathrm{RBS}} := X \cup \coprod_{\mathbf{P}} e(\mathbf{P}), \tag{5.3.2}$$

with the topology as follows. Recall, for each $\mathbf{Q} > \mathbf{P}$, (5.2.5)

$$X_{\mathbf{Q}} \simeq N_{P'}(\mathbb{R}) \times A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ \times X_{\mathbf{P}}.$$

So $e(\mathbf{P})$ is attached to $e(\mathbf{Q})$ at infinity (here, we use the isomorphism $A_{\mathbf{P}}(\mathbb{R})^+ \simeq \mathbb{R}^r_{>0}$ which is componentwise the inverse of (5.1.1)). In particular for $\mathbf{Q} = \mathbf{G}^{\mathrm{der}}$, we retain the horospherical decomposition $X \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}}$. Now for any $z \in e(\mathbf{P}) = X_{\mathbf{P}}$, a basis of neighborhood system of of z in $\overline{X}^{\mathrm{RBS}}$ is given by

$$(N_P(\mathbb{R}) \times A_{\mathbf{P},t} \times W) \cup \coprod\nolimits_{\mathbf{Q}>\mathbf{P}} (N_{P'}(\mathbb{R}) \times A_{\mathbf{P},\mathbf{Q},t} \times W)$$

with W a neighborhood of z in $e(\mathbf{P})$ and $t > 0$. Observe that if W is open, then the union above is the interior of the closure of $N_P(\mathbb{R}) \times A_{\mathbf{P},t} \times W$ in $\overline{X}^{\mathrm{RBS}}$.

Similarly to the discussion on Borel–Serre compactifications, we have:

Theorem 5.3.1. *$\overline{X}^{\mathrm{RBS}}$ is Hausdorff, and the $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action on X extends continuously to $\overline{X}^{\mathrm{RBS}}$.*

The quotient $\Gamma\backslash\overline{X}^{\mathrm{RBS}}$ is a compact Hausdorff space containing $\Gamma\backslash X$ as an open dense subset. If we let $\{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$ be a set of representatives of the Γ-conjugacy classes of proper parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$, then we have

$$\Gamma\backslash\overline{X}^{\mathrm{RBS}} = \Gamma\backslash X \cup \coprod_{j=1}^{m} \Gamma_{\mathbf{M}_{\mathbf{P}_j}} \backslash X_{\mathbf{P}_j} \tag{5.3.3}$$

with $\Gamma_{\mathbf{M}_{\mathbf{P}_j}} := \Gamma \cap \mathbf{M}_{\mathbf{P}_j}(\mathbb{Q})$.

We also use $\overline{\Gamma\backslash X}^{\mathrm{RBS}}$ to denote $\Gamma\backslash\overline{X}^{\mathrm{RBS}}$.

Theorem 5.3.2. *The identity map on X extends to a continuous surjective $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-equivariant map $\overline{X}^{\mathrm{BS}} \to \overline{X}^{\mathrm{RBS}}$.*

The identity map on $\Gamma\backslash X$ extends to a continuous map $\overline{\Gamma\backslash X}^{\mathrm{BS}} \to \overline{\Gamma\backslash X}^{\mathrm{RBS}}$.

Chapter 6

Baily–Borel compactification

Let $(\mathbf{G}, X)$ be a Shimura datum. By abuse of notation use X to denote a connected component. Let $\Gamma < \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup.

Throughout the whole chapter, we will assume $\mathbf{G}$ to be quasi-simple, *i.e.* $\mathbf{G}^{\mathrm{der}}$ is a simple group. For the purpose of compactifying $\Gamma\backslash X$ we can easily reduce to this case. Notice that $\mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$ may not be simple as an $\mathbb{R}$-group, so that X is not necessarily irreducible.

We also fix a maximal $\mathbb{Q}$-split torus $\mathbf{S}$ of $\mathbf{G}^{\mathrm{der}}$, and a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$ which contains $\mathbf{S}$.

6.1 Baily–Borel compactification as a topological space

Consider the Harish–Chandra embedding

$$X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \simeq \mathbb{C}^N$$

with $\overline{\mathcal{D}}$ the closure of $\mathcal{D}$ in $\mathfrak{m}^+$. Let $F \neq X$ be an analytic boundary component of X, with normalizer $N(F) = \{g \in \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+ : gF = F\}$.

Recall (5.1.3) the fundamental set Σ constructed from Siegel sets associated with the minimal rational parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$.

Let $\overline{\Sigma}$ be the closure of $\Sigma \subseteq X \simeq \mathcal{D} \subseteq \mathfrak{m}^+$. Then $\overline{\Sigma} \subseteq \overline{\mathcal{D}}$, with an induced topology.

Theorem 6.1.1. *The followings are equivalent:*

(1) $\Gamma F \cap \overline{\Sigma} \neq \emptyset$,

(2) *F is a rational analytic boundary component (i.e. $N(F)$ equals $\mathbf{P}_F(\mathbb{R})$ for a parabolic subgroup $\mathbf{P}_F$ of $\mathbf{G}^{\mathrm{der}}$), and $\mathbf{P}_F$ is a maximal proper parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$.*

Theorem 6.1.1 indicates that we can do the following compactification of $\Gamma\backslash X$:

(i) Define $\overline{X}^{\mathrm{BB}} := \mathcal{D} \cup \bigsqcup_{\mathbf{P}} F_{\mathbf{P}} \subseteq \overline{\mathcal{D}}$, where $\mathbf{P}$ runs over all *maximal* proper parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$ and $F_{\mathbf{P}}$ is the rational analytic boundary component $F_{\mathbf{P}}$.

(ii) Endow $\overline{X}^{\mathrm{BB}}$ with the Satake topology.

(iii) The space $\Gamma\backslash\overline{X}^{\mathrm{BB}}$ is then a compact Hausdorff space containing $\Gamma\backslash X$ as an open dense subset.

Then $(\Gamma\backslash\overline{X})^{\mathrm{BB}} := \Gamma\backslash\overline{X}^{\mathrm{BB}}$ is called the *Baily–Borel compactification* of $\Gamma\backslash X$, and

$$(\Gamma\backslash\overline{X})^{\mathrm{BB}} = \Gamma\backslash X \cup \bigsqcup_{j=1}^{m} \Gamma_{F_j}\backslash F_j, \tag{6.1.1}$$

where $F_1, \ldots, F_m$ are rational analytic boundary components such that $\{\mathbf{P}_{F_1}, \ldots, \mathbf{P}_{F_m}\}$ is a set of representatives of Γ-conjugacy classes of maximal proper parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$, with $\Gamma_{F_j} := \Gamma \cap \mathbf{P}_{F_j}(\mathbb{Q})$.

6.1.1 Satake topology on $\overline{X}^{\mathrm{BB}}$

The *Satake topology* on $\overline{X}^{\mathrm{BB}}$ is defined as follows. For each $x \in \overline{X}^{\mathrm{BB}} \subseteq \overline{\mathcal{D}}$, the neighborhoods of any point $x \in X^*$ is the saturations of the neighborhoods of the corresponding points in $\overline{\Sigma}$ under the action of $\Gamma_x := \{\gamma \in \Gamma : \gamma \cdot x = x\}$. More precisely, a fundamental system of neighborhoods of x is given by all subsets $U \subseteq \overline{\mathcal{D}}$ such that

$$\Gamma_x \cdot U = U,$$

and such that $\gamma U \cap \overline{\Sigma}$ is a neighborhood of $\gamma \cdot x$ in $\overline{\Sigma}$ whenever $\gamma \cdot x \in \overline{\Sigma}$.

Proposition 6.1.2. *The Satake topology is the unique topology on $\overline{X}^{\mathrm{BB}}$ such that the followings hold:*

(i) it induces the original topologies on $\overline{\Sigma}$ and on X,

(ii) the $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action on $\overline{X}^{\mathrm{BB}}$ is continuous,

(iii) for any $x \in \overline{X}^{\mathrm{BB}}$, there exists a fundamental system of neighborhoods $\{U\}$ of x such that

$$\gamma U = U \text{ for all } \gamma \in \Gamma_x \quad \text{and} \quad \gamma U \cap U = \emptyset \text{ for all } \gamma \notin \Gamma_x,$$

(iv) if $x, x' \in \overline{X}^{\mathrm{BB}}$ are not in one Γ-orbit, then there exist neighborhoods U of x and U' of x' such that

$$\Gamma U \cap U' = \emptyset.$$

Corollary 6.1.3. $\Gamma\backslash\overline{X}^{\mathrm{BB}}$ *is compact and Hausdorff.*

6.1.2 $\mathbb{Q}$-roots vs $\mathbb{R}$-roots, and $\mathbb{Q}$-polydisc

Let K_∞ be a maximal compact subgroup of $\mathbf{G}_{\mathbb{R}}^{\mathrm{der}}$ such that $\mathrm{Lie}K_\infty \cap \mathrm{Lie}\mathbf{S}_{\mathbb{R}} = 0$, for the maximal $\mathbb{Q}$-split torus $\mathbf{S} < \mathbf{P}_0$.[1] Then there exists $x_0 \in X$ such that $\mathrm{Stab}_{\mathbf{G}^{\mathrm{der}}(\mathbb{R})}(x_0) = K_\infty$.

In §4.6.2 and §4.6.3, we discussed the complex roots and real roots respectively. Now we need to analyze the $\mathbb{Q}$-roots of $\mathbf{G}^{\mathrm{der}}$. First, we can make an appropriate choice of K_∞ such that $\mathbf{S}_{\mathbb{R}} < A$ with A from §4.6.3.

Let ${}_{\mathbb{R}}\Psi = \{\gamma_1, \ldots, \gamma_r\}$ be as in (4.6.13); it is obtained from a set of strongly orthogonal complex roots $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ from (4.6.4). If $\mathbf{G}_{\mathbb{R}}^{\mathrm{der}}$ is simple, we described the real roots ${}_{\mathbb{R}}\Phi = \Phi(A, \mathbf{G}_{\mathbb{R}}^{\mathrm{der}})$ in terms of $\gamma_1, \ldots, \gamma_r$ in Proposition 4.6.12. It turns out that one can also do this for the rational roots ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\mathrm{der}})$ when $\mathbf{G}^{\mathrm{der}}$ is simple.

[1]Even strongly, $\mathrm{Lie}K_\infty$ is orthogonal to $\mathrm{Lie}\mathbf{S}_{\mathbb{R}}$ for the Killing form on $\mathrm{Lie}\mathbf{G}_{\mathbb{R}}^{\mathrm{der}}$.

Proposition 6.1.4. *Let $s = \dim \mathbf{S}$. There is a partition*

$$\{1, \ldots, r\} = I_0 \cup I_1 \cup \cdots \cup I_s \tag{6.1.2}$$

such that

$$\begin{aligned} X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q} &\simeq \frac{X(A) \otimes_{\mathbb{Z}} \mathbb{Q}}{\{\text{subspace spanned by } \gamma_i \text{ for } i \in I_0 \text{ and } \gamma_i - \gamma_j \text{ for } i, j \in I_\ell,\ \ell > 0\}} \\ &\simeq \sum_{\ell=1}^{s} \mathbb{Q}\beta_\ell, \qquad \text{where } \beta_\ell = \text{image of any } \gamma_j \text{ with } j \in I_\ell. \end{aligned}$$

In particular, $\mathbf{S} = \{\gamma_i = 1 \text{ for } i \in I_0;\ \gamma_i = \gamma_j \text{ for } i, j \in I_\ell, \text{ where } \ell > 0\}$.

Corollary 6.1.5. *Recall our assumption that $\mathbf{G}^{\mathrm{der}}$ is simple. One of the two cases occurs:*

- (Type C_s) ${}_{\mathbb{Q}}\Phi = \{\pm\frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j,\ \pm\frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j\}$.
- (Type BC_s) ${}_{\mathbb{Q}}\Phi = \{\pm\frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j,\ \pm\frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j,\ \pm\frac{1}{2}\beta_i\}$.

If we order the roots such that $\beta_1 > \ldots > \beta_s$, then the set of simple roots is:

- (Type C_s) ${}_{\mathbb{Q}}\Delta = \{\nu_1 := \frac{1}{2}(\beta_1 - \beta_2), \ldots, \nu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \nu_s := \beta_s\}$.
- (Type BC_s) ${}_{\mathbb{Q}}\Delta = \{\nu_1 := \frac{1}{2}(\beta_1 - \beta_2), \ldots, \nu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \nu_s := \frac{1}{2}\beta_s\}$.

The proof goes as follows: We have the group-theoretic result that $\mathbf{G}^{\mathrm{der}} = \mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}'$ for some absolutely simple k-group $\mathbf{G}'$ with k a totally real number field. Then $\mathbf{G}^{\mathrm{der}}_{\mathbb{R}} = \prod_{\sigma\colon k \hookrightarrow \mathbb{R}} \mathbf{G}'_\sigma$ with each $\mathbf{G}'_\sigma$ a group defined over $\sigma(k) \subseteq \mathbb{R}$. Then one analyzes each factor and use the Galois action.

Next we turn to the $\mathbb{Q}$-polydisc. Recall from the Polydisc Theorem (Theorem 4.6.7) that we have a totally geodesic embedding $D^r \to X$ (with $D = \{z \in \mathbb{C} : |z| < 1\}$ the Poincaré unit disc) arising from a group morphism

$$\varphi\colon \mathrm{SL}_2(\mathbb{R})^r \to \mathbf{G}^{\mathrm{der}}(\mathbb{R}), \tag{6.1.3}$$

and $X = K_\infty \cdot D^r$. This embedding gives rise to the analytic boundary components as in the diagram (4.6.15). Let us rephrase it here. Recall $\mathfrak{H} \simeq D$ with the Cayley transformation sending $\sqrt{-1} \mapsto 0$ and $\infty \mapsto 1$. Then we have the diagram

$$\begin{tikzcd} \mathfrak{H}^r \arrow[r, "f_1"] \arrow[d, "\subseteq"'] & \mathcal{D} \arrow[d, "\subseteq"] \\ (\mathbb{P}^1)^r \arrow[r, "f_3"] & X^\vee \end{tikzcd} \tag{6.1.4}$$

where f_1 is the natural composite $\mathfrak{H}^r \simeq D^r \to X \simeq \mathcal{D}$, with $D^r \to X$ the geodesic embedding as above and $X \simeq \mathcal{D}$ the Harish–Chandra realization, and $\mathcal{D} \subseteq X^\vee$ from (4.6.3). Then for any subset $S \subseteq \{1, \ldots, r\}$, the unique standard analytic boundary component containing the point $f_3((\sqrt{-1})_{j \notin S}, (\infty)_{j \in S})$ is F_S. In general, an analytic boundary component of X is of the form $g \cdot F_S$ for some $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{R})$.

We wish to do this discussion and obtain the relevant results over $\mathbb{Q}$. First of all, any rational analytic boundary component is easily seen to be of the form $g \cdot F_S$, with $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and F_S rational. Next we prove the following lemma.

Lemma 6.1.6. *For $S \subseteq \{1, \dots, r\}$, the standard analytic boundary component F_S is rational if and only if $S = I_{\ell_1} \cup \cdots \cup I_{\ell_t}$, where $1 \le \ell_1 < \cdots < \ell_t \le s$, for the partition* (6.1.2).

Proof. For the proof, it is more convenient to use the description of parabolic subgroups given by Theorem 4.2.5. In §4.6.6, we explained that the normalizer $P_{F_S} = P(\lambda_S)$, with $\lambda_S \colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$ sending

$$t \mapsto \varphi(\underbrace{\cdots, 1, \cdots}_{j \notin S}, \underbrace{\cdots, \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}, \cdots}_{j \in S}),$$

with φ from (6.1.3). By Proposition 6.1.4, λ is defined over $\mathbb{Q}$ if and only if $S = I_{\ell_1} \cup \cdots \cup I_{\ell_t}$ for some $1 \le \ell_1 < \cdots < \ell_t \le s$. We are done. $\square$

With this lemma in hand, we obtain the $\mathbb{Q}$-version of (6.1.4)

$$\begin{tikzcd} \mathfrak{H}^s \arrow[r, "f_{1,\mathbb{Q}}"] \arrow[d, "\subseteq"'] & \mathcal{D} \arrow[d, "\subseteq"] \\ (\mathbb{P}^1)^s \arrow[r, "f_{3,\mathbb{Q}}"] & X^\vee \end{tikzcd} \tag{6.1.5}$$

arising from

$$\varphi_{\mathbb{Q}} \colon \mathrm{SL}_2(\mathbb{R})^s \to \mathbf{G}^{\mathrm{der}}_{\mathbb{R}} \tag{6.1.6}$$

such that $\varphi_{\mathbb{Q}}$(diagonal matrices) is the maximal $\mathbb{Q}$-split torus $\mathbf{S}$ of $\mathbf{G}^{\mathrm{der}}$. We can renumber the factors of $\mathfrak{H}^s$ and $\mathrm{SL}_2(\mathbb{R})^s$ such that: For the $\beta_1, \dots, \beta_s \in {}_{\mathbb{Q}}\Phi$ from Proposition 6.1.4, we have

$$\beta_\ell \colon \varphi_{\mathbb{Q}} \left(\begin{bmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{bmatrix}, \dots, \begin{bmatrix} t_s & 0 \\ 0 & t_s^{-1} \end{bmatrix} \right) \mapsto t_\ell^2. \tag{6.1.7}$$

Now for each subset $S_{\mathbb{Q}} \subseteq \{1, \dots, s\}$, the unique standard analytic boundary component which contains the point

$$f_{3,\mathbb{Q}}(\underbrace{\cdots, \sqrt{-1}, \cdots}_{\ell \notin S_{\mathbb{Q}}}, \underbrace{\cdots, \infty, \cdots}_{\ell \in S_{\mathbb{Q}}})$$

is F_S with $S = \bigcup_{\ell \in S_{\mathbb{Q}}} I_\ell$. In particular, F_S is rational.

Proof of Theorem 6.1.1. Assume F meets $\overline{\Sigma}$.

Order the roots such that $\beta_1 > \ldots > \beta_s$, then $\mathbf{S}(\mathbb{R})^+$ consists of

$$\varphi_{\mathbb{Q}} \left(\begin{bmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{bmatrix}, \dots, \begin{bmatrix} t_s & 0 \\ 0 & t_s^{-1} \end{bmatrix} \right)$$

where $t_1 \ge \ldots \ge t_s \ge 1$. Hence

$$\overline{\mathbf{S}(\mathbb{R})^+ x_0} = f_{3,\mathbb{Q}} \left(\{ (\sqrt{-1} x_1, \dots, \sqrt{-1} x_s) : \infty \ge x_1 \ge \cdots \ge x_s \ge 1 \} \right).$$

Hence $\overline{\mathbf{S}(\mathbb{R})^+ x_0}$ meets exactly the standard boundary components $F_1, \dots, F_s$ with

$$f_{3,\mathbb{Q}}(\infty, \sqrt{-1}, \dots, \sqrt{-1}) \in F_1,\ f_3(\infty, \infty, \sqrt{-1}, \dots, \sqrt{-1}) \in F_2,\ \dots,\ f_3(\sqrt{-1}, \sqrt{-1}, \dots, \sqrt{-1}) \in F_s.$$

So $F = F_\ell$ for some $\ell \in \{1, \dots, s\}$. We can compute the normalizer of each F_ℓ as in Theorem 4.6.19, and get

$$N(F_\ell) = \mathbf{P}_{{}_{\mathbb{Q}}\Delta \setminus \{\nu_\ell\}}(\mathbb{R})$$

for each $\ell \in \{1, \dots, s\}$. Hence we are done. $\square$

6.2 First step towards the complex structure

6.2.1 A general criterion for a topological space to be complex analytic

Assume V is a compact Hausdorff space which can be written as a disjoint union

$$V = V_0 \bigsqcup V_1 \bigsqcup \cdots \bigsqcup V_m$$

with each V_j an irreducible normal complex analytic space. Assume that $\dim V_0 > \dim V_j$ for all $j \geq 1$, and that V_0 is open dense in V.

Define a sheaf $\mathcal{F}$ of $\mathcal{A}$-functions on V as follows. For any open subset $U \subseteq V$, a complex-valued continuous function on U is an $\mathcal{A}$-function if its restriction to each $U \cap V_j$ $(0 \leq j \leq m)$ is complex analytic.

Proposition 6.2.1. *Assume:*

(i) For each integer $d \geq 1$, the union $V_{(d)} := \bigcup_{\dim V_j \leq d} V_j$ is closed.

(ii) Any $v \in V$ has a countable fundamental set of open neighborhoods $\{U_\ell\}$ such that $U_\ell \cap V_0$ is connected for all ℓ.

(iii) The restriction to V_j of local $\mathcal{A}$-functions define the structure sheaf of V_j, for all $j \geq 0$.

(iv) Any $v \in V$ has a neighborhood U_v whose points are separated by the $\mathcal{A}$-functions defined on U.

Then V is an irreducible normal complex analytic space with structure sheaf $\mathcal{F}$. For each $d \leq \dim V_0$, the union $V_{(d)}$ is an analytic subspace of V with dimension $\max\{\dim V_j : V_j \subseteq V_{(d)}\}$.

6.2.2 Application to the Baily–Borel compactification

We shall apply Proposition 6.2.1 to the Baily–Borel compactification (6.1.1) (which is compact Hausdorff space by Corollary 6.1.3), with $V_0 = \Gamma\backslash X$ and $V_j = \Gamma_{F_j}\backslash F_j$ for $1 \leq j \leq m$.

Conditions (i) and (ii) can be shown to hold by checking with the Satake topology from §6.1.1.

To check condition (iii), we define the projection

$$\pi_F \colon X \to F \tag{6.2.1}$$

for each analytic boundary component F. We focus on the rational ones. The example of the Siegel case will be presented in Example 6.3.6.

Recall our choice of a maximal $\mathbb{Q}$-split torus $\mathbf{S}$ (from §6.1.2) in our minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$ (see above Theorem 6.1.1), and the basis ${}_{\mathbb{Q}}\Delta = \{\nu_1, \ldots, \nu_s\}$ (see Corollary 6.1.5) of the relative root system ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\mathrm{der}})$. The root ν_s is called the *distinguished root* because it has different length.

Over $\mathbb{R}$, we explained the relation between F and the boundary symmetric domain associated with $P_F = N(F)$; see below Theorem 4.6.19. The discussion can be carried over $\mathbb{Q}$.

Let F be a rational analytic boundary component which meets $\overline{\Sigma}$. We have shown in the proof of Theorem 6.1.1 that $\mathbf{P}_F = \mathbf{P}_{{}_{\mathbb{Q}}\Delta\setminus\{\nu_\ell\}}$ for some $\ell \in \{1, \ldots, s\}$. Let $I_h := \{\nu_{\ell+1}, \ldots, \nu_s\}$ and $I_l := \{\nu_1, \ldots, \nu_{\ell-1}\}$. We thus have the refined rational horospherical decomposition

$$X \simeq N_{P_F}(\mathbb{R}) \times A_{\mathbf{P}_F}(\mathbb{R})^+ \times X_{\mathbf{P}_{I_h}} \times X_{\mathbf{P}_{I_l}}.$$

Moreover, the proof of Theorem 6.1.1 exhausts the possibilities of all F's, and hence implies that F can be identified with the boundary symmetric domain associated with $\mathbf{P}_{I_h}$. Thus the refined rational horospherical decomposition above becomes

$$X \simeq N_{P_F}(\mathbb{R}) \times A_{\mathbf{P}_F}(\mathbb{R})^+ \times F \times X_{\mathbf{P}_{I_l}}, \tag{6.2.2}$$

and it induces a natural projection $X \to F$, which is our desired π_F. Although the decomposition is only real semi-algebraic, the projection π_F is also holomorphic.

If F is contained in $\overline{F'}$ for another rational boundary component F', then F is a rational boundary component of F', and one gets a projection $\pi_{F',F}\colon F' \to F$. It is not hard to check that π_F is the composite of $\pi_{F',F} \circ \pi_{F'}$.

Now to check condition (iii) of Proposition 6.2.1, we only need to work locally and hence on the universal covering. But now for any rational boundary component F of X, any complex analytic function near F can be extended to an $\mathcal{A}$-function on a neighborhood of F in $\overline{X}^{\mathrm{BB}}$ by the discussion above. This establishes (iii).

Proving condition (iv) is the hardest part. We need to realize X as a Siegel domain of the third kind[2] and define the Poincaré–Eisenstein series.

6.3 X as a Siegel domain of the third kind

Continue to use the notation from §6.1.2. In particular, we have the relative root system ${}_{\mathbb{Q}}\Phi = \Phi(\mathbf{S}, \mathbf{G}^{\mathrm{der}})$, the roots $\beta_1 > \cdots > \beta_s$ which arise from the set of strongly orthogonal roots, and the basis ${}_{\mathbb{Q}}\Delta = \{\nu_1 = \frac{1}{2}(\beta_1 - \beta_2), \ldots, \nu_{s-1} = \frac{1}{2}(\beta_{s-1} - \beta_s)\} \bigcup \{\nu_s\}$; ν_s is the distinguish roots which is either β_s or $\frac{1}{2}\beta_s$.

Let $\mathbf{P}$ be a standard maximal proper parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$. Then $\mathbf{P} = \mathbf{P}_{{}_{\mathbb{Q}}\Delta\setminus\{\nu_\ell\}}$ for some $\ell \in \{1, \ldots, s\}$. We have seen in the proof of Theorem 6.1.1 that $\mathbf{P}(\mathbb{R})$ is the normalizer of the standard rational analytic boundary component $F = F_S$ with $S = I_1 \bigcup \ldots \bigcup I_\ell$.

The non-standard maximal proper parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$ are $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-conjugates of the standard ones, and the rational analytic boundary components are all of the form $g \cdot F_S$ with $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and F_S as above. So all the discussion in this section applies to an arbitrary rational analytic boundary component by applying suitable $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-conjugation.

6.3.1 Rational 5-group decomposition and refined horospherical decomposition

Let $I_h := \{\nu_{\ell+1}, \ldots, \nu_s\}$ and $I_l := \{\nu_1, \ldots, \nu_{\ell-1}\}$.

Above Example 4.6.20, we defined several subgroups of $\mathbf{P}_{\mathbb{R}}$. We can also define the following $\mathbb{Q}$-subgroups of $\mathbf{P}$ similarly:

- $\mathbf{W}(F) := \mathcal{R}_u(\mathbf{P})$,
- $\mathbf{L}(F)$ is a suitable Levi subgroup of $\mathbf{P}$ which we shall define later using Lie algebra,
- $\mathbf{G}_h(F)$ is the semi-simple group whose $\mathbb{Q}$-root system is spanned by I_h. It is normal subgroup of $\mathbf{L}(F)$, has no compact $\mathbb{Q}$-factors, and F is a $\mathbf{G}_h(F)(\mathbb{R})^+$-orbit,

[2]In the original paper of Baily–Borel, this was done using partial Cayley transformation and the map η from Theorem 2.3.5. We will directly introduce the more explicit version using the 5-decomposition of the parabolic subgroup. This more explicit version will be crucial for the toroidal compactifications.

- $\mathbf{G}_l(F)\mathbf{M}(F)$ is the normal reductive subgroup of $\mathbf{L}(F)$ which is the complement to $\mathbf{G}_h(F)$, with $\mathbf{G}_l(F)$ with no compact $\mathbb{Q}$-factors and $\mathbf{M}(F)(\mathbb{R})$ compact.

Then $\mathbf{P} = \mathbf{W}(F) \rtimes (\mathbf{G}_h(F) \cdot \mathbf{G}_l(F) \cdot \mathbf{M}(F))$.

Remark 6.3.1. *Let us compare them with the subgroups of $\mathbf{P}_\mathbb{R}$ defined above Example 4.6.20. We have $W(F) = \mathbf{W}(F)_\mathbb{R}$ and $L(F) = \mathbf{L}(F)_\mathbb{R}$ by definition. Later on, we will see that $G_l(F) = \mathbf{G}_l(F)_\mathbb{R}$ in Corollary 6.3.4. However, in general $\mathbf{G}_h(F)_\mathbb{R}$ is not $G_h(F)$. In other words, $G_h(F)$ may not be defined over $\mathbb{Q}$, in which case we have $G_h(F) = \mathbf{G}_h(F)_\mathbb{R} \cdot M'$ for some factor M' of $\mathbf{M}(F)_\mathbb{R}$.*

Denote by $\mathfrak{g} := \mathrm{Lie}\mathbf{G}^{\mathrm{der}}$. Then one can compute that

$$\mathrm{Lie}\mathbf{P} = \mathfrak{g}^{\mathbf{S}} \oplus \sum_{\substack{\varphi = \frac{\pm\beta_i \pm \beta_j}{2} \text{ or } \frac{\pm\beta_i}{2} \\ \ell+1 \le i,j \le s}} \mathfrak{g}_\varphi \oplus \sum_{\substack{\varphi = \frac{\gamma_i \pm \gamma_j}{2} \text{ or } \frac{\gamma_i}{2} \\ 1 \le i \le \ell}} \mathfrak{g}_\varphi.$$

Now we define $\mathbf{L}(F)$ to be the subgroup of $\mathbf{P}$ whose Lie algebra is the direct sum of the first two factors. Then by construction, $\mathbf{L}(F)$ is defined over $\mathbb{Q}$ and $\mathbf{L}(F)_\mathbb{R}$ is precisely the $L(F)$ defined above Example 4.6.20.

The Lie algebra of $\mathbf{W}(F) = \mathcal{R}_u(\mathbf{P})$ is, by computation, the direct sum of

$$\mathfrak{u} := \sum_{\substack{\varphi = \frac{\gamma_i + \gamma_j}{2} \\ 1 \le i,j \le \ell}} \mathfrak{g}_\varphi$$

and

$$\mathfrak{v} := \sum_{\substack{\varphi = \frac{\gamma_i \pm \gamma_j}{2} \text{ or } \frac{\gamma_i}{2} \\ 1 \le i \le \ell,\ \ell+1 \le j \le s}} \mathfrak{g}_\varphi.$$

Let $\mathbf{U}(F)$ be the exponent of $\mathfrak{u}$. One can prove:

Lemma 6.3.2. *$\mathbf{U}(F)$ is the center of $\mathbf{W}(F)$, and $\mathbf{V}(F) := \mathbf{W}(F)/\mathbf{U}(F)$ is a vector group (i.e. abelian and diffeomorphic to its Lie algebra).*

Clearly, $\mathrm{Lie}\mathbf{V}(F)$ can be canonically identified with $\mathfrak{v}$.

Write $U(F) = \mathbf{U}(F)_\mathbb{R}$ and $V(F) = \mathbf{V}(F)_\mathbb{R}$. Then the refined rational horospherical decomposition (6.2.2) can be furthermore refined to be

$$X \simeq U(F)(\mathbb{R}) \times V(F)(\mathbb{R}) \times A_\mathbf{P}(\mathbb{R})^+ \times F \times X_{\mathbf{P},l}. \tag{6.3.1}$$

Notice that the $\mathbb{R}$-split torus $\mathbf{A_P}$ is contained in $\mathbf{G}_l(F)$ by definition of $\mathbf{G}_l(F)$, and $X_{\mathbf{P},l} \simeq \frac{(\mathbf{G}_l(F)/\mathbf{A_P})(\mathbb{R})^+}{\text{maximal compact}}$. Denote by $K_{l,\infty}$ this maximal compact subgroup of $\mathbf{G}_l(F)(\mathbb{R})^+$, then we have

$$X_{\mathbf{P},l} \simeq \mathbf{G}_l(F)(\mathbb{R})^+/K_{l,\infty}A_\mathbf{P}(\mathbb{R})^+. \tag{6.3.2}$$

6.3.2 Cone in $U(F)(\mathbb{R})$

We start with the following proposition. Let us go back to $\mathbb{R}$-groups, and recall the subgroups $G_h(F)$, $G_l(F)$, $M(F)$ of $\mathbf{P}_\mathbb{R}$ defined above Example 4.6.20. We have $\mathbf{L}(F)_\mathbb{R} = G_h(F) \cdot G_l(F) \cdot M(F)$ as almost direct product.

The group $\mathbf{L}(F)$ acts naturally on $\mathbf{W}(F)$, and hence on $\mathbf{U}(F)$. So $G_h(F)$, $G_l(F)$, and $M(F)$ act on $U(F)$.

Proposition 6.3.3. *The centralizer of the action of* $L(F) = G_h(F) \cdot G_l(F) \cdot M(F)$ *on* $U(F)$ *is* $G_h(F) \cdot M(F)$.

Proof. The subgroups $G_h(F)$, $G_l(F)$, and $M(F)$ can all be defined using the (real) root decomposition (4.6.12), for example (4.6.14) for $G_h(F)$. Hence we can prove this proposition by checking roots and direct computation. □

This proposition immediately yields the following proposition:

Corollary 6.3.4. *The group* $G_l(F)$ *is defined over* $\mathbb{Q}$, *and hence is precisely* $\mathbf{G}_l(F)_{\mathbb{R}}$.

Proof. Since $U(F)$ is defined over $\mathbb{Q}$ and $L(F)$ is defined over $\mathbb{Q}$, the centralizer of the action of $L(F)$ on $U(F)$ is also defined over $\mathbb{Q}$. So $G_h(F) \cdot M(F)$ is defined over $\mathbb{Q}$ by Proposition 6.3.3. But $G_l(F)$ is defined to be the complement of $G_h(F) \cdot M(F)$ in $L(F)$. So we are done. □

For the morphism $\varphi_{\mathbb{Q}} \colon \mathrm{SL}_2(\mathbb{R})^s \to \mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$ from (6.1.6), take the point

$$\Omega_F := \varphi_{\mathbb{Q}} \left(\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\ell\text{-components}}, 1, \ldots, 1 \right) \in U(F)(\mathbb{R}).$$

Consider the action of $G_l(F)(\mathbb{R})^+$ on $U(F)(\mathbb{R})$.

Proposition 6.3.5. $\mathrm{Stab}_{G_l(F)(\mathbb{R})^+}(\Omega_F) = K_{l,\infty}$.

The orbit

$$C(F) := \{ g\Omega_F g^{-1} : g \in G_l(F)(\mathbb{R})^+ \}$$

is an open symmetric homogeneous cone in $U(F)(\mathbb{R})$.

By (6.3.2), we have $C(F) \simeq A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P},l}$. Hence (6.3.1) becomes

$$X \simeq U(F)(\mathbb{R}) \times C(F) \times V(F)(\mathbb{R}) \times F. \tag{6.3.3}$$

Denote by

$$\Phi_F \colon X \to C(F) \tag{6.3.4}$$

the natural projection.

Example 6.3.6. *In the Siegel case,* $s = r$ *(i.e. the* $\mathbb{Q}$*-rank equals the* $\mathbb{R}$*-rank), and the partition* (6.1.2) *is simply* $I_0 = \emptyset$ *and* $I_\ell = \{\ell\}$.

Take $\mathbf{P} = \mathbf{P}_{{}_\mathbb{Q}\Delta\setminus\{\nu_{d'}\}}$. *Then as in Example 4.6.20, we have*

$$\mathbf{P} = \left\{ \begin{bmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & (u^{\mathrm{t}})^{-1} \end{bmatrix} \in G : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d',\mathbb{Q}},\ u \in \mathrm{GL}_{d-d',\mathbb{Q}} \right\},$$

$$\mathbf{W}(F) = \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & n \\ m^{\mathrm{t}} & I_{d-d'} & n^{\mathrm{t}} & b \\ 0 & 0 & I_{d'} & -m \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : n^{\mathrm{t}}m + b = m^{\mathrm{t}}n + b^{\mathrm{t}} \right\},$$

$$\mathbf{G}_h(F) = \left\{ \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & I_{d-d'} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d',\mathbb{Q}} \right\} \simeq \mathrm{Sp}_{2d',\mathbb{Q}},$$

$$\mathbf{G}_l(F) = \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & I_{d'} & 0 \\ 0 & 0 & 0 & (u^{\mathrm{t}})^{-1} \end{bmatrix} : u \in \mathrm{GL}_{d-d',\mathbb{Q}} \right\} \simeq \mathrm{GL}_{d-d',\mathbb{Q}},$$

$$\mathbf{M}(F) = \{\pm I_{2d}\}.$$

Moreover,

$$\mathbf{U}(F) = \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & 0 \\ 0 & I_{d-d'} & 0 & b \\ 0 & 0 & I_{d'} & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : b = b^{\mathrm{t}} \right\} \simeq \{b \in \mathrm{Mat}_{(d-d')\times(d-d')}(\mathbb{Q}) : b = b^{\mathrm{t}}\} \ni \Omega_F = I_{d-d'},$$

and

$$C(F) = \{b \in \mathrm{Mat}_{(d-d')\times(d-d')}(\mathbb{R}) : b = b^{\mathrm{t}},\ b > 0\}.$$

Notice that $F \simeq \mathfrak{H}_{d'}$ *in this case. The projections* (6.2.1) *and* (6.3.4) *are*

$$\pi_F \colon \mathfrak{H}_d \to F \simeq \mathfrak{H}_{d'}, \qquad \begin{bmatrix} \tau' & \tau_0 \\ \tau_0^{\mathrm{t}} & \tau'' \end{bmatrix} \mapsto \tau'$$
$$\Phi_F \colon \mathfrak{H}_d \to C(F), \qquad \begin{bmatrix} \tau' & \tau_0 \\ \tau_0^{\mathrm{t}} & \tau'' \end{bmatrix} \mapsto \mathrm{Im}\tau'' - (\mathrm{Im}\tau_0)^{\mathrm{t}}(\mathrm{Im}\tau')^{-1}(\mathrm{Im}\tau_0). \tag{6.3.5}$$

6.3.3 Fibered structure

Recall the Harish–Chandra embedding together with the Borel embedding $X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \subseteq X^\vee$, with $X^\vee$ a $\mathbf{G}^{\mathrm{der}}(\mathbb{C})$-orbit.

Define

$$\mathcal{D}(F) := \mathbf{U}(F)(\mathbb{C}) \cdot \mathcal{D} = \bigcup_{g \in \mathbf{U}(F)(\mathbb{C})} g \cdot \mathcal{D} \subseteq X^\vee. \tag{6.3.6}$$

Then $\mathcal{D}(F)$ has a natural complex structure, and $U(F)(\mathbb{C})$ acts holomorphically on $\mathcal{D}(F)$. So the quotient

$$\mathcal{D}'(F) := \mathbf{U}(F)(\mathbb{C})\backslash\mathcal{D}(F) \tag{6.3.7}$$

has a complex structure. Moreover since $\mathcal{D}'(F) \simeq V(F)(\mathbb{R}) \times F$ real semi-algebraically, we have that $\mathcal{D}'(F) \to F$ is a complex vector bundle, *i.e.* each $x \in F$ determines a complex structure on $V(F)(\mathbb{R})$.

Now we have a holomorphic isomorphism $\mathcal{D}(F) \simeq U(F)(\mathbb{C}) \times \mathcal{D}'(F) = (U(F)(\mathbb{R}) \oplus \sqrt{-1}U(F)(\mathbb{R})) \times \mathcal{D}'(F)$. The cone $C(F)$ should be seen as a cone in $\sqrt{-1}U(F)(\mathbb{R})$.

Theorem 6.3.7. *The projection Φ_F in* (6.3.4) *extends to $\Phi_F \colon \mathcal{D}(F) \to U(F)(\mathbb{R})$ such that $X \simeq \mathcal{D} = \Phi_F^{-1}(C(F))$.*

We have the following $\mathbf{P}(\mathbb{R})$-equivariant commutative diagram of holomorphic maps

$$\begin{tikzcd}
C(F) & \subseteq & U(F)(\mathbb{R}) . \\
X \simeq \mathcal{D} \arrow[u, "\Phi_F"] \arrow[dr] \arrow[ddr, "\pi_F"'] & \subseteq & \mathcal{D}(F) \arrow[u, "\Phi_F"'] \arrow[dl, "\pi'_F"', "\mathrm{mod}\, U(F)(\mathbb{C})"] \\
& \mathcal{D}'(F) \arrow[d, "p_F"] & \\
& F &
\end{tikzcd}$$

The map p_F is a holomorphic vector bundle with each fiber $\simeq V(F)(\mathbb{R})$ (real semi-algebraically).

Example 6.3.8. *Continue with the Siegel case in Example 6.3.6. We have*

$$\mathcal{D}(F) \simeq \left\{ \tau = \begin{bmatrix} \tau' & \tau_0 \\ \tau_0^{\mathrm{t}} & \tau'' \end{bmatrix} \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}},\ \mathrm{Im}\tau' > 0 \right\}.$$

The map $\Phi_F \colon \mathcal{D}(F) \to U(F)(\mathbb{R})$ is defined with the same formula as (6.3.5). *The map π'_F is* $\mathrm{mod}\tau''$, *and the map $p_F \circ \pi'_F$ is* $\begin{bmatrix} \tau' & \tau_0 \\ \tau_0^{\mathrm{t}} & \tau'' \end{bmatrix} \mapsto \tau'$.

For $\tau' \in F \simeq \mathfrak{H}_{d'}$, the complex structure on $V(F)(\mathbb{R}) = W(F)(\mathbb{R})/U(F)(\mathbb{R})$ determined by τ' is

$$\begin{bmatrix} I_{d'} & 0 & 0 & n \\ m^{\mathrm{t}} & I_{d-d'} & n^{\mathrm{t}} & b \\ 0 & 0 & I_{d'} & -m \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} \mathrm{mod}\, b \mapsto \tau' m + n.$$

6.4 Poincaré–Eisenstein series and complex algebraic structure on $\overline{\Gamma\backslash X}^{\mathrm{BB}}$

6.4.1 Bounded realization of the Poincaré series

Consider the Harish–Chandra realization $X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \simeq \mathbb{C}^N$, where $\mathfrak{m}^+$ is identified with the holomorphic tangent space of $o \in X$.

For each $g \in \mathbf{G}(\mathbb{R})$, we have a map $g\cdot \colon \mathcal{D} \to \mathcal{D}$. Denote by $J_g \colon \mathcal{D} \to \mathbb{C}^\times$, sending each $z \in \mathcal{D}$ to the determinant of the Jacobian of the action $g\cdot$ on $\mathcal{D}$ at z. In fact $J_g(z)$ can be computed as follows; for simplicity we only write the formula for $z = o$. Denote by abuse of notation

$K_\infty := \mathrm{Stab}_{\mathbf{G}^{\mathrm{der}}(\mathbb{R})}(o)$, and let $M^+ \times K_{\infty,\mathbb{C}} \times M^- \to \mathbf{G}_{\mathbb{C}}$ be as in Theorem 2.3.5. The image of this map contains $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$. So each $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{R})$ can be decomposed into $m^+k(g)m^-$ in a unique way. Then we have

$$J_g(o) = \mathrm{Ad}_{\mathfrak{m}^+} k(g)^{-1}. \tag{6.4.1}$$

One can then prove that $|J_g(o)|$ is bounded on $\mathcal{D}$.[3]

Lemma 6.4.1. *The function $g \mapsto |J_g(o)|^m$ is in $L^1(\mathbf{G}^{\mathrm{der}}_{\mathbb{R}})$ for any $m \geq 2$, i.e.*

$$\int_{\mathbf{G}^{\mathrm{der}}(\mathbb{R})} |J_g(o)|^m \mathrm{d}g < \infty.$$

Proof. By (6.4.1), the function $g \mapsto |J_g(o)|^m$ is left and right invariant under K_∞, and in particular can be viewed as a function on $\mathcal{D} \simeq \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+/K_\infty$. We have

$$\int_{\mathbf{G}^{\mathrm{der}}(\mathbb{R})} |J_g(o)|^m \mathrm{d}g = \int_{\mathcal{D}} |J_x(o)|^m \mathrm{d}x$$

with $\mathrm{d}x$ a suitable invariant volume form, which up to a positive factor is $|J_x(o)|^{-2}\omega$ for the Euclidean volume form ω on $\mathfrak{m}^+$.

Now we can conclude because $\mathcal{D}$ is bounded and $|J_g(o)|$ is bounded. $\square$

Now we are ready to define the *Poincaré series* associated with any polynomial f on $\mathcal{D}$.

Definition 6.4.2. *Let $m \geq 2$. The* **Poincaré series** *of weight m is $P_{f,m}\colon \mathcal{D} \to \mathbb{C}$ defined by*

$$P_{f,m}(z) = \sum_{\gamma \in \Gamma} J_\gamma(z)^m f(\gamma z).$$

The series converges absolutely uniformly on compact sets by Lemma 6.4.1, and it satisfies the modularity condition by the chain rule. Indeed, $P_{f,m}$ is a holomorphic automorphic form of weight m.

Theorem 6.4.3. *Suppose Γ is torsion-free. For any Γ-inequivalent points $z_1, \ldots, z_n \in \mathcal{D}$ and any complex numbers $b_1, \ldots, b_n$, there exists a polynomial f on $\mathcal{D}$ such that*

$$P_{f,m}(z_1) = b_1, \ldots, P_{f,m}(z_n) = b_n$$

for all $m \gg 1$.

Proof. Fix $0 < u < 1$. The set $\Gamma_u := \{\gamma \in \Gamma : |J_\gamma(z_i)| \geq u\}$ is finite by Lemma 6.4.1. Thus we can take a polynomial f such that $f(z_j) = b_j$ and $f(\gamma z_j) = 0$ for all $j \in \{1, \ldots, n\}$ and all $\gamma \in \Gamma_u$. It is not hard to check that $|b_j - P_{f,m}(z_j)| = O(u^m)$, and hence $P_{f,m}(z_j) \to b_j$ with $m \to \infty$ because $0 < u < 1$. Therefore the image of the linear map $f \mapsto (P_{f,m}(z_1), \ldots, P_{f,m}(z_n))$ contains a basis of $\mathbb{C}^n$ and hence is surjective. Now we are done. $\square$

[3]This follows from the KAK-decomposition of $\mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$ and explicit computation for $g \in A$.

6.4.2 Poincaré–Eisenstein series with respect to an analytic boundary component

Now let F be a rational analytic boundary component and let $\pi_F \colon X \to F$ be the holomorphic projection (6.2.1).

Through the whole subsection, we will identify F with its Harish–Chandra realization and identify X as the Siegel domain of the third kind, fibered over F, as in Theorem 6.3.7.

For each $\gamma \in \Gamma$, denote by $J_\gamma^F \colon X \to \mathbb{C}^\times$ the map sending each $x \in X$ to the determinant of the Jacobian of the the action $\gamma\cdot$ on X at x.

Let $\Gamma_0 := \Gamma \cap (\mathbf{W}(F)\mathbf{G}_l(F))(\mathbb{Q})$ (see §6.3.1 for the notation). By (4.6.19) and Corollary 6.3.4, $\gamma_0 \cdot z = z$ for all $\gamma_0 \in \Gamma_0$ and $z \in F$.

Definition 6.4.4. *For any polynomial f on F and any $m \geq 2$, define the associated* **Poincaré–Eisenstein series** *of weight m to be*

$$E_{f,m}(x) := \sum_{\gamma \in \Gamma/\Gamma_0} f(\pi_F(\gamma x)) \cdot J_\gamma^F(x)^m.$$

For this definition to make sense, we need to check that every term in the sum of the right hand side is Γ_0-invariant. This is true for the first term by the discussion above, and is true for the second term for all m dividing a certain fixed integer m_0. From now on, we will take these m.

We also need to settle the convergence of the series defining $E_{f,m}(x)$. The key is the following proposition. Denote by $j_g \colon F \to \mathbb{C}^\times$, sending each $z \in F$ to the determinant of the Jacobian of the action $g\cdot$ on F at z.

Proposition 6.4.5. *For any $g \in P_F(\mathbb{R})$, there are rational numbers n and $q > 0$ such that*

$$|J_g^F(x)| = |\chi(g)|^n |j_g(\pi_F(x))|^q$$

where χ is a rational character of $\mathbf{P}_F$.

This settles the convergence issue: the series defining $E_{f,m}(x)$ is absolutely uniformly convergent on compact sets. Hence $E_{f,m}$ is a holomorphic function on X. Better, it is a holomorphic automorphic form.

6.4.3 Analytic structure on $\overline{\Gamma\backslash X}^{\mathrm{BB}}$

We need to prove the separation property (Proposition 6.2.1.(iv)) for $\overline{\Gamma\backslash X}^{\mathrm{BB}}$. The key theorem to prove is:

Theorem 6.4.6. *Let F be a rational analytic boundary component of X. The Poincaré–Eisenstein series $E_{f,m}$ associated with F extends to a holomorphic function on $\overline{X}^{\mathrm{BB}}$ (which by abuse of notation we still denote by $E_{f,m}$) with the following properties:*

(i) the restriction of $E_{f,m}$ to F is a Poincaré series on F,

(ii) $E_{f,m}$ vanishes on any rational analytic boundary component F' if $\dim F' \leq \dim F$ and $F' \not\subseteq \Gamma F$.

Moreover, all Poincaré series on F can be obtained as restrictions of such extensions of $E_{f,m}$.

The "Moreover" part of the theorem immediately implies the separation property because Poincaré series separate points on each boundary component (Theorem 6.4.3). So

Theorem 6.4.7. *$\overline{\Gamma\backslash X}^{\mathrm{BB}}$ carries a structure of complex analytic space, compatible with the complex structure on $\Gamma\backslash X$. In other words, $\overline{\Gamma\backslash X}^{\mathrm{BB}}$ is a compactification of $\Gamma\backslash X$ in the category of complex analytic varieties.*

6.4.4 Algebraic structure on $\overline{\Gamma\backslash X}^{\mathrm{BB}}$

Denote by $\overline{S}^{\mathrm{BB}} := \overline{\Gamma\backslash X}^{\mathrm{BB}}$. Consider the canonical line bundle (in the complex analytic category)

$$\omega_{\overline{S}^{\mathrm{BB}}}.$$

Poincaré–Eisenstein series defined before are global sections of $\omega_{\overline{S}^{\mathrm{BB}}}^{\otimes m}$. Since $\overline{S}^{\mathrm{BB}}$ is compact, it satisfies the descending chain condition for closed complex analytic subsets. So there exist finitely many global sections $E_0, \ldots, E_{N'}$ of $\omega_{\overline{S}^{\mathrm{BB}}}^{\otimes m}$ which separate points. Thus we get an injective analytic map

$$\varphi = [E_0 : \cdots : E_{N'}] \colon \overline{S}^{\mathrm{BB}} \longrightarrow \mathbb{P}^{N'}.$$

Theorem 6.4.8. *This map φ endows $\overline{S}^{\mathrm{BB}}$ with the structure of a normal complex projective variety. In particular, $\overline{\Gamma\backslash X}^{\mathrm{BB}}$ carries a structure of normal projective complex varieties which induces the complex analytic structure in Theorem 6.4.7.*

This theorem gives $\Gamma\backslash X$ a complex algebraic structure. Moreover, the complex algebraic structure on $\Gamma\backslash X$ is unique by the following theorem. Recall $D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc and let $D^{\times} := D \setminus \{0\}$ be the punctured disc.

Theorem 6.4.9. *Assume Γ is torsion-free. Then any holomorphic map $D^a \times (D^{\times})^b \to \Gamma\backslash X$ extends to a holomorphic map $D^a \times (D^{\times})^b \to \overline{\Gamma\backslash X}^{\mathrm{BB}}$.*

Chapter 7

Toroidal compactification

7.1 Background knowledge on toric varieties

Let k be a field. All varieties are defined over k.

Let T be a k-split algebraic torus.

Definition 7.1.1. *A* **toric variety with torus** T *is a variety V equipped with an open immersion $\phi\colon T \to V$ and an action T on V such that $t \cdot \phi(t') = \phi(tt')$ for all $t, t' \in T$.*

7.1.1 Affine toric varieties

Assume V is affine. The action of T on V induces an action of T on the ring of regular functions $k[V]$. So $k[V]$ (as a vector space) is a representation of T. For each $\chi \in X^*(T)$, define

$$k[V]_\chi := \{f \in k[V] : t \cdot f = \chi(t)f\} = \{f \in k[V] : f(\chi(t)v) = \chi(t)f(v), \text{ for all } v \in V \text{ and } t \in T\}.$$

Then $k[V] = \bigoplus_{\chi \in X^*(T)} k[V]_\chi$.

Lemma 7.1.2. *$k[V]_\chi \neq 0$ if and only if χ extends to a regular function on V.*

Proof. This lemma is clearly true because $k[T]$ equals the group algebra $k[X^*(T)]$. □

Corollary 7.1.3. *$S(V) := \{\chi \in X^*(T) : k[V]_\chi \neq 0\}$ is a semi-group, with the identity being the trivial character χ_0.*

Proof. It is easy to check $k[V]_{\chi_0} = k$, so $\chi_0 \in S(V)$. For $\chi_1, \chi_2 \in S(V)$, by Lemma 7.1.2 both χ_1 and χ_2 extend to a regular functions on V, and so does the product $\chi_1\chi_2$. So $\chi_1\chi_2 \in S(V)$ by Lemma 7.1.2. □

The following theorem is then easy to check.

Theorem 7.1.4. *The following categories are equivalent:*

(i) sub-semi-groups S of $X^*(T)$ of finite type which generate $X^*(T)$ as a group,

(ii) affine toric varieties with torus T.

For (i) to (ii), S is sent to $\mathrm{Spec} k[S]$, with $k[S] = \{\sum a_s s : a_s \in k,\ s \in S\}$. For (ii) to (i), V is sent to $S(V)$.

Among the sub-semi-groups of $X^*(T)$, the *saturate* ones (*i.e.* $(S \otimes \mathbb{Q}) \cap X^*(T) = S$) give rise to normal affine toric varieties.

Next, we want to turn to the *cocharacters* of T. Denote for simplicity by $X_* := X_*(T)$. Use $X_{*,\mathbb{Q}}$ (resp. $X_{*,\mathbb{R}}$) to denote $X_*(T) \otimes \mathbb{Q}$ (resp. $X_*(T) \otimes \mathbb{R}$).

Definition 7.1.5. *A subset $\sigma \subseteq X_{*,\mathbb{R}}$ is called a* **(rational) polyhedral cone** *if it satisfies one of the two equivalent conditions:*

- *σ is the intersection of finitely many rational semi-spaces, i.e. there exist $\lambda_1, \ldots, \lambda_m \in X^*(T) \otimes \mathbb{Q}$ such that $\sigma = \{x \in X_{*,\mathbb{R}} : \lambda_j(x) \geq 0 \text{ for all } j \in \{1, \ldots, m\}\}$.*
- *there exist $x_1, \ldots, x_m \in X_{*,\mathbb{Q}}$ such that $\sigma = \{\sum_{j=1}^m \alpha_j x_j : \alpha_j \in \mathbb{R}_{\geq 0}\}$.*

For any polyhedral cone σ, its dual is $\sigma^\vee = \{\lambda \in X^*(T) \otimes \mathbb{R} : \lambda(x) \geq 0 \text{ for all } x \in \sigma\}$. So σ contains a line in $X_{*,\mathbb{R}}$ if and only if $\sigma^\vee$ is contained in a hyperplane of $X^*(T) \otimes \mathbb{R}$.

Definition 7.1.6. *A* **face** *of a polyhedral cone σ is a subset of the form $\{x \in \sigma : \lambda(x) = 0\}$ for some $\lambda \in \sigma^\vee$.*

The intersection of two faces of σ is still a face, because $\{x \in \sigma : \lambda_1(x) = 0\} \cap \{x \in \sigma : \lambda_2(x) = 0\} = \{x \in \sigma : (\lambda_1 + \lambda_2)(x) = 0\}$.

Theorem 7.1.7. *The map $\sigma \mapsto V_\sigma := \mathrm{Spec} k[\sigma^\vee \cap X^*(T)]$ defines a bijection between:*

- *polyhedral cones in $X_{*,\mathbb{R}}$ which do not contain lines,*
- *isomorphic classes of normal affine toric varieties with torus T.*

Moreover, we have:

(1) For $\mu \in X_*$, we have $\mu \in \sigma \Leftrightarrow \lim_{t \to 0} \mu(t) \in V_\sigma$.

(2) V_σ is smooth if and only if $\sigma \cap X_*$ is generated by part of a $\mathbb{Z}$-bases of X_* (in which case $V_\sigma \simeq \mathbb{G}_{\mathrm{m}}^\bullet \times \mathbb{G}_{\mathrm{a}}^\bullet$).

(3) If $\sigma_1 \subseteq \sigma_2$, then there exists a morphism $V_{\sigma_1} \to V_{\sigma_2}$. This morphism is an open immersion if and only if σ_1 is a face of σ_2.

Example 7.1.8. *Consider the simplest example $T = \mathbb{G}_{\mathrm{m},k}$. Then $X_{*,\mathbb{R}} \simeq \mathbb{R}$. Polyhedral cones in $X_{*,\mathbb{R}}$ which do not contain lines are $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$, and $\{0\}$. In the first two cases $V_\sigma \simeq \mathbb{G}_{\mathrm{a},k}$ and in the third case $V_\sigma \simeq \mathbb{G}_{\mathrm{m},k}$.*

7.1.2 General toric varieties

Definition 7.1.9. *A* **fan** *Σ in $X_{*,\mathbb{R}}$ is a collection $\{\sigma\}$ of polyhedral cones such that:*

(i) If $\sigma \in \Sigma$ and $\sigma' \subseteq \sigma$ is a face, then $\sigma' \in \Sigma$;

(ii) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma' \in \Sigma$ and is a common face of σ and of σ'.

The **trivial fan** *consists of only the trivial cone.*

Each fan Σ gives arise to a toric variety V_Σ as follows: To each $\sigma \in \Sigma$ we associate V_σ as in Theorem 7.1.7, and then glue V_σ and $V_{\sigma'}$ along the common open subset $V_{\sigma \cap \sigma'}$.

Theorem 7.1.10. *The map $\Sigma \mapsto V_\Sigma$ defines a bijection between*

- *fans in $X_{*,\mathbb{R}}$ whose polyhedral cones do not contain lines,*
- *isomorphic classes of normal toric varieties with torus T.*

Moreover, V_Σ is a complete variety if and only if $X_{*,\mathbb{R}} = \bigcup_{\sigma\in\Sigma}\sigma$.

Example 7.1.11. *Continue with Example 7.1.8. If the fan $\Sigma = \{\mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}, \{0\}\}$, then we get $V_\Sigma \simeq \mathbb{P}^1_k$ by glueing $\mathbb{G}_{\mathrm{a},k}$ and $\mathbb{G}_{\mathrm{a},k}$ along their intersection $\mathbb{G}_{\mathrm{m},k}$.*

Definition 7.1.12. *A* **refinement** *of a fan Σ is a fan Σ' such that*

(i) each $\sigma' \in \Sigma'$ is contained in some $\sigma \in \Sigma$,

(ii) each $\sigma \in \Sigma$ is a finite union of some $\{\sigma'\} \subseteq \Sigma'$.

Let Σ and Σ' be two fans in $X_{*,\mathbb{R}}$. Condition (i) above implies that there exists a T-equivariant morphism $V_{\Sigma'} \to V_\Sigma$. Then the valuative criterion of properness implies: this morphism is proper if and only if Σ' is a refinement of Σ.

Theorem 7.1.13. *Each fan Σ admits a refinement Σ' such that $V_{\Sigma'}$ is a resolution of singularities of V_Σ. If V_Σ is complete, then we can find such an Σ' that $V_{\Sigma'}$ is smooth and projective.*

7.2 Toroidal compactifications of $\Gamma\backslash X$

Let $(\mathbf{G}, X)$ be a Shimura datum. By abuse of notation use X to denote a connected component. Let $\Gamma < \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup.

7.2.1 The algebraic torus associated with a rational analytic boundary component

Take a rational analytic boundary component F whose normalizer is $\mathbf{P}$.

Recall the diagram from Theorem 6.3.7, with $C(F)$ a cone in $U(F)(\mathbb{R})$ such that $X \simeq \mathcal{D} = \Phi_F^{-1}(C(F))$:

$$\begin{tikzcd}
C(F) & \subseteq & U(F)(\mathbb{R}) \\
X \simeq \mathcal{D} \arrow[u, "\Phi_F"] \arrow[dr] \arrow[ddr, "\pi_F"'] & \subseteq & \mathcal{D}(F) \arrow[u, "\Phi_F"'] \arrow[dl, "\pi'_F"', "\text{mod } U(F)(\mathbb{C})"] \\
& \mathcal{D}'(F) \arrow[d, "p_F"] & \\
& F &
\end{tikzcd} \tag{7.2.1}$$

Let $\Gamma_U := \Gamma \cap \mathbf{U}(F)(\mathbb{Q})$, and let

$$T_F := \Gamma_U \backslash U(F)(\mathbb{C}). \tag{7.2.2}$$

Then T_F is an algebraic torus, and $X_*(T_F) = \Gamma_U$ and $X_*(T_F)_\mathbb{R} = U(F)(\mathbb{R})$. Thus $C(F)$ is a cone in $X_*(T_F)_\mathbb{R}$.

7.2.2 The fibration on each rational analytic boundary

Take a rational analytic boundary component F whose normalizer is $\mathbf{P}$. It is tempting to take the quotient of $\mathcal{D}(F)$ by $\Gamma_F := \Gamma \cap \mathbf{P}(\mathbb{Q})$. It turns out that Γ_F too large! Instead, we consider the following short exact sequence

$$1 \to \Gamma_F^\circ \to \Gamma_F \to \overline{\Gamma}_F \to 1, \tag{7.2.3}$$

where $\Gamma_F^\circ := \{\gamma \in \Gamma_F : \gamma u \gamma^{-1} = u \text{ for all } u \in U(F)(\mathbb{R})\}$. We will do the quotient in two steps: quotient by Γ_F° and then by $\overline{\Gamma}_F$.

By Lemma 6.3.2 and Proposition 6.3.3, we have

$$\Gamma_F^\circ = \Gamma \cap (\mathbf{W}(F)\mathbf{G}_h(F)\mathbf{M}(F))(\mathbb{Q}) = \Gamma \cap (W(F)G_h(F))(\mathbb{R}).$$

Hence $\overline{\Gamma}_F$ is canonically isomorphic to (a finite-indexed subgroup of)

$$\Gamma_{l,F} := \Gamma \cap \mathbf{G}_l(F)(\mathbb{Q}) = \Gamma \cap G_l(F)(\mathbb{R}).$$

Quotient by Γ_F°

From (7.2.1) we obtain

$$\Gamma_F^\circ \backslash \mathcal{D}(F) \longrightarrow \mathcal{A}_\Gamma \longrightarrow \Gamma_F \backslash F =: S_F, \tag{7.2.4}$$

where $\mathcal{A}_\Gamma = (\Gamma_F^\circ / \Gamma_U) \backslash \mathcal{D}'(F)$ is an abelian scheme over S_F (which is an algebraic variety since it is a connected component of a Shimura variety).

The fibration $\Gamma_F^\circ \backslash \mathcal{D}(F) \longrightarrow \mathcal{A}_\Gamma$ is easily seen to be a T_F-torsor. We can "compactify" T_F using a fan in $X_*(T_F)_\mathbb{R} = U(F)(\mathbb{R})$ as in §7.1.2 (in particular Theorem 7.1.10 and 7.1.13). In our case, this fan must satisfy some properties so that we can do the quotient by $\overline{\Gamma}_F$.

$\Gamma_{l,F}$-admissible polyhedral decomposition of $C(F)$ and further quotient by $\Gamma_{F,l}$

Definition 7.2.1. *A* **$\Gamma_{l,F}$-admissible polyhedral decomposition of** $C(F)$ *is a fan* Σ_F *in* $X_*(T_F)_\mathbb{R} = U(F)(\mathbb{R})$ *satisfying the following properties:*

(i) Each polyhedral cone in Σ_F is contained in $\overline{C(F)}$ and is strongly convex.

(ii) $C(F) \subseteq \bigcup_{\sigma \in \Sigma_F} \sigma$, i.e. $C(F) = \bigcup_{\sigma \in \Sigma_F} (C(F) \cap \sigma)$.

(iii) For any $\gamma \in \Gamma_{l,F}$ and any cone $\sigma \in \Sigma_F$, we have $\gamma\sigma \in \Sigma_F$.

(iv) There are only finitely many classes of cones in Σ_F modulo $\Gamma_{l,F}$.

Now take Σ_F to be a $\Gamma_{l,F}$-admissible polyhedral decomposition of $C(F)$. By Theorem 7.1.10, we get a toric variety V_{Σ_F} which torus T_F. Consider

$$(\Gamma_F^\circ \backslash \mathcal{D}(F)) \times^{T_F} V_{\Sigma_F} \tag{7.2.5}$$

which is the quotient of $(\Gamma_F^\circ \backslash \mathcal{D}(F)) \times V_{\Sigma_F}$ by the diagonal action of T_F. Finally set

$$(\Gamma_F^\circ \backslash \mathcal{D}(F))_{\Sigma_F}$$

to be the interior of the closure of $\Gamma_F^\circ \backslash \mathcal{D}(F)$ in (7.2.5). Now $X \simeq \mathcal{D} = \Phi_F^{-1}(C(F))$ allows us to define

$$(\Gamma_F^\circ \backslash \mathcal{D})_{\Sigma_F} \subseteq (\Gamma_F^\circ \backslash \mathcal{D}(F))_{\Sigma_F}. \tag{7.2.6}$$

Finally, the $\Gamma_{l,F}$-admissibility of Σ_F allows to do the quotients

$$(\Gamma_F \backslash \mathcal{D})_{\Sigma_F} := \frac{(\Gamma_F^\circ \backslash \mathcal{D})_{\Sigma_F}}{\Gamma_{l,F}} \subseteq \frac{(\Gamma_F^\circ \backslash \mathcal{D}(F))_{\Sigma_F}}{\Gamma_{l,F}}. \tag{7.2.7}$$

7.2.3 Final conclusion

Definition 7.2.2. *A* **Γ-admissible polyhedral decomposition** *is a collection* $\{\Sigma_F\}_F$ *of* $\Gamma_{l,F}$*-admissible polyhedral decomposition of* $C(F)$ *for all rational analytic boundary components* F *satisfying the following properties:*

(i) If $F_1 = \gamma \cdot F_2$ *for* $\gamma \in \Gamma$*, then* $\Sigma_{F_1} = \gamma\Sigma_{F_2}$*.*

(ii) If F_2 *is contained in the boundary of* F_1 *(i.e.* $F_2 \subseteq \overline{F_1}$ *which implies* $C(F_1) \subseteq \overline{C(F_2)}$*), then* $\Sigma_{F_1} = \{\sigma \cap \overline{C(F_1)} : \sigma \in \Sigma_{F_2}\}$*.*

Now take a Γ-admissible polyhedral decomposition $\{\Sigma_F\}_F$, and set

$$\overline{\Gamma\backslash X}^{\mathrm{tor}}_{\Sigma} := \frac{\coprod(\Gamma_F\backslash\mathcal{D})_{\Sigma_F}}{\sim}. \tag{7.2.8}$$

Here the equivalence $\sim$ is defined as follows: Two points

$$x_1 \in (\Gamma_{F_1}\backslash\mathcal{D})_{\Sigma_{F_1}} \quad \text{and} \quad x_2 \in (\Gamma_{F_2}\backslash\mathcal{D})_{\Sigma_{F_2}}$$

are equivalent (*i.e.* $x_1 \sim x_2$) if and only if

(a) there exists a rational analytic boundary component F and some $\gamma \in \Gamma$ such that

$$F_1 \subseteq \overline{F} \quad \text{and} \quad \gamma F_2 \subseteq \overline{F};$$

(b) there exists a point $x \in (\Gamma_F\backslash\mathcal{D})_{\Sigma_F}$ which projects to x_1 and x_2 respectively under the natural projections.

Theorem 7.2.3. $\overline{\Gamma\backslash X}^{\mathrm{tor}}_{\Sigma}$ *is a compactification of* $\Gamma\backslash X$*, which dominants* $\overline{\Gamma\backslash X}^{\mathrm{BB}}$*. More precisely, there exists a natural morphism*

$$\overline{\Gamma\backslash X}^{\mathrm{tor}}_{\Sigma} \longrightarrow \overline{\Gamma\backslash X}^{\mathrm{BB}}$$

which is identity on $\Gamma\backslash X$*.*

Moreover, there exists a refinement of Σ *such that the morphism above is a resolution of singularities.*